UNIT 20 YIELD LINE ANALYSIS OF SLABS

Structure

20.1 INTRODUCTION

Reinforced Concrete (RC) slabs are plate elements forming floors and roofs of buildings and carrying distributed loads primarily by flexure. Slabs may sometimes be carrying point loads (wheel loads) especially in bridges or line load (weight of wall, parapet or railing etc.). Slabs may be rectangular, circular or of other shapes. Inclined slabs may be used in staircases, ramps in hospitals or in multistorey car parks etc. but these will not be covered in this unit. A slab may be supported on beams, walls or columns. If the beams are very stiff, the beam deflections may be negligible, and the slab supports become relatively unyielding, similar to wall supports. However, if the beams are relatively flexible, the beam deflections are no longer negligible and will influence the slab behaviour.

A slab may be simply supported or continuous over one or more supports and is classified according to the manner it is supported as follows :

(a) one-way slabs spanning in one direction (Figures 20.1 and 20.2),

(b) two-way slabs spanning in both directions (Figures 20.1 and 20.2),

(c) circular slabs,

(d) grid floor and ribbed slabs (Figure 20.3), and

(e) flat slabs resting directly on columns with no beams (Figure 20.4).

A slab is called one-way slab when it bends only in one direction and two-way when it bends in two directions. When a rectangular slab of any length and width is supported on two opposite edges, the slab bends only in one direction; hence it is called a one-way slab. When a rectangular slab is supported on all four sides, and its length and breadth are comparable, the slab bends in two directions; hence it is called a two-way slab. However, if the length is greater than about twice the width, the bending along the long span becomes negligible in comparison with that along the short span, and the resulting slab action is effectively one-way. This discussion assumes uniformly distributed loads on the slab, but when the loading is for example wheel load as in the case of bridge slabs, a rectangular slab (whatever be its aspect ratio) supported on all four edges or even if supported on two opposite edges will bend in both directions. In this unit only the uniformly distributed loads will be considered. The last two types of slabs are not dealt herewith.

