UNIT 16  NON-LINEAR DIFFERENCE EQUATIONS*

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16.0  OBJECTIVES

After going through this Unit, you will be able to:

• Differentiate between linear and non-linear difference equations;
• Explain the concept of a phase diagram and the important role that phase diagrams play in understanding non-linear difference equations;
• Discuss how qualitative analysis of non-linear difference equations is carried out;
• Describe the procedure for linearising non-linear difference equations; and
• Discuss the application of non-linear difference equation to the Solow growth model and to Cycles and Chaos theory.

16.1  INTRODUCTION

In the previous unit, we had discussed linear difference equations. In practice, several economic models give rise to non-linear dynamic relations. The introduction of non-linearity does not alter the essence of a difference equation. Let us consider a simple economic system. Suppose that the economic system under discussion can be described by a single variable \( x \in X \), where \( X \subseteq \mathbb{R} \) is a non-empty interval of the real number line. You have studied about real number line and subsets in the unit on sets. The variable \( x \) is referred to as the system variable and \( X \) is referred to as the system domain. The system domain consists of all possible values of the system variable. In our context, we think of the economic system as a dynamic one, that is, that the values of \( x \) change over time. We assume that time is measured in discrete periods \( t \in \{0,1,2,3..\} \). The variable \( x \) changes only once in one period. The set \( \mathbb{Z}^+ \) (i.e. a set of positive integers)is the set of all time periods and is called the time domain or time horizon. Time is

* Contributed by Shri Saugato Sen & Chetali Arora
indicated by a subscript on the variable \( x \). Thus \( x_t \) is the value of \( x \) in time period \( t \). All difference equations describe the evolution of a variable over time. The dynamic evolution of the system is described by a function \( f \), which shows how \( x_t \) is dependent on past values of \( x \). We write

\[
x_{t+1} = f(x_t)
\]

Here the function \( f \) is called the law of motion or the system dynamics. In the previous unit, we were concerned with difference equations that could be described by a linear equation like

\[
x_{t+1} = a + bx_t
\]

The non-linear form includes the linear form as a special case, but has the advantage of allowing a much broader range of varieties of time paths to emerge.

Actually the non-linear difference equation of the above type can be expressed as \( x_{t+1} = f(x_t, t) \). Here \( t \) itself appears as an argument, that is, an independent variable. However, we consider only autonomous difference equation where time does not appear as an independent variable. Hence we restrict ourselves to the study of equations of the type

\[
x_{t+1} = f(x_t)
\]

Notice that \( x_{t+1} \) depends only on the value of \( x \) in the period immediately preceding to it, that is, on \( x_t \) and not on \( x_{t-1} \) and other past values of \( x \). Thus the value of \( x \) in any period depends only on the value of \( x \) in the immediately preceding period and not on the values of \( x \) in distant past periods. This type of difference equation is called a \textit{first-order difference equation}. In this unit, we will only consider first-order difference equation.

A general form of linear difference equation would be

\[
x_{t+1} = f(x_t, x_{t-1}, x_{t-2}, \ldots, x_{t-k+1}, t)
\]

where \( k \) is a positive integer called the \textit{order} of the difference equation. The above equation also considers time as an independent variable. As we saw above, if \( t \) does not enter as a variable itself, then the difference equation is autonomous.

The solution to the autonomous difference equation of degree one, \( x_{t+1} = f(x_t) \), is a sequence \( \{x_t\}_{t=0}^{\infty} \), such that \( x_t \in X \) and the equation \( x_{t+1} = f(x_t) \) holds. Such a sequence is referred to as the \textit{trajectory} of the equation \( x_{t+1} = f(x_t) \).

Let us look again at our difference equation, \( x_{t+1} = f(x_t) \). Notice that we can shift it by a period and write \( x_t = f(x_{t-1}) \).

Given an initial condition \( x_0 \) at \( t = 0 \), we can work out a solution sequence as follows:

\[
x_1 = f(x_0)
\]

\[
x_2 = f(x_1) = f[f(x_0)] = f^2(x_0)
\]

\[
x_3 = f(x_2) = f[f^2(x_0)] = f^3(x_0)
\]
\[
x_t = f(x_{t-1}) = f(f^{t-1}(x_0)) = f^{t}(x_0)
\]

Where, \(f^t\) denotes the \(t\)th iterate of \(f\) not its power. The function \(x(t, x_0) = f^t(x_0)\) is called the flow of the system \(x_t = f(x_{t-1})\).

