UNIT 2 CONVERGENCE

Structure

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2.1 INTRODUCTION

In the last unit we defined metric spaces and discussed some related notions. In this unit we generalise the notion of convergent sequences and Cauchy sequences in the context of a metric space. You are already familiar with the notion of convergence. If you look closely at this concept for **R**, you will notice that we can define convergence of sequences in a metric space. The presence of metric (distance) defined on a metric space helps to extend this notion. In Sec. 2.2, we do this. Here, we also discuss the relationship between continuity and convergence. Then we talk about subsequences and bounded sequences.

In Sec. 2.3, we consider Cauchy sequences and discuss its relationship with convergence. Then we define the notion of completeness in a metric space and prove certain results which characterise complete metric spaces. We also discuss two important results for complete metric spaces known as Cantor's intersection and Baire category theorem.

Objectives

After studying this unit, you should be able to

- check whether a given sequence is convergent in a metric space;
- check when a given sequence is bounded;
- use the notion of continuity to check the convergence of a sequence;
- check whether a given sequence is Cauchy or not;
- state the relationship between Cauchy sequences and convergent sequences and apply it to check whether a sequence is convergent or not;
- define complete metric spaces and verify whether a metric space is complete or not; and
- explain Cantor's intersection theorem and Baire category theorem.

2.2 SEQUENCES AND CONVERGENCE

In this section, we define a sequence in metric space and discuss its convergence. You will see that the definitions and results for sequences of real numbers apply with proper modification to any metric space. We shall begin with some definitions.

Definition 1: Let X be any non empty set. A sequence in X is a function $s : N \to X$ where N is the set of all natural numbers. We denote the value of the function s at n by x_n rather than by s(n). We call x_n, the nth term of the sequence. The sequence itself is denoted by $\{x_n\}_{n \in N}$ or simply by $\{x_n\}$.

You are already familiar with sequences in **R**.

Can you give an example of a sequence in $\mathbf{R}^2$, $\mathbf{R}^3$ etc.?

For example, the sequence $\left\{\left(\frac{1}{n}, \frac{1}{n}\right), n \in \mathbf{N}\right\}$ is a sequence in $\mathbf{R}^2$ which is given by

$$(1,1), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{3}\right), \ldots$$

Here is an example of a sequence in C[0, 1].

For any $n \in \mathbf{N}$, define f_n on [0, 1] by

$$f_n(x) = x^n, x \in [0, 1]$$

Then $\{f_n\}$ is a sequence in C[0, 1].

Now we define convergence of a sequence.

Definition 2: Let $\{x_n\}$ be a sequence in a metric space (X,d) and x be an element of X. We say that $\{x_n\}$ converges to x in X if given any $\epsilon > 0$, there is a positive integer m (depending on ϵ) such that

$d(x_n, x) < \epsilon$ for all $n \geq m$

From the definition it is clear that the convergence of a sequence depends on the metric. So, the correct terminology would be x_n converges to x in X w.r.t. the metric. But we shall rather use the shorter form as given in the Definition 2.

Note that the condition $d(x_n, x) < \epsilon$ for all $n \geq m$ can be expressed as $x_n \in B(x, \epsilon)$ for all $n \geq m$, that is, from a certain stage onwards, all terms of the sequence lie in the ball $B(x, \epsilon)$.

This will be more clear to you if you observe that x_n converges to x is the same as the sequence of real numbers $d(x_n, x)$ converges to 0 in the context of **R**. We carry this comparison further.

Let us consider an example.

Example 1: Let us consider the sequence $\left\{\left(\left(\frac{1}{n}, \frac{1}{n}\right), n \in \mathbf{N}\right)\right\}$ in $\mathbf{R}^2$. We shall discuss the convergence of this sequence for different metrics on $\mathbf{R}^2$.

i) $X = \mathbf{R}^2$ with the standard metric d.
Under this metric, we have

$$\lim_{n\to\infty} d\left(\left(\frac{1}{n},\frac{1}{n}\right),(0,0)\right) = \lim_{n\to\infty}\sqrt{\frac{1}{n^2}+\frac{1}{n^2}}$$
$$= \lim_{n\to\infty}\frac{\sqrt{2}}{n}$$
$$= 0$$

This shows that $\left\{\left(\frac{1}{n},\frac{1}{n}\right)\right\}$ converges to (0, 0).

ii) $X = R^2$ with the discrete metric d. Then

$$\lim_{n\to\infty} d\left\{\left(\frac{1}{n},\frac{1}{n}\right),(0,0)\right\} = \lim_{n\to\infty} 1 = 1.$$

This shows that $\left\{\left(\frac{1}{n},\frac{1}{n}\right)\right\}$ does not converge to (0, 0). In fact we can see that it does not converge to any $(x, y) \in R^2$.

iii) $X = R^2$ with the taxicab metric d_1 (see Sec 1, Unit 1). The sequence $\left\{\left(\frac{1}{n},\frac{1}{n}\right)\right\}$ converges to (0, 0) w.r.t. this metric.

iv) $X = R^2$ with the metric d_∞ (see Sec.1, Unit 1). The sequence $\left\{\left(\frac{1}{n},\frac{1}{n}\right)\right\}$ converges to (0, 0) in X.