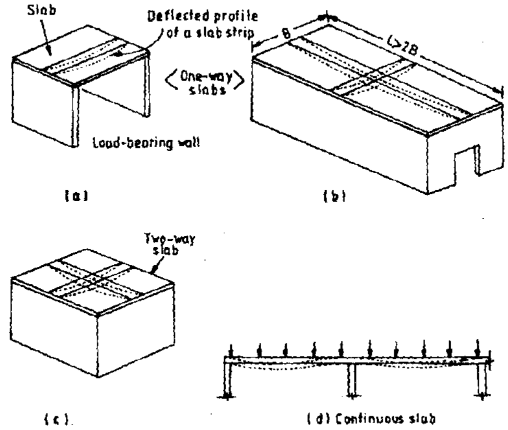


Figure 20.1 : Wall-supported Slab Systems

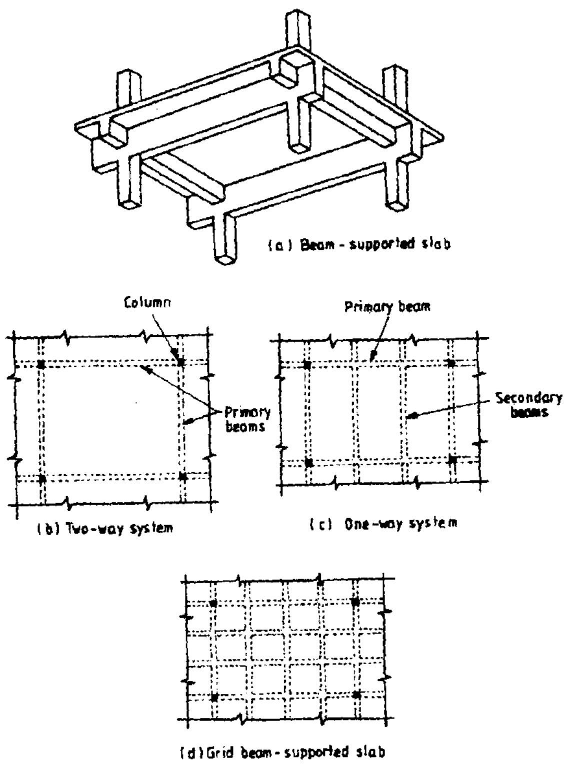
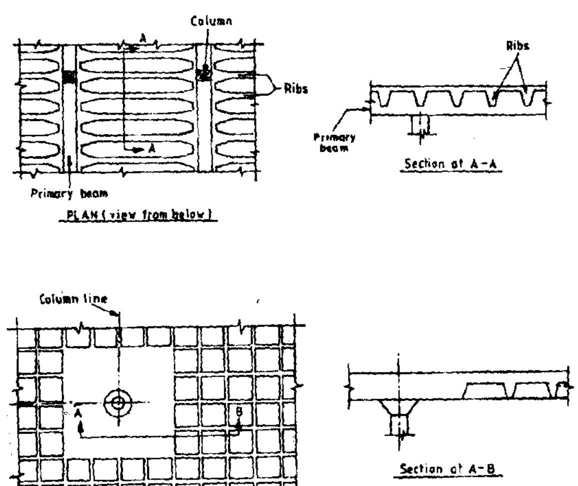
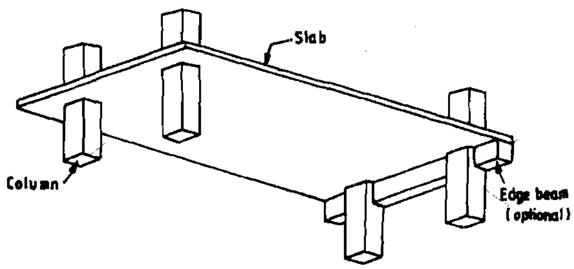
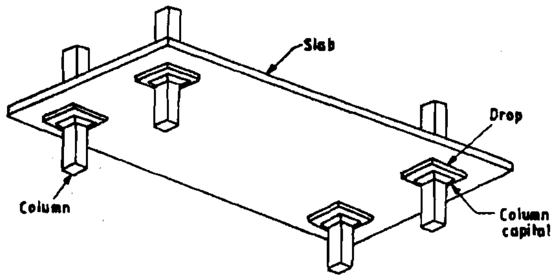


Figure 20.2 : Beam-supported Slab Systems





(a) Without drop and column capital



(b) With drop and column capital

Figure 20.4 : Flat Slabs

If a comparison is made between the percentage of steel provided in the three basic structural elements of a RC framed building : beam, slab and column, it will be seen that the percentage of steel is usually maximum in columns than in beams and the least in slabs. On the other hand, volume of concrete and, hence, dead load is large in case of slabs. A small reduction in the thickness of slab, therefore, leads to considerable economy. But care has to be taken to see that its performance (serviceability) is not affected due to excessive deflection, intolerable vibrations, visible cracking or ineffective sound and heat insulation. The distinction between a beam and a slab can be made as follows :

(i) A beam can bend only in one plane, whereas a slab may bend in two transverse planes.

(ii) Slabs are analyzed and designed considering a strip of unit width (usually 1 m in practical design).

(iii) Compression reinforcement is used only in exceptional cases in a slab.

(iv) Shear stresses are usually very low and shear reinforcement is always avoided in preference to increase in depth over the region critical in shear. The shear does, however, become a controlling factor in flat slabs.

(v) Distribution steel or temperature and shrinkage steel is invariably provided at right angles to the main steel, if there happens to be no steel in both directions.

(vi) Slabs are usually much thinner than beams.

Slabs are designed by using the same theory of bending and shear as are used for beams. The following methods of analysis are available :

(a) elastic analysis – idealization into strips or beams,

(b) semi-empirical coefficients as given in the code,

(c) yield line theory, and

(d) finite element and finite difference methods requiring the use of computers.

In this unit the design of slabs by Yield Line theory has been discussed. Analysis procedure of different types of slabs is followed by the design examples.

Objectives

After studying this unit, you should be able to

- draw yield line patterns developed in slabs of different shapes and with different edge conditions, and
- determine the collapse load for one-way and two-way slab for different end

20.2 YIELD LINE ANALYSIS

The yield line analysis is an ultimate load analysis of RC slabs. The method was innovated by Ingerslav (1923) and was later extended and advanced by Johansen, with the result that it is commonly known as Johansen's yield line analysis. This method of analysis enables the determination of failure moment in slabs of rectangular as well as irregular shapes for different support conditions and loading. The strength of slab is assumed to be governed by flexure alone, other effects such as shear and deflection are only required to be checked, if necessary.