There is very simple but powerful method of analysing the dynamics of the above equation. To use this technique, we must draw the graph of the function \(f\), as well as a 45° line into a \(\{x, x_{t+1}\}\) plane. We take up the discussion of trying to arrive at a solution of first-order difference equation by making use of such diagrams (called phase diagrams) in the next section.

After that, the unit will discuss how non-linear difference equations can be approximated linearly in small neighbourhoods around specific points. Following this the unit will discuss a few applications of non-linear difference equations in section 16.4. First, the unit will discuss the growth model propounded by Solow. Finally the unit will discuss periodic cycles as well as aperiodic behavior of economic systems. This aperiodic behavior is called chaos.

### 16.2 PHASE DIAGRAMS AND QUALITATIVE ANALYSIS

We draw phase diagrams in the case of non-linear equations to help us get a qualitative ‘solution’ of a difference equation. To understand phase diagrams, consider a difference equation

\[x_{t+1} = f(x_t), \text{ where } t = 0, 1, 2, 3\ldots\]

**Case I:** \(0 < f'(x) < 1 \text{ and } f''(x) < 0\)

We first consider the phase diagrams of the non-linear difference equation by assuming that the function is upward sloping, that is \(f'(x) > 0\) and that the function flattens as \(x\) increases, that is, \(f''(x) < 0\).

*A general non-linear, first order difference equation is given by \(x_{t+1} = f(x_t, t), \text{ where } t = 0, 1, 2\ldots\). Here, we are considering an equation which does not explicitly depends on time, that is \(x_{t+1} = f(x_t)\), where \(t = 0, 1, 2, 3\ldots\). This is called nonlinear, first-order, autonomous difference equation.*

If we plot this curve on a graph which has \(x_{t+1}\) on the vertical axis and \(x_t\) on the horizontal axis, we get a picture as in Figure 16.1. Let us denote this curve (also called the phase line) equation by

\[
x_{t+1} = f_a(x_t)
\]

We have also plotted a 45° line in diagram 16.1. The intersection of the curve \(f_a\) with the 45° line marks an equilibrium point where \(x_{t+1} = x_t\). Given an initial value \(x_0\) on the horizontal axis, from our equation 1, we get the vertical-coordinate value as \(x_1\) [where \(x_1 = f_a(x_0)\)], this is what is mapped by the Phase line \(f_a\) in figure 16.1; considering the other way round, \(x_0\) on the
horizontal axis, marked straight up to the phase line at point A gives the value of $x_1$. Next we map another pair given by $x_1$ and $x_2$ [where, $x_2 = f_a(x_1)$]. Note that $x_1$ is transplotted from vertical axis to the horizontal axis with the help of $45^\circ$ line (having slope = 1). Steps are repeated in similar fashion to plot subsequent pairs. This process is what we learnt in previous unit, called the Iteration method.

On similar lines, the Graphic iterations discussed above will be applicable to the following three cases as well:

**Case II:** $f'(x) > 1$ and $f''(x) < 0$

Let Phase line equation be give by

$$x_{t+1} = f_b(x_t)$$

*Figure 16.2: Phase Diagram for $f'(x) > 1$ and $f''(x) < 0$*
**Case III: \(-1 < f' (x) < 0\)**

Let Phase line equation be give by \(x_{t+1} = f_c(x_t)\)

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**Case IV: \(f'(x) < -1\)**

Let Phase line equation be give by \(x_{t+1} = f_d(x_t)\)

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The above four cases illustrate four basic types of Phase lines, each indicating a different time path. The intertemporal (where ‘intertemporal’ refers to ‘across the time’) equilibrium value of \(x\) denoted by \(\bar{x}\) is given by the intersection of the phase lines in each diagram with the respective 45° line, labeled as point T. At point T, \(x_{t+1} = x_t\)
Our next task is to see whether given an initial value \( x_0 \neq \bar{x} \), the pattern of change as indicated by the phase line will lead consistently towards \( \bar{x} \) — which is the case of Convergence, or away from it — the case of Divergence. In Case I above, phase line \( f_a \) leads us from \( x_0 \) to \( \bar{x} \) in a steady path, without oscillation. Even when \( x_0 \) is placed to the right of \( \bar{x} \), the movement towards \( \bar{x} \) will be steady and in the left direction. This is referred to as convergence to the equilibrium. Under Case II, phase line \( f_b \) with a slope exceeding 1, showcases a divergent path. Here beginning from an initial value \( x_0 \) greater than \( \bar{x} \), we are lead steadily away from the equilibrium value to higher and higher values of \( x \). An initial value, lower than \( \bar{x} \), will give rise to similar steady divergent movements in the opposite direction.