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Thus, the convergence of a sequence depends on what metric we use. The following result explains the convergence of sequences in discrete metric spaces.

Proposition 1: Let (X, d) be a discrete metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ converges to x w.r.t. the discrete metric if and only if there exists $m \in N$ such that $x_n = x$ for all $n \geq m$.

The proof of this result is clear from the definition of convergence and that of the metric.

The above result says that the sequence in a discrete metric space must be eventually constant in order to converge.

Now, we prove a fundamental result about convergent sequences.

Theorem 1: Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X. Suppose that there are two points $x, y \in X$ such that $\{x_n\}$ converges to x and $\{x_n\}$ also converges to y in X. Then we have $x = y$.

Proof : We have to show that $x = y$. Suppose $x \neq y$. Then $d(x, y) > 0$.

Let ϵ be a positive real number less than or equal to $\frac{1}{2}d(x, y)$.

Since $\{x_n\}$ converges to x there exists a positive real number m such that $d(x_n, x) < \epsilon$ if $n \geq m$. Further since $\{x_n\}$ converges to y there exists a positive

real number p such that $d(x_n, y) < \epsilon$ if $n \geq p$. Choose $n = \max\{m, p\}$. Then $d(x_n, x) < \epsilon$, $d(x_n, y) < \epsilon$. So $d(x, y) \leq d(x, x_n) + d(x_n, y)$
$= d(x_n, x) + d(x_n, y) < \epsilon + \epsilon = 2\epsilon \leq d(x, y)$

This is a contradiction. Hence x = y. □

In view of the above theorem, we introduce the following notation:

Note: If a sequence $\{x_n\}$ converges to x in a metric space (X, d), then we denote it by $x_n \to x$. We also write $\lim_{n\to\infty} x_n = x$. Then x is called the limit of the sequence $\{x_n\}$.

Now we define a subsequence of a sequence.

Definition 3: Let $\{x_n\}$ be a sequence in X. Let $n_1, n_2, n_3 \ldots$ be natural numbers such that $n_1 < n_2 < n_3 \ldots$. For $k \in \mathbf{N}$, let $y_k = x_{n_k}$. Then the sequence $\{y_k\}_{k\in\mathbf{N}} = \{x_{n_k}\}_{k\in\mathbf{N}}$ is called the subsequence of the sequence $\{x_n\}_{n\in\mathbf{N}}$.

In short, we may say that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. We notice that the sequence $\{n_k\}$ in the above definition satisfies $n_k \geq k$ for all $k \in \mathbf{N}$. Also $\{n_k\}$ may be thought of as a subsequence of $\{n\}$ of natural numbers.

Theorem 2: If $\{x_n\}$ converges to x in X, then every subsequence of $\{x_n\}$ also converges to x.

Proof : Let $\{x_n\}$ be a convergent sequence in a metric space X with limit x. Let $\{x_{n_k}\}$ be a subsequence of this sequence. Let $\epsilon > 0$ be given. Then there exists a positive integer m such that $d(x_n, x) < \epsilon$ for all $n \geq m$. Since $n_k \geq k$ for every positive integer k, we get that $d(x_{n_k}, x) < \epsilon$ for all $k \geq m$. Hence $x_{n_k} \to x$. □

The converse of this theorem is not true. A sequence may not be convergent but may have a convergent subsequence. For example, $\{(-1)^n\}$ is not a convergent sequence in **R** but has many convergent subsequences. For instance, the subsequence obtained by taking $n_k = 2k, k \in \mathbf{N}$ in Definition 3 has all the terms equal to 1 and, therefore, converges to 1.

We discuss some interesting examples below.

Example 2: Let $\{x_n\}$ be a sequence in $\mathbf{R}^k$. Let, for $n \in \mathbf{N}$, $x_n = (x_n^1, x_n^2, \ldots, x_n^k)$. Then for each $j = 1, 2, \ldots, k$, $\{x_n^j\}$ is a sequence in **R**. We shall show that $\{x_n\}$ converges in R^k if and only if $\{x_n^j\}$ converges in **R** for each $j = 1, 2, \ldots, k$.

Let $\{x_n\}$ converge to $x \in \mathbf{R}^k$. Let $x = (x^1, x^2, \ldots, x^k)$. Let us fix a j, say j_0 with $1 \leq j_0 \leq k$. Let $\epsilon > 0$ be given. Then there exists a positive integer m such that $d(x_n, x) < \epsilon$ for all $n \geq m$. Then

$$\left|x_n^{j_0} - x^{j_0}\right| \leq \left(\sum_{j=1}^{k} |x_n^j - x^j|^2\right)^{1/2} = d(x_n, x) < \epsilon \text{ if } n \geq m$$

This shows that $\{x_n^{j_0}\}$ converges to x^{j_0}. Since j_0 is arbitrary, we get that $\{x_n^j\}$ converges to x^j for all $j = 1, \ldots k,$.