When a slab is loaded with increasing loads, the stresses in the reinforcement and the concrete increase more or less proportionately upto load level corresponding to yield stress in the reinforcement. If the load is increased further, excessive deformations and increase in the strain will result. These deformations will be elastoplastic upto a load called *limit load*. When limit state is reached, the slab will continue to deform without any additional load, leading to total collapse. At this stage, a pattern of cracks will form a set of lines known as *yield lines*, resulting into a mechanism leading to total collapse of the slab. A *yield line* is defined as *a line in the plane of slab across which reinforcing bars have yielded and about which excessive deformation (plastic rotations) under constant limit moment (ultimate moment) continues to occur leading to failure.*

20.2.1 Assumptions

The yield line analysis of RC slab is based on the following assumptions :

(a) The reinforcing steel yields fully along the yield lines.

(b) The bending and twisting moments are uniformly distributed along the yield lines and have the maximum values provided by the ultimate moment capacities.

(c) The slab deforms plastically at failure and is separated into segments by the yield lines. These individual segments of the slab behave elastically.

(d) The elastic deformations are negligible as compared with the plastic deformations. Thus the entire deformations take place only in the yield lines, and thus the individual segments of the slab are plane segments at collapse.

20.2.2 Location of Yield Lines

In the yield line analysis of slabs, it is required to first postulate the possible yield line patterns. There may be more than one possible yield line patterns for a slab. The properties of yield lines, assumptions made, and the observed experimental phenomenon, help to form guidelines for predicting the possible yield line patterns which are detailed below :

(i) The yield lines are straight lines since the intersections between the inclined planes are straight lines.

(ii) A yield line ends at a slab boundary. If this is not so, the failure mechanism will not be complete.

(iii) A yield line (or its extension) between two slab segments passes through the intersection of the axes of rotation of the two adjacent slab segments.

(iv) The axes of rotation generally lie along line of supports and pass over columns, if present.

In a one-way simply supported slab, the yield line will form at the centre of slab at the bottom. In a one-way continuous slab, the yield lines will occur at the supports in addition to those near the mid-span.

The yield line due to sagging BM will be referred to as positive yield line, whereas due to hogging BM will be called as negative yield line. The symbols used for depicting boundary conditions are shown in Figure 20.5. Some typical yield line patterns for slabs of various shapes and boundary conditions loaded by uniformly distributed load are shown in Figure 20.6.

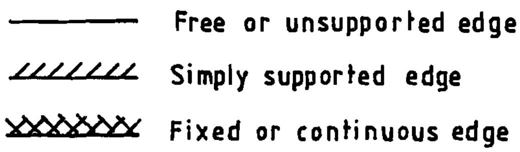


Figure 20.5 : Convention for Showing the Edges of Slabs

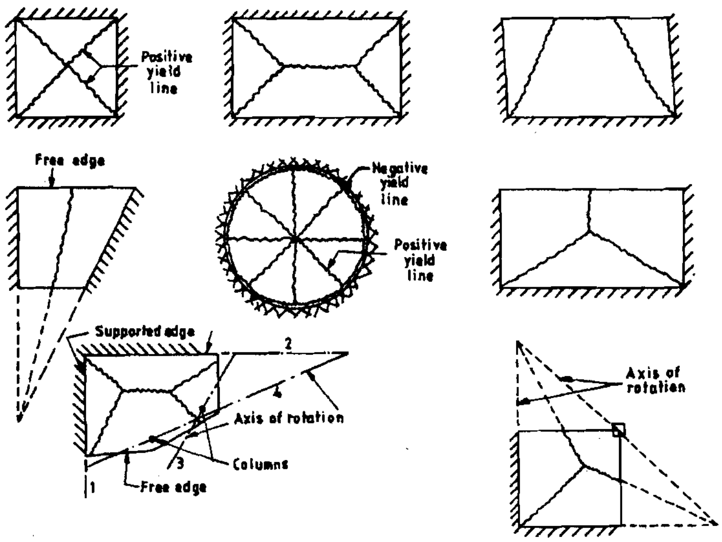


Figure 20.6 : Some Typical Yield Line Patterns

20.2.3 Methods of Analysis

There are two methods of yield line analysis of slab :

(i) Virtual Work Method, and

(ii) Equilibrium Method.