In Cases III and IV, we encounter steady oscillatory movements, along with the phenomenon of overshooting the equilibrium mark. Phase line \( f_c \) with absolute slope less than 1, will result in convergence, along with the extent of overshooting (given by \( x_0 \) leading to \( x_1 \) which exceeds \( \bar{x} \) to be followed by \( x_2 \) falling short of \( \bar{x} \)) diminishing in successive periods. In case of phase line \( f_d \) whose absolute slope exceeds 1, there is an opposite scenario of a divergent time path.

Thus, two basic rules can be drawn from above discussion:

i) A first-order, autonomous, non-linear difference equation will have a locally stable steady-state equilibrium point (\( \bar{x} \)) if the absolute value of the derivative (which is nothing but the slope of the phase line), given by \( f'(\bar{x}) \) is less than 1, whereas will be unstable if the absolute value of the derivative is greater than 1 at that point (i.e. at \( \bar{x} \)).

ii) A first-order, autonomous, non-linear difference equation will lead to oscillations in \( x \), if \( f'(x) \) is negative for all \( x > 0 \), whereas there will be monotonic movement in \( x \), if \( f'(x) \) is positive for all \( x > 0 \).

Check Your Progress 1

1) What is a non-linear difference equation?

2) What do you understand by a phase diagram? What is it used for?
3) What is the basic difference in the equilibrium of a linear difference equation and that of a non-linear one?

16.3 LINEARISING NON-LINEAR DIFFERENCE EQUATIONS

After obtaining equilibrium point(s), that is, the fixed point(s) of a difference equation, an important task left is to explore the stability of that equilibrium. For non-linear equations, we can only explore stability of equilibrium point(s) in a small neighbourhood. This is done by linearising the non-linear function at equilibrium and then testing the slope of the linear approximation. In fact, this is just an extension of what we did when we studied phase diagrams in the previous section: we looked at the slope of \( f(x) \) function in the phase diagram to judge about the stability of the equilibrium. We investigated the slope of \( f(x) \) in the region of the equilibrium.

You have seen in an earlier unit how we use Taylor’s series expansion to linearise a non-linear function. Here too, we shall use Taylor’s series expansion. We attempt to linearise \( f(x) \), i.e. our Non-linear difference function by finding a first-order Taylor series expansion of this function, with equilibrium as the point of expansion. Although, a first-order Taylor series expansion produces a linear approximation to a non-linear function, the problem is that the approximation is good only for a limited range around the point of expansion. Also, the greater the degree of curvature of the original function, the smaller that range will be. Of course, our expansion need not be limited to first-order only, and we can take the expansion to as high an order as we like. Moreover, if the original curve is very greatly curved, we may require higher orders of expansion to approximate the original curve, but the higher orders of expansion introduce non-linear elements into the expansion, and since the idea is to take a linear expansion, we stop at the first-order, that is, at a linear expansion.

To take a first-order Taylor series approximation to a general function \( f(x) \), we begin by choosing the value of \( x \) which would determine the point around which we will construct a linear approximation to the non-linear function. Let us denote this value of \( x \) by \( x^* \), which means that the value of the function \( f(x) \) at the point of approximation is \( f(x^*) \). Then we can write, as the approximation to the function \( f(x) \) at some arbitrary point \( x^* \):

\[
 f(x) \approx f(x^*) + \frac{df(x^*)}{dx}(x - x^*)
\]

Note that the derivative on the right-hand side of the equation is also evaluated at \( x^* \). The closer \( x \) is to \( x^* \), the closer the value of the approximation.
Now we apply this approximation to a first-order difference equation. Remember we are using the following equation as our first-order difference equation:

\[ x_{t+1} = f(x_t) \]

and using \( x^* \) or \( \bar{x} \) to denote an equilibrium of the system. Approximating the function close to the equilibrium gives

\[ x_{t+1} = f(x_t) = f(x^*) + \frac{df(x^*)}{dx}(x_t - x^*) \]