Conversly let $\{x_n^j\}$ converge to $x^j \in \mathbf{R}$ for each $j = 1, 2, \ldots, k$. Let $\epsilon > 0$ be given. Then for each $j = 1, 2, \ldots, k$ there exists a positive integer m_j such that

$|x_n^j - x^j| < \frac{\epsilon}{\sqrt{k}}$ if $n \geq m_j$. Let $m = \max(m_1, m_2, \ldots, m_k)$ and let $x = (x^1, x^2, \ldots, x^k$. Then $x \in \mathbf{R}^k$. Also, for $n \geq m$

$$d(x_n, x) = \left(\sum_{j=1}^{k} |x_n^j - x^j|^2\right)^{\frac{1}{2}} < \left(\sum_{j=1}^{k} \frac{\epsilon^2}{k}\right)^{\frac{1}{2}} = \epsilon$$

This shows that $x_n \to x$.

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Next, we shall consider convergent sequences in $C[0, 1]$. Here we make an important observation.

Let $\{f_n\}$ be a sequence in $(C[0, 1], d_\infty)$. What does it mean to have $f_n \to f$? By definition it means that for $\epsilon > 0$ $\exists$ $m \in \mathbf{N}$ such that $d(f_n, f) < \epsilon$ for $n \geq m$ i.e.

$$\sup |f_n(x) - f(x)| < \epsilon \quad \text{for } n \geq m.$$

But this is same as saying that $f_n \to f$ uniformly on [0, 1].

In particular if $f_n \to f$ in $(C[0, 1], d_\infty)$, then for each x in [0, 1], we have $f_n(x) \to f(x)$ in $\mathbf{R}$.

Example 3: Consider the metric space $(C[0, 1], d_\infty$ (Refer Example 10 in Sec.1 of Unit 1).

Consider the sequence $\{f_n\}$ in $C[0, 1]$, given by

$$f_n(x) = x^n, \; x \in [0, 1].$$

We will show that $\{f_n\}$ is not convergent in C[0, 1] with respect to the sup metric. Let, if possible, $\{f_n\}$ is convergent in $C[0, 1]$, say to f. Then for $x \in [0, 1]$,

$$f_n(x) \to f(x).$$

We observe that if $0 \leq x < 1$, then for each x, $f_n(x) = x^n \to 0$ as $n \to \infty$. Also $f_n(1) = 1$ for all n, so that $f_n(1) \to 1$ as $n \to \infty$. Thus,

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}$$

But f is not continuous at x = 1. Therefore $f \notin C[0, 1]$. This is a contradiction. Hence $\{f_n\}$ is not convergent in C[0, 1].

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Example 4: Consider C[0, 1] with the metric defined by the Riemann integral.

Consider the sequence $\{f_n\}$ which we have discussed in Example 3 above.

Take $f(x) = 0, \forall x \in [0, 1]$. Then $f \in C[0, 1]$.

We claim that $f_n \to f$ in C[0, 1]. For that we note that

$$d(f_n, f) = \int_0^1 |f_n(x) - f(x)| dx = \int_0^1 x^n \, dx = \frac{1}{n+1} \to 0.$$

Hence $f_n \to f$ in $C[0, 1]$.

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In the above two examples we could check the convergence without much difficulty. Are we able to do the same for other sequences in other metric spaces? We need to look for a criteria for checking the convergence which saves us from the laborious process of checking for each $x \in X$. In the next section, we shall discuss this.

Next, we will give connection between continuity and convergence.

Theorem 3: Let (X, d) and (Y, d') be metric spaces and f be a function from X to Y. Let $x \in X$. Then the following are equivalent

a) f is continuous at a point $x_0 \in X$.

b) For every sequence (x_n) in X converging to x_0, we have $f(x_n) \to f(x_0)$ in Y.

Proof: We shall first show that (a) $\Rightarrow$ (b).

Consider a sequence $\{x_n\}$ in X such that $x_n \to x_0$ and f is a function continuous at $x_0 \in X$. Continuity of f at x_0 implies that, for a given $\epsilon > 0$ there exists a δ such that

$$d'(f(x), f(x_0)) < \epsilon \text{ whenever } d(x, x_0) < \delta. \quad (1)$$

For this δ since $x_n \to x_0$, there exists an $n \in N$ such that for all $n \geq N, d(x_n, x_0) < \delta$. Then from (1) we have

$$d'(f(x_n), f(x_0)) < \epsilon \text{ for all } n \geq N$$

This shows that $\{f(x_n)\}$ converges to $f(x_0)$ in Y.

Next we show that (b) $\Rightarrow$ (a). Suppose that f is not continuous at x_0. Then $\exists$ an $\epsilon > 0$ such that for every $\delta > 0$ and $x \in X, d(x, x_0) < \delta$, but

$$d'(f(x), f(x_0)) \geq \epsilon.$$

For each $n \in N$, and $\delta = 1/n$. Then since $x_n \to x$, we can find $\{x_n\}$ in X such that $d(x_n, x_0) < \delta = \frac{1}{n}$ and $d'(f(x_n), f(x_0)) \geq \epsilon$.