These two methods should give exactly the same results because they are based on the same basic assumptions for the yield line theory. In either method, a yield line pattern is assumed so that a collapse mechanism is produced. The next step is to determine the exact location of the yield lines. At this point, one may select either the virtual work method or the equilibrium method.

Virtual Work Method

According to the principle of virtual work "*If a rigid body that is in static equilibrium under the action of a set of forces is given a virtual displacement, the sum of the virtual work done by external forces and by the internal actions is equal to zero*".

In this method, a work equation is used to determine the location of the yield line and the collapse loads. Equation is formed by equating the total external work done by the collapse loads during simultaneous rigid body rotations of the slab segments (while maintaining deflection compatibility) to the total internal work done by the bending and twisting moments on all the yield lines. The work done can be written in general terms :

$$\sum W \Delta = \sum m \theta L \qquad \ldots (20.1)$$

where, W = collapse load,

Δ = vertical deflection through which collapse load moves,

m = moment of resistance of slab unit length,

θ = rotation of the slab segment compatible with the deflection, and

L = length of yield line.

Virtual work method gives upper bound to the true collapse load. Different yield line patterns must be investigated to determine the minimum collapse load by differential calculus.

Equilibrium Method

In the equilibrium method, an equation of equilibrium is used to determine the location of the yield line and the collapse load. Equations are formed by equating the external loads to the internal forces while maintaining deflection compatibility. *The equilibrium method gives lower bound to the true collapse load.* The equilibrium condition is set in such a way that mathematical differentiation is avoided as required in the virtual work method.

In the following sections, both the virtual work method and the equilibrium method are used to illustrate the analysis of one-way slab and two-way slabs.

SAQ 1

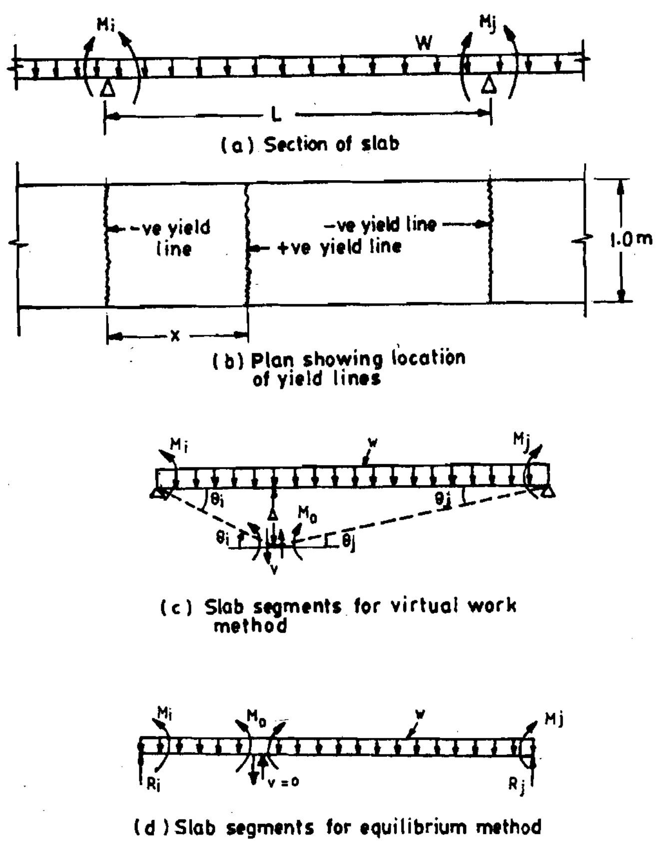
(a) What is a yield line? State the assumptions made in the yield line analysis of slabs.

(b) It is justified to use the yield line method of analysis for the design of RC slabs? What are the conditions that shall be satisfied?

(c) Explain the relative merits and demerits of using the virtual work and equilibrium method of analysis.

20.2.4 Analysis of One-way Slabs

Consider a 1 m wide strip of a continuous one-way slab as shown in Figure 20.7(a) and 20.7(b). Let, M_o = ultimate moment of resistance per unit length for mid span for positive bottom reinforcement; M_i, M_j = ultimate moment of resistance per unit length for negative reinforcement at supports i and j respectively; w = collapse load per unit length of slab; L = span of slab between two successive supports; θ_i, θ_j = rotations of slab segment at supports i and j in clockwise and anticlockwise direction; and, Δ = vertical deflection at positive yield line as shown in Figure 20.7(c).