Now note that since \( x^* \) is an equilibrium point, \( f(x^*) = x^* \). Hence we can write the above equation as

\[ x_{t+1} = x^* + \frac{df(x^*)}{dx}(x_t - x^*) \quad \ldots (2) \]

Let us define a new variable \( x^D_t \), as the deviation of the current value of \( x \) from its equilibrium value \( x^* \). Thus, \( x^D_t = x_t - x^* \) and \( x^D_{t+1} = x_{t+1} - x^* \). Hence, we can write the equation (2) as

\[ x^D_{t+1} = \frac{df(x^*)}{dx}x^D_t \quad \ldots (3) \]

In interpreting equation 3, we must remember that we have evaluated the first derivative \( \frac{df(x^*)}{dx} \) at a single point (here the equilibrium point), thus it is a constant. Given this, equation 3 becomes a homogeneous equation in \( x^D \), with constant coefficient. This makes it a linear first-order homogeneous difference equation. Since it is a homogeneous equation, it means that its equilibrium is at \( x^D = 0 \), but since \( x^D \) is the deviation of the original untransformed variable from its equilibrium, it means when \( x^D = 0 \), \( x = x^* \). So if equation 2 is stable with \( x^D \) converging to its equilibrium, then \( x \) too must converge to its own equilibrium.

**Check Your Progress 2**

1) Why do we need to linearise non-linear difference equations?

2) Suggest a method by which we can approximate a first-order non-linear difference equation near a small neighbourhood of its equilibrium point.
16.4 APPLICATIONS OF NON-LINEAR DIFFERENCE EQUATIONS

In this section we discuss some applications of non-linear difference equations. Probably the most familiar among the various applications are some models of economic growth. Even several consumption models display non-linear relations, especially when the utility function is not very restrictive. However we are not discussing consumption functions here, but taking up other non-linear dynamic processes. Specifically we shall take up for discussion a non-linear growth model suggested by Robert Solow, and Cycles and Chaos.

16.4.1 The Solow Model

Let us see what we understand by a growth model. Since we are discussing dynamics, we would want to see how a variable evolves over time. Here time is expected to pass in discrete levels. In the theory of economic growth, we see how the aggregate economic output or income grows over time. Your “Principles of Microeconomics” course in this semester will introduce you to the concept of a production function, where output is a function of labour and capital, in your. Here, we will think of an aggregate production function for the economy as a whole. Your “Principles of Macroeconomics” course will introduce you to the study of the economy in aggregate, that is, the Macroeconomics aspect of an economy. By combining our understanding from both the courses, here we are basically examining how the aggregate output or what we call the Gross Domestic Product (GDP) of an economy grows as a result of growth of total labour and capital (machines and equipment) over time.

Professor Robert Solow in 1956 propounded a model of economic growth. This model has become the central growth model in economic analysis, and has served as the baseline growth model upon which further work on growth has been carried out and new models suggested. Prof. Solow went on to receive the Nobel Prize in Economics in 1987 for this model. The economic growth model of Robert Solow which we consider here is also known as the Neoclassical growth model. This model contains difference equations for two variables, but by a trick common to growth models we are able to reduce it to a single difference equation model.

We begin with an aggregate production function:

$$ Y_t = f(K_t, L_t) $$

Where $Y$ is aggregate output, $K$ is aggregate capital and $L$ is aggregate labour. The time subscripts on each variable indicate that there are no lags in the production process.

For simplicity, labour (which is here assumed to be identical to population; that is, the labour force participation rate is 100 per cent) is assumed to grow at an exogenous proportional rate $n$. In other words, we get a difference equation:

$$ L_t = (1 + n)L_{t-1} $$

Capital grows as a result of net investment, which is given by gross investment minus an allowance for depreciation. Thus, Net investment is defined as the change in the stock of capital in two successive time periods.
This is a neoclassical model, so all savings are invested in productive physical capital, hence investments are considered to be equal to the savings. Consider the following equation that gives the value of capital at period $t$ ($K_t$) as a function of capital at period $t - 1$ ($K_{t-1}$):

$$K_t = sF(K_{t-1}, L_{t-1}) + (1-\delta)K_{t-1}$$

Here, $s$ is average total saving per output, $\frac{S}{Y}$. So $S = sY = sF(K, L)$. We are also considering saving $S$ equals investment $I$. So we can write, $I = sF(K, L)$.