That means $x_n \to x_0$, but $\{f(x_n)\}$ does not converges to $f(x_0)$, which is a contradiction. Hence the result.

Here are some exercises for you.

E1) A sequence $\{x_n\}$ in a metric space (X, d) is said to be a bounded sequence if the set $\{x_n : n \in N\}$ is a bounded set. Show that every convergent sequence in a metric space is a bounded sequence. Is the converse true?

E2) Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a metric space converging to x and y respectively. Show that $d(x_n, y_n) \to d(x, y)$.

E3) Let E be a subset of a metric space X and $x \in X$. Show that $x \in \overline{E}$ if and only if there exists a sequence $\{x_n\}$ in E converging to x.

In the next section, we shall discuss two other important concepts related to convergence.

2.3 CAUCHY SEQUENCES AND COMPLETENESS

We shall start with a definition.

Definition 4: A sequence $\{x_n\}$ in a metric space X is said to be a Cauchy sequence if given $\epsilon > 0$, there exists a positive integer m (depending on ϵ) such that $d(x_n, x_k) < \epsilon$ for all $n, k \geq m$.

Example 5: Consider X = C[0, 1] with the metric defined by Riemann integral. For each positive integer n let

$$f_n(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq \frac{1}{2} \\ -2^n\left(x - \frac{1}{2}\right) + 1, & \text{if } \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{2^n} \\ 0, & \text{if } \frac{1}{2} + \frac{1}{2^n} \leq x \leq 1 \end{cases}$$

Then $\{f_n\}$ is a sequence in X. Let n and p be positive integers such that $n > p$. Then $d(f_n, f_p) = \int_0^1 |f_n(x) - f_p(x)|dx = \frac{1}{2^{p+1}} - \frac{1}{2^{n+1}}$. Now, given $\epsilon > 0$ we can find a positive integer N such that $\left|\frac{1}{2^{p+1}} - \frac{1}{2^{n+1}}\right| < \epsilon$ for all $n, p \geq N$. This shows that $\{f_n\}$ is a Cauchy sequence in X.

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Next, we shall prove some useful theorems about Cauchy sequences.

Theorem 4: Every Cauchy sequence is a bounded sequence.

Proof : Let us take $\epsilon = 1$. Since $\{x_n\}$ is Cauchy, there exists a positive integer N such that $d(x_n, x_m) < 1$ for $n, m \geq N$. In particular, $x_n \in B(x_N, 1)$ for all $n \geq N$. Let

$$R = \max\{1, d(x_j, x_N) : 1 \leq j \leq N\}$$

Then $x_n \in B(x_N, R)$ for all $n \in N$. This shows that $\{x_n\}$ is bounded. □

Remark 1: The converse is not true. For example, $\{(-1)^n\}$ is a bounded sequence in **R** but is not a Cauchy sequence in **R**.

Theorem 5: Every convergent sequence is a Cauchy sequence.

Proof : Suppose that (x_n) is a sequence in X which converges to a point x of X, and let $\epsilon > 0$ be given. There exists a positive integer N such that $x_n \in B\left(x, \frac{\epsilon}{2}\right)$

for all $n \geq N$. Thus, if $m, n \geq N$ then $d(x, x_n) < \frac{\epsilon}{2}$ and $d(x, x_m) < \frac{\epsilon}{2}$ so that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x) + d(x, x_m) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence (x_n) is a Cauchy sequence in X. □

Remark 2: The converse of the above theorem is not true. For example, consider $X = \mathbf{R}^+$, the set of all positive real numbers with the usual metric. In this metric space the sequence $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence but not a convergent sequence.

The following example shows that there are spaces in which every Cauchy sequence converges.

Example 6: Let $\{x_n\}$ be a Cauchy sequence in a discrete metric space. Choose $\epsilon = \frac{1}{2}$. Then there exists a positive integer N such that $d(x_n, x_k) < \frac{1}{2}$ for all $n, k \geq N$. Since d is discrete metric, we have $x_n = x_k$ for all $n, k \geq N$. In particular $x_n = x_N$ for all $n \geq N$. Thus $\{x_n\}$ converges to x_N.

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The above example shows that in a discrete metric space every Cauchy sequence is a convergent sequence.

Example 7: Let $X = C[0, 1]$. Consider $\{f_n\}$ discussed in Example 5. There we have shown that the sequence is Cauchy. Now we will show that the sequence is not convergent.

We shal first split the domain into $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$. Then we first consider f_n's on the domain $\left[0, \frac{1}{2}\right]$. Since $\{f_n\}$ is Cauchy in $C[0, 1]$, it is Cauchy $C\left[0, \frac{1}{2}\right]$. Therefore, it is convergent in $C\left[0, \frac{1}{2}\right]$. But

$$f_n(x) = 1, \ \forall\, x \in \left[0, \frac{1}{2}\right].$$

Therefore the limit, say g, of the $\{f_n\}$, will be such that

$$g(x) = 1, \ \forall\, x \in \left[0, \frac{1}{2}\right].$$

Next, we consider f_n's on the domain $\left[\frac{1}{2}, 1\right]$.