It must be clear that, $I_{t-1} = K_t - K_{t-1}$. This means $K_t = I_{t-1} + K_{t-1}$. If we subtract depreciation at rate $\delta$, we get

$$K_t = I_{t-1} + (1-\delta)K_{t-1}.$$  Substituting for $I_{t-1}$ as $sF\left(K_{t-1}, L_{t-1}\right)$, we can write

$$K_t = sF\left(K_{t-1}, L_{t-1}\right) + (1-\delta)K_{t-1}$$

Note that saving is done in period $t - 1$ for new capital to emerge in period $t$. The above equation tells us that the current period’s capital ($K_t$) is equal to the part of last period’s capital left after deducting for depreciation $[(1-\delta)K_{t-1}]$ plus saving (equaling investment) done out of last year’s income (output) which shows up in new capital goods and equipment in the current period, i.e., $[sF(K_{t-1}, L_{t-1})]$.

Now let us make an assumption about the production function. Let us assume that the production function displays Constant Returns to Scale (CRS). This means that if labour and capital are scaled up by a factor $\lambda$, the output will increase by the same factor $\lambda$.

That is, CRS implies

$$\lambda Y_t = F(\lambda K_t, \lambda L_t)$$

Let us assume $\lambda = \frac{1}{L_t}$. Then,

$$\frac{F\left(K_t, L_t\right)}{L_t} = F\left(\frac{K_t}{L_t}, 1\right)$$

where $\frac{K_t}{L_t}$ is the current capital-labour ratio. Right-hand side of the equation shows the amount a single worker would produce if he or she had available a capital stock equal to the current capital-labour ratio. The left-hand side of this equation is output per worker and is denoted $y_t$. We can write the above equation as $y_t = f(k_t)$ where, $k_t = \frac{K_t}{L_t}$. Here, we have depicted output per capita as a function of capital per capita.

Recall the equation $K_t = sF\left(K_{t-1}, L_{t-1}\right) + (1-\delta)K_{t-1}$. Let us divide both sides of the above equation by $L_t$. We get

$$\frac{K_t}{L_t} = s\frac{F\left(K_{t-1}, L_{t-1}\right)}{L_t} + \frac{(1-\delta)K_{t-1}}{L_t}$$

In this equation, the right hand side has terms pertaining to both time periods $t - 1$ and $t$. To resolve this, let us multiply and divide all terms on the right hand side by $L_{t-1}$. We get

$$\frac{K_t}{L_t} \left(\frac{L_{t-1}}{L_t}\right) = s\left(\frac{F\left(K_{t-1}, L_{t-1}\right)}{L_{t-1}}\right)\left(\frac{L_{t-1}}{L_t}\right) + \left(\frac{(1-\delta)K_{t-1}}{L_{t-1}}\right)\left(\frac{L_{t-1}}{L_t}\right)$$

... (4)
The term $\frac{F(K_{t-1}, L_{t-1})}{L_{t-1}}$ in above equation is output per worker in time period $t-1$ and the term $\frac{K_{t-1}}{L_{t-1}}$ is capital-labour ratio in period $t-1$. We had seen above that $L_t = (1+n)L_{t-1}$. From this we get $\frac{L_{t-1}}{L_t} = \frac{1}{1+n}$. Using the notation for per capita quantities we had developed earlier, we can write equation (4) as,

$$k_t = \frac{sf(k_{t-1})}{1+n} + \left(1 - \frac{\delta}{1+n}\right)k_{t-1}$$

...(5)

Here $n$ and $\delta$ are exogenous (i.e., given and fixed from outside the model). Hence, we get a first-order difference equation given by equation 5, where $k_t$ is a function of $k_{t-1}$. We have not specified a precise functional form for $f(k)$, so we limit ourselves to a qualitative, phase diagram analysis of equation (5). We can see that per capita production function $y_t = f(k_t)$ has usual production function properties. Also $f'(k) > 0$, $f''(k) < 0$, considering these assumptions, we can draw a phase diagram of the type given in figure 16.5.

![Figure 16.5: Phase Diagram for a Neoclassical Growth Model](image)

At equilibrium (that is at point E), $k_{t-1} = k_t = k^*$, say. Then, even though we may not have a precise definition of $f(k)$, we can see that equation (B) must satisfy the relationship

$$\frac{f(k^*)}{k^*} = \frac{(n+\delta)}{s}$$

16.4.2 Cycles and Chaos

Till now we have considered only those non-linear difference equations for which the slope of the function $f(.)$, does not change sign, that is, the graph of $x_{t+1}$ against $x_t$, or $x_t$ against $x_{t-1}$, does not change sign. In other words, the graph of $x_t$ against $x_{t-1}$ in phase diagram, is either monotonically increasing or
monotonically decreasing, but never has the shape of a hill (inverted U), or valley (U-shaped). In this sub-section, we consider non-linear difference equations that generate hill-shaped curves in the phase diagram. This kind of difference equation generates interesting dynamic behavior, such as Cycles which repeat themselves every two or three periods, or even dynamic processes in which there is irregularity in the behavior of $x_t$. This type of an irregular process is called Chaos. It is beyond the scope of this unit to provide a detailed analysis of such cycles as well as chaotic behavior. In this subsection, we give a simple exposition of the topic.

Consider the first-order, nonlinear, autonomous difference equation

$$x_{t+1} = A x_t (1 - x_t), \text{ where } t = 0, 1, 2, \ldots \quad \ldots (6)$$

The equilibrium values $\bar{x}$ are obtained by solving

$$\bar{x} - A \bar{x} (1 - \bar{x}) = 0 \Rightarrow A \bar{x}^2 - A \bar{x} + \bar{x} = 0$$

$$\Rightarrow A \bar{x}^2 + \bar{x} (1 - A) = 0 \Rightarrow \bar{x}^2 + \frac{(1 - A)}{A} = 0$$

This gives

$$\bar{x} \left[ \left( \frac{1 - A}{A} \right) + \bar{x} \right] = 0$$

The two steady points are $\bar{x} = 0$ and $\bar{x} = \left( \frac{A - 1}{A} \right)$.

From the second steady-state point, we can say that a strictly positive steady-state equilibrium exists only if $A > 1$. If $A \leq 1$, then the steady states are zero or negative. These we do not discuss as they do not have much relevance in Economics. Recall from our discussion of phase diagrams that a steady-state equilibrium point of any first-order autonomous, non-linear difference equation is locally stable, if the absolute value of its slope, that is its derivative is less than 1 at that point. The value of the derivative of equation 6 above at the two steady points $\bar{x} = 0$ and $\bar{x} = \left( \frac{A - 1}{A} \right)$ we found above is given as follows:

$$\frac{dx_{t+1}}{dx_t} = A - 2Ax_t$$

At $\bar{x} = 0$, $\frac{dx_{t+1}}{dx_t} = A$ and at $\bar{x} = \left( \frac{A - 1}{A} \right)$, $\frac{dx_{t+1}}{dx_t} = 2 - A$

The above result implies that the steady point $\bar{x} = 0$ is unstable (as we assumed $A > 1$). The point $\bar{x} = \left( \frac{A - 1}{A} \right)$ will be locally stable only if absolute value of $2 - A$, that is $|2 - A| < 1 \Rightarrow 1 < A < 3$. This was what concluded the basic rule (i) under section 16.2.

Since the above equation is a first-order nonlinear difference equation, we can draw a phase diagram for the equation. The intersection of the phase line with the 45° line will occur at our steady points, that is at 0 and $\left( \frac{A - 1}{A} \right)$. The graph will peak at $x = \frac{1}{2}$, where the slope given by $\frac{dx_{t+1}}{dx_t} = 0$. Second derivative of the function, given by $\frac{d^2x_{t+1}}{dx_t^2} = -2A$, which is negative, implying that the graph of the equation will be inverted U shape. (refer Figure 16.6)
Note yourself from the graph that the intersection of the phase line with the 45° line at point will happen to the left of the peak, if $1 < A < 2$. This implies that the slope is positive at the stable-steady point $\bar{x} = \frac{A-1}{A}$. Whereas, when $2 < A < 3$, the intersection will be on the right of the steady point. This satisfies the condition of local stability, as the slope of the phase line will be negative at the stable steady-state point.