Define a function f on $\left[\frac{1}{2}, 1\right]$ such that

$$f(x) = 0 \ \forall\, x \in \left[\frac{1}{2}, 1\right].$$

Then we have

$$\int_{\frac{1}{2}}^{1} |f_n(x) - f(x)|dx = \frac{1}{2^{n+1}} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

This shows that $d(f_n, f) \to 0$ as $n \to \infty$. Therefore $f_n \to f$ in $\left[\frac{1}{2}, 1\right]$.

Thus we got that $f_n \to g$ in $C\left[0, \frac{1}{2}\right]$ and $f_n \to f$ in $C\left[\frac{1}{2}, 1\right]$ and $f \neq g$. This is not possible since $\{f_n\}$ is convergent in $C[0, 1]$ and, therefore, cannot have different limits in its subspaces.

Thus, we conclude that $\{f_n\}$ is not convergent in $C[0, 1]$.

Hence the result.

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The following theorem gives a criteria for Cauchy sequences to converge.

Theorem 6: A Cauchy sequence in a metric space is convergent if and only if it has a convergent subsequence.

Proof : Let $\{x_n\}$ be a Cauchy sequence in a metric space X. If it is convergent then we can take it as convergent subsequence.

Conversely suppose $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ convergent to $x \in X$. Let $\epsilon > 0$ be given. Since $\{x_n\}$ is a Cauchy sequence, there exists a positive integer m_0 such that $d(x_n, x_p) < \epsilon/2$ for all $n, p \geq m_0$. Since $x_{n_k} \to x$, there exists a positive integer m_1 such that $d(x_{n_k}, x) < \epsilon/2$ for all $k \geq m_1$. Let $m = \max(m_0, m_1)$. Since $m \geq m_1$, $d(x_{n_m}, x) < \epsilon/2$. Since $n_m \geq m \geq m_0$, $d(x_n, x_{n_m}) < \epsilon/2$ for all $n \geq m_0$. From these it follows that

$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_{n_m}) + d(x_{n_m}, x) \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

for all $n \geq m_0$

Hence $x_n \to x$. □

In Theorem 1, we saw that the continuous image of a convergent sequence is a convergent sequence. But the continuous image of a Cauchy sequence may not be Cauchy. The next example illustrates this point.

Example 8: Consider **R** with the standard metric. Consider the map $f : (0, 1] \to \mathbf{R}$ defined by

$$f(x) = \frac{1}{x}, 0 < x \leq 1.$$

Then f is continuous.

For each positive integer n, put $x_n = \frac{1}{n}$. Then $\{x_n\}$ is a Cauchy sequence in $(0, 1]$ and $f(x_n) = n$ for each positive integer n. So $\{f(x_n)\}$ is not a Cauchy sequence in **R**.

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Next theorem shows that the uniformly continuous image of a Cauchy sequence is always a Cauchy sequence.

Theorem 7: Let(X, d_1) and (Y, d_2) be two metric spaces and $f : X \to Y$ be a uniformly continuous function. If $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Y.

Proof : Let $\epsilon > 0$ be given. Since f is uniformly continuous, $\exists$ a $\delta > 0$ such that

$$d_2(f(x), f(y)) < \epsilon \tag{2}$$

for all x, y in X such that $d_1(x, y) < \delta$.

Since $\{x_n\}$ is a Cauchy sequence in X, there exists a positive integer N such that

$$d_1(x_n, x_m) < \delta, \text{ for all } m, n \geq N.$$

Therefore by (2), we have

$d_2(f(x_n), f(x_m)) < \epsilon$ for all $m, n \geq N$. Hence $\{f(x_n)\}$ is a Cauchy sequence in Y. □

The metric spaces in which every Cauchy sequence converges are given a special name as given in the following definition.

Definition 5: A metric space X is said to be a **complete metric space** if every Cauchy sequence in X is convergent in X.

You may recall from your undergraduate course (**IGNOU course MTE-09**) that **R** is complete. Further, we have shown in Example 6 above that a discrete metric space is complete.

Let us see some more examples.

Example 9: Consider the space of all positive real numbers with the standard metric, In this space $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence but is not a convergent sequence. Therefore, this space is not complete.

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In the next theorem we show that the Euclidean space $\mathbf{R}^n$ is complete.

Theorem 8: $\mathbf{R}^n$ is complete.

Proof : (Sketch of the proof): We have shown in Example 2, Section 1 that a sequence in $\mathbf{R}^n$ is convergent if and only if the corresponding coordinate sequences converge in **R**. We can modify the argument to show that sequence (x_n) is Cauchy in $\mathbf{R}^k$ iff for each $j = 1, \ldots, k$ the coordinate sequences $\{x_n^j\}_{j=1}^{\infty}$ is Cauchy in **R**. We can then use the completeness of **R** to complete the proof. □

Theorem 9: Let X be a complete metric space and $E \subseteq X$. Then E is a complete metric space with the induced metric if and only if E is a closed subset of X.