A negative slope with less than 1 absolute value will mean that $x_t$ will converge to $\bar{x}$ from either direction within a neighbourhood, but the path of approaching it will be oscillatory. Refer the figure. Starting from $x_0$, the slope is positive, with $x_t$ increasing monotonically in the initial few periods. However as $x_t$ approaches the neighbourhood of $\bar{x}$, the slope becomes negative with $x_t$ becoming oscillating in the neighbourhood before it converges to the steady state.

What will be the behavior of the phase line when $A \geq 3$? Firstly, $\bar{x} = \frac{A-1}{A}$ will no longer be a stable steady state, rather will be unstable. Secondly, the hill-shaped phase line possesses a peculiar characteristic which does not hold for a monotonically phase diagram—that is, $x_t$ will not diverge endlessly to 0 or infinity, but will be oscillating within a bound range, though it will not be converging to the steady state, but could converge to regular periodic behavior.

When a non-linear difference equation throws up this kind of a inverted U-shape phase diagram, there arise threshold in the behavior of the function in the sense that small changes in the value of $A$ can lead to dramatic changes in the behavior of $x$ and in its trajectory. For instance, in our above example, if the value of $A$ lies between 2 and 3, $x$ follows a simple convergent alternations, that is, $x$ takes on alternate values but moves towards convergence. As the value of $A$ becomes 3 or above, the trajectories become very complicated. For some values between 3 and 4, $x_t$ settles into periodic alternations, or what is called a limit cycle. This basically means that $x$ takes

Figure 16.6: Phase Diagram for the first-order, nonlinear, autonomous difference equation $x_{t+1} = Ax_t(1-x_t)$.
values only within a certain range, and alternates values within this range. It can be shown that when the value of \( A \) is 3.2, if the dynamic system runs long enough, the system will settle down to a pattern that is called a period 2 cycle, getting back and forth between a value of 0.513 and 0.799. This is an example of the alternation version of the limit cycle. Limit cycles produce oscillations but these appear mainly in the case of higher order difference equations. The alternations, however, give the essence of a limit cycle. This is an example of a stable limit cycle.

However, there can be cases when the limit cycles are unstable. These cycles have the same basic properties as an unstable equilibrium. The basic feature is that if we start from a value in the limit cycle, we will stay in that same cyclical path always, neither converging nor diverging; if we start from a value on either side of the cycle, we will diverge from it.

If we increase the value of \( A \) steadily, different results emerge. At \( A = 3.4 \), the range of the interval within which the value of \( x \) lies is bigger, but it is still between two values. If \( A = 3.5 \), the system follows a period 4 cycle, that is going from \( x = 0.382 \) to \( x = 0.827 \) to \( x = 0.501 \) to \( x = 0.875 \). If \( A \) is slightly higher at \( A = 3.84 \), the system is back to a period three cycle.

The interesting thing about values of \( A \), and this is where chaos enters, is that the periodicity of alternations is not always smooth. If we set \( A = 3.58 \), we will find that the system alternates around an upper equilibrium, but never repeats itself. It does not display any pattern which repeats over and over. It becomes aperiodic. In other words, the system is chaotic.

In what way is the study of chaos useful in economics? First of all, chaos is very hard to distinguish from a random process. And what is a random process? You will study about uncertainty and probability in the course on Statistics in the next semester. Some of you may already be familiar with probability. Random processes are associated with probability. A random variable is a variable whose value is determined by chance. Non-random or non-probability-based processes are called deterministic. The dynamic processes that we have been studying using non-linear (or even linear in the previous unit) difference equation is a deterministic process. The significance of chaos is that chaos depicts a situation where a deterministic process mimics the pattern of a random process. It is very difficult to distinguish a random process from a chaotic process. A fundamental difference between a random process and a chaotic process is that a random process cannot be predicted. However in a chaotic system, given the parameters, future values can fairly easily be predicted.

**Check Your Progress 3**

1) Explain the fundamental difference equation in the Solow growth model.
Difference Equations

2) a) What is the basic shape of the phase diagram of a chaotic system.

b) What do you understand by a limit cycle?

3) How is chaos important in economics? In what way is a chaotic process different from a random process?