Proof : Let E be a complete metric space with the induced metric. Let x be a limit point of E. Then for each positive integer n, $B\left(x, \frac{1}{n}\right)$ must contain a point of E say x_n other than x. Obviously the sequence $\{x_n\}$ converges to x in X and $\{x_n\}$ is a Cauchy sequence in E. Since E is complete metric space, $\{x_n\}$ must converge to an element in E. Hence $x \in E$. This shows that E contains all its limit points. So E is a closed subset of X.

Conversely, let E be a closed subset of X. Let $\{x_n\}$ be a Cauchy sequence in E. It is then a cauchy sequence in X also. As X is a complete metric space, $\{x_n\}$

must converge to x in X. Suppose $x \notin E$, then $x \in E^c$. As E^c is an open subset of X (because E is a closed subset of X), $B(x,r) \subseteq E^c$ for some $r > 0$. As $x_n \to x$, there exists a positive integer m such that $x_n \in B(x,r)$ for all $n \geq m$. In particular $x_m \in B(x,r) \subseteq E^c$, a contradiction as $x_m \in E$. Hence $x \in E$ and therefore $\{x_n\}$ is convergent in E. This shows that E is a complete metric space. □

Here are some exercises for you.

E4) Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in a metric space X. Show that the sequence $\{d(x_n, y_n)\}$ converges in R.

E5) A Cauchy sequence is convergent if and only if it has a convergent subsequence.

Next, we define certain sets in general metric spaces. We are discussing these sets here because of their connection with completeness. In fact these sets arose as an extension of the property of **R** that the set of rationals Q is dense in **R** and it is not complete.

Definition 6: A subset A of a metric space (X, d) is said to be **dense** in X if $\overline{A} = X$.

As we stated above we have Q is a dense in **R**.

You will see that in the space $C[0, 1]$ with sup metric, the set P consisting of all the real polynomials restricted to $[0, 1]$ is dense in $C[0, 1]$.

Now, we shall discuss a theorem.

Theorem 10: (Cantor's Intersection Theorem): Let $\{F_n\}$ be a sequence of non-empty closed subsets of a complete metric space X such that $F_n \supseteq F_{n+1}$ for each positive integer n and $d(F_n) \to 0$. Let $F = \cap_{n=1}^{\infty} F_n$. Then F is a singleton, i.e., it contains exactly one element of X.

Proof: Let $F = \bigcap_{n=1}^{\infty} F_n$. Since $F \subseteq F_n$ we have $d(F) \leq d(F_n)$ for each positive integer n. As $d(F_n) \to 0$, we get $d(F) \leq 0$. So F cannot contain more than one element. Thus, the theorem is proved if we show that $F \neq \phi$. Since F_n is not empty, we can choose an element $x_n \in F_n$. We, thus, get a sequence $\{x_n\}$ in X. Let $\epsilon > 0$ be given. Since $d(F_n) \to 0$, there exists a positive integer m such that $d(F_n) < \epsilon$ if $n \geq m$. Let $n \geq k \geq m$. Then

$$F_n \subseteq F_k \Rightarrow \{x_n, x_k\} \subset F_k \Rightarrow d(x_n, x_k) \leq d(F_k) < \epsilon.$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. As X is a complete metric space, $\{x_n\}$ converges to some $x \in X$. We shall now show that $x \in F_n$ for every n, which in turn will give $x \in F$. Let, if possible, $x \notin F_k$ for some fixed positive integer k. Then $x \in F_k^c$. Since F_k is a closed subset of X, F_k^c is an open subset of X, so there exists $r > 0$ such that $B(x,r) \subseteq F_k^c$. As $x_n \to x$, there exists a positive integer p such that $d(x_n, x) < r$ if $n \geq p$, that is, $x_n \in B(x,r)$ if $n \geq p$. Choose $m = \max(p, k)$. Then $x_n \in B(x,r) \subseteq F_k^c$ and $x_n \in F_n \subseteq F_k$. This is a contradiction. This proves that $x \in F_k$. Since this is true for all k, we get that $x \in F_n$ for all n. Thus $x \in F$. Hence the result.

Example 10: In this example, we show that the set F in the Cantor's intersection theorem may be empty if the hypothesis $d(F_n) \to 0$ is dropped.

* * *

Example 11: Let $X = \mathbf{R}$ and $F_n = \{x \in \mathbf{R} : x \geq n\}$. Then X is a complete metric space and $\{F_n\}$ is a decreasing sequence of non-empty closed subsets of X. Also $d(F_n) = \infty$ for each n so that the condition $d(F_n) \to 0$ is not satisfied here. Also we have $F = \bigcap_{n=1}^{\infty} F_n = \phi$. Hence the claim.