16.5 LET US SUM UP

This unit followed unit 15 which was on linear difference equation. Like the previous unit, this unit dealt with discrete dynamic processes, that is, processes where the value of a variable depends on the past values of the same variable. The variables change values in discrete steps rather than continuously. The unit was concerned with general difference equations, of which linear difference equations were of one special type. Hence, the term 'non-linear' in the title is to be interpreted as general difference equations, rather than being restricted to the case of difference equations that are not linear.

The Unit began with explaining the general nature of difference equations. We saw that linear difference equations are a specific type of difference equations. The unit restricted itself to first-order difference equations, that is, to equations in which the value of the variable in period \( t \) depends only on value of the variable in period \( t - 1 \), and not on the values in time periods \( t - 2, t - 3 \), and so on. The value of the variable thus depends on the immediate past value of the variable and not on the values of the variable in the distant past. We saw that we can shift the time period of the equation and still maintain the first order of the equation by considering that the value of the variable in period \( t + 1 \) depends only on the value of the variable in period \( t \).
In the next section, the Unit introduced you to phase diagrams (or phase line) as a way to depict first-order, nonlinear, autonomous difference equations, and say something about its possible solutions. The phase diagram is constructed by showing the current value of the variable on the horizontal axis and the future value of the variable on the vertical axis; or depicting the past value of the variable on the horizontal axis and the current value on the vertical axis. We saw that unlike linear difference equations of the first degree, non-linear difference equations of the first degree can intersect the 45° line at more than one place. If the curve depicting the difference equation is upward sloping then the equilibrium point (fixed point) can be converging (attractor) or diverging (repellor). A downward sloping curve leads to oscillations. Since the unit restricted itself to first-order difference equations, the unit relied mainly on graphic solution methods and hence phase diagrams became a very important tool of analysis.

In the next section, we took a look at linearising non-linear difference equations. To this end, we utilised first-order Taylor Series expansion of the function \( f(x) \) at the relevant point \( x^* \). In the subsequent section, the unit took up for discussion two applications of non-linear difference equations: the Solow growth model, and a simple look at periodic Cycles as well as periodic Chaos. The Solow growth model showed the dynamics of how output per worker evolves over time as a function of capital per worker. A suitable phase diagram was provided with capital-worker ratio in one period shown on the vertical axis, and output-worker ratio shown on the horizontal axis. We showed the dynamics with the help of a non-linear difference equation relating capital per worker of the current period to the capital per worker of the previous period.

Finally, the unit discussed dynamic processes where the phase diagram of the non-linear function is inverted U-shaped. Important features and properties of such phase diagrams were discussed depending on whether the maximum point of the function lay to the left of the 45° line, or to its right. From such a quadratic difference equation, we studied the properties of limit cycles, as well as aperiodic cycles. The latter situation is called chaos, and at the end we saw that chaotic behavior mimics the behavior of random processes.

**16.6 ANSWERS/HINTS TO CHECK YOUR PROGRESS EXERCISES**

**Check Your Progress 1**

1) A non-linear difference equation is an equation that relates the value of a variable in one period to the value(s) of the same variable in past periods, where the shape of the function depicting the equation is not limited to certain specific shape. A linear difference equation is a special case of a non-linear difference equation.

2) A phase diagram plots the values of a variable on the vertical axis and the values of the same variable in the previous period on the horizontal axis. It is used to depict first-order difference equations. There is a 45° line, and the intersection of the function with the 45° line helps us know about equilibrium points.

3) (a) A linear difference equation has one equilibrium point; a non-linear equation has multiple equilibria. (b) an attractor is a stable equilibrium whereas a repellor is an unstable equilibrium.
Check Your Progress 2

1) We linearise a non-linear difference equation to be able to make statements about the stability of equations.

2) We can use first-order Taylor expansion to linearise a non-linear difference equation around its equilibrium value

Check Your Progress 3

1) The fundamental non-linear difference equation in the Solow model relates the capital-labour ratio to its value in the previous period. This is the basic function whose phase diagram is drawn.

2) (a) The basic shape of a chaotic dynamic system is quadratic with a hill shaped curve, though not all quadratic functions have chaotic behavior. (b) A limit cycle shows a periodic regular cyclical behavior with the interval within which the function can take its values being limited.

Chaos is important in Economics because it shows how deterministic equations can give rise to irregular aperiodic cyclical behavior. Chaos is different from a random process in that a random process’ future values cannot be predicted (that is the nature of the variable being random!) while a chaotic dynamic process’ future values can be predicted.