* * *

Example 12: Here we give an example to show that the set F in the Cantor's intersection theorem may be empty if the hypothesis that each F_n is a closed subset of X is dropped.

Let $X = \mathbf{R}$ and $F_n = \{x \in \mathbf{R} : 0 < x \leq \frac{1}{n}\}$

Then X is a complete metric space and $\{F_n\}$ is a decreasing sequence of nonempty subsets such that of X such that $d(F_n) = \frac{1}{n} \to 0$. But F_n is not a closed subset of R. Now we have $F = \bigcap_{n=1}^{\infty} F_n = \phi$. Hence the claim.

* * *

Example 13: In this example, we show that the set E in the Cantor's intersection theorem may be empty if the condition that X is a complete metric space is dropped.

Let X be the metric space of all positive real numbers. Then X is not complete. Let $F_n = \left\{x \in \mathbf{R} : 0 < x \leq \frac{1}{n}\right\}$. Then $\{F_n\}$ is a decreasing sequence of non-empty closed subset of X and $d(F_n) = \frac{1}{n} \to 0$. Here also we have $F = \bigcap_{n=1}^{\infty} F_n = \phi$. Hence the claim.

* * *

Next we shall prove another theorem related to completeness property.

Theorem 11: Let (X, d) be a complete metric space and $\{U_n\}$ a countable collection of dense open subsets of X. Then $\bigcap U_n$ is not empty.

Proof : We first note that if $x \in X$ and $0 < r < s$, then

$$\overline{B(x, r)} \subseteq B[x, r] \subseteq B(x, s,) \tag{3}$$

We shall often use this in the proof.

Since U_1 is dense in X $U_1 \neq \phi$ and $x_1 \in O_1$. Then $\exists\, r_1 > 0$, $r_1 < \frac{1}{2}$ such that $B_1 = B(x_1, r_1) \subseteq U_1$. Since U_1 is dense, there must be a point x_2 in $U_2 \cap B$. Clearly

$$s_1 = r_1 - d(x_1, x_2) > 0$$

Since U_2 is open $\exists\ s > 0$ such that $B(x_2, s) \subseteq U_2$. Let $0 < r_2 < \min\left\{\frac{1}{2^2}, s\right\}$ and take $B_2 = B(x_2, r_2)$. Then by (3)

$$\overline{B_2} \subseteq B(x_2, s) \subseteq U_2.$$

Also by (3)

$$\overline{B_2} \subseteq B(x_2, s_1) \subseteq B(x_1, r_1) = B_1 \subseteq U_1.$$

Further $d(\overline{B_2}) \leq \frac{1}{2}$.

Proceeding inductively, we get a $\{B_n\}$ such that $\overline{B_n} \subset B_{n-1}, B_n \subseteq U_n$ and $d(B_n) \leq \frac{1}{2^{n-1}}$. Let $F_n = \overline{B_n}$ for $n \in N$. Then by Cantor intersection theorem, $\cap F_n$ contain exactly one element, say x. Then $x \in U_n \forall n$ and, therefore, $\cap U_n$ is not empty. Hence the result. □

A dense set is sometimes called an everywhere dense set.

In the next definition, we consider an opposite concept, namely of a **nowhere dense** set.

Definition 7: A set E in a metric space X is said to be nowhere dense if $\overline{E}^0 = \phi$.

Note that this is equivalent to saying that $\overline{E}$ contains no open ball and, therefore, $\overline{E}^C$ is dense in X. For example, the set of integers is nowhere dense in R.

In terms of nowhere dense sets, we have an interesting corollary of Theorem 10, which in fact is only a restatement of Theorem 10. This theorem is known as Baire Category Theorem.

Theorem 12: (Baire Category Theorem): A complete metric space is not the union of a countable collection of nowhere dense sets.

Proof : Let $\{E_n\}$ be a countable collection of nowhere dense sets. Then $U_n = \overline{E}_n$ is a countable collection of dense open sets, and, therefore, by Theorem 11 there is a point $x \in \bigcap U_n$. But this means that $x \notin \bigcup E_n$ □

Why don't you try these exercises now.

E6) What are the dense subsets of a discrete metric space?

E7) Show that a closed set is nowwhere dense if and only if it contains no open set.

E8) Check whether the set $\left\{1, \frac{1}{2}, \frac{1}{3} \ldots\right\}$ is nowhere dense or not?

E9) Give an example of a set which is neither dense nor nowhere dense.

The two theorems which we have covered in this section are important theorems in Mathematical Analysis.

With this we come to an end of this unit. We shall summarise the points covered in this unit.

2.4 SUMMARY

In this unit, we have covered the following points:

1. We defined a sequence in a metric space (X, d) and discussed its convergence.
2. We defined subsequences of a sequence and have shown the relationship between convergence of a sequence and its subsequences.
3. We have shown the connection between continuity and convergence. "f is continuous iff $x_n \to x$ implies that $f(x_n) \to f(x)$".
4. We defined Cauchy sequences and explained the connection between Cauchy sequences and convergence.

   A Cauchy sequence is convergent if and only if it has a convergent subsequence.
5. We defined complete metric spaces.

   A metric space (X, d) is complete if every Cauchy sequence in X is convergent in X.
6. We discussed two important theorems and explained the importance of them.
   1) Cantor's Intersection Theorem
   2) Baire Category Theorem

2.5 HINTS/SOLUTIONS

E1) Let $\{x_n\}$ be a sequence in X such that $x_n \to x$. Take $\epsilon = \frac{1}{2}$. Then given $\epsilon > 0, \exists$ N such that

$$d(x_n, x) \leq \epsilon \; \forall \; n \geq N.$$

For any element x_n in X we have

$$\begin{aligned} d(x_n, x_N) &\leq d(x_n, x) + d(x, x_N) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \; \forall \; n, \geq N \\ &< \epsilon \end{aligned} \tag{4}$$

Consider the elements $x_1, x_2, \ldots, x_N$ in X.

Let $M = \max\{\epsilon, d(x_i, x_N), i = 1, \ldots, N-1\}$

Then,

$$d(x_i, x_N) \leq M \; \forall \; i = 1, \ldots, N-1. \tag{5}$$

From (4) and (5), we get that

$$d(x_i, x_N) \leq M \; \forall \; i < N \tag{6}$$

Thus

$$x_n \in B[x_N, M]$$

Hence $\{x_n\}$ is bounded

The converse is not true. For example, consider the sequence $\{x_n\}$ where $x_n = \frac{1}{n}$ in $X =]0, 1]$. Then $\{x_n\}$ is a bounded sequence and is not convergent in]0, 1] since $x_n \to 0$ and $0 \notin]0, 1]$.

E2) Put $a_n = d(x_n, y_n)$. Then we note that

$$\begin{aligned} |a_n - a_m| &= |d(x_n, y_n) - d(x_m, y_m)| \\ &= |d(x_n, y_n) - d(x_m, y_n)| + |d(x_m, y_n) - d(x_m, y_m)| \quad (7) \\ &\leq d(x_n, x_m) + d(y_n, y_m) \end{aligned}$$

Now let $\epsilon > 0$ be given. Since $\{x_n\}$ is a Cauchy sequence, $\exists$ a positive integer N_1 such that

$$d(x_n, x_m) < \frac{\epsilon}{2} \text{ for all } m, n \geq N_1. \quad (8)$$

Similarly, $\exists$ a positive integer N_2 such that

$$d(y_n, y_m) < \frac{\epsilon}{2} \text{ for all } m, n \geq N_2 \quad (9)$$

Let N = Maximum of $\{N_1, N_2\}$.

Combining (7), (8) and (9) we get that,

$$|a_n - a_m| \leq d(x_n, x_m) + d(y_n, y_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for all } n, m \geq N.$$

This shows that $\{d(x_n, y_n)\}$ is Cauchy sequence in **R**. Since **R** is complete, $\{d(x_n, y_n)\}$ is convergent.

E3) Let $x \in \overline{E}$. By definition, every open ball with centre x intersects E. In particular

$$B\left(x, \frac{1}{n}\right) \cap E \neq \phi \; \forall \, n.$$

Let $x_n \in B\left(x, \frac{1}{n}\right) \cap E$ for each n. Then $\{x_n\}$ is a sequence in E. Also

$$d(x_n, x) < \frac{1}{n} \; \forall \, n$$

This implies that $d(x_n, x) \to 0$ or $x_n \to x$.

Conversely, let $\{x_n\}$ be a sequence in E such that $x_n \to x$. We have to show that $x \in \overline{E}$.

Let B(x, r) be an open ball with centre x. Since $x_n \to x$, given $r > 0 \; \exists \; m \in \mathbf{N}$ such that $d(x_n, x) < r \; \forall \, n \geq m$. This implies that $x_n \in B(x, r) \; \forall \, n \geq m$. That is $B(x, r) \cap E \neq \phi$. Hence $x \in \overline{E}$.

E4) **Hint:** This follows from the definition.

E5) **Hint:** Let $\{x_n\}$ be a Cauchy sequence in X. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ which converges to $x \in X$.

Let $\epsilon > 0$ be given. Choose $M \in N$ such that

$$d(x_n, x_m) < \frac{\epsilon}{z}$$

for all $n, m \geq M$ (using the fact that $\{x_n\}$ is a Cauchy sequence). Then chose $p \in N$ such that $n_p \geq M$ and

$$d\,(x_{n_i}) < \frac{\epsilon}{z}$$

for all $n_i \geq n_p$ (using the fact that $n_1 < n_2 < \ldots$ is an increasing sequence of integers and x_{n_i} converges to x). Then we get that

$$d(x_n, x) \leq d(x_n, x_{n_i}) + d(x_{n_i}, x) < \epsilon$$

E6) The only dense subset is the whole set X.

E7) **Hint:** This follows directly from the definition and the fact that E is closed iff $\overline{E} = E$.

E8) It is nowhere dense since its closure contains no open intervals.

E9) $[0, 1]$ is an example.