Analysis of Slab

The slab will now be analysed by both the methods, i.e. virtual work method and equilibrium method.

(a) *Virtual Work Method*

External Work Done (EWD) $= wx\left(\dfrac{\Delta}{2}\right) + w\,(L - x)\,\dfrac{\Delta}{2} = wL\,\dfrac{\Delta}{2}$

Internal Work Done (IWD) $= M_i\,\theta_i + M_o\left(\theta_i + \theta_j\right) + M_j\,\theta_j$

From Figure 20.7(c) $\quad \theta_i = \dfrac{\Delta}{x}$ and $\theta_j = \dfrac{\Delta}{L - x}$

Therefore, IWD $= M_i\,\dfrac{\Delta}{x} + M_o\left(\dfrac{\Delta}{x} + \dfrac{\Delta}{L - x}\right) + M_j\left(\dfrac{\Delta}{L - x}\right)$

For equilibrium, the total EWD must be equal and opposite to the total IWD in going through a rigid body displacement Δ

$$\frac{w\,L\,\Delta}{2} = M_i\frac{\Delta}{x} + M_o\left(\frac{\Delta}{x} + \frac{\Delta}{L - x}\right) + M_j\left(\frac{\Delta}{L - x}\right)$$

$$w = \frac{2\,M_i}{Lx} = \frac{2\,M_o}{x\,(L - x)} + \frac{2\,M_j}{L\,(L - x)} \qquad \ldots (20.2)$$

Minimum value of w is obtained by differentiating Eqn. (20.2) w.r.t. x and setting the derivative to zero,

$$\frac{dw}{dx} = \frac{2\,M_i}{L\,x^2} + 2\,M_o\left[\frac{1}{x\,(L - x)^2} - \frac{1}{x^2\,(L - x)}\right] + \frac{2\,M_j}{L\,(L - x)^2} = 0$$

or, $\left(M_j - M_i\right)x^2 + 2\left(M_i + M_o\right)Lx - \left(M_i + M_o\right)L^2 = 0 \qquad \ldots (20.3(a))$

(b) *Equilibrium Method*

The plastic hinge will occur at the section of maximum BM and SF will be zero at this section. This condition may be used to determine the two unknowns x and w. Referring to Figure 20.7(d),

$$R_i = \frac{wL}{2} + \left(\frac{M_i - M_j}{L}\right),\quad R_j = \frac{wL}{2} - \left(\frac{M_i - M_j}{L}\right)$$

The two equations of equilibrium are : $\sum V = 0$ and $\sum M = 0$

i.e., $R_i - wx = 0$ or, $\dfrac{wL}{2} + \left(\dfrac{M_i - M_j}{L}\right) - wx = 0$

or, $$w = \frac{2\,(M_j - M_i)}{L\,(L - 2x)} \qquad \ldots (i)$$

and, $$R_i\,x - M_i - \frac{wx^2}{2} - M_o = 0$$

or, $$\left(\frac{wL}{2} + \frac{M_i - M_j}{L}\right)x - M_i - \frac{wx^2}{2} - M_o = 0 \qquad \ldots (ii)$$

On eliminating w between Eqs. (i) and (ii) we get

$$(M_j - M_i)\,x^2 + 2\,(M_i + M_o)\,Lx - (M_i + M_o)\,L^2 = 0 \qquad \ldots (20.3(b))$$

This equation is exactly the same as Eq. (20.3(a))

Special Cases

(a) *Case I : Simply Supported Slab*

End moments M_i and M_j are zero. The value of w and x can be obtained from Eqs. (20.2) and (20.3) as :

$$x = \frac{L}{2} \text{ and } w = \frac{2M_o}{x(L - x)} \text{ or, } w = \frac{8M_o}{L^2}$$

(b) *Case II : Fixed End Beam with Equal Moment Capacities*

$$M_i = M_j = 2M_o \text{ (say)}$$

The values of w and x can be obtained from Eqs. (20.2) and (20.3)

$$6M_o Lx - 3M_o L^2 = 0 \text{ or, } x = \frac{L}{2}$$

$$\text{and, } w = \frac{24M_o}{L^2}$$

Example 20.1

The factored moment capacities of a one-way continuous slab are : $M_i = 12$ kNm, $M_j = 20$ kNm, and $M_o = 16$ kNm. If the span of the slab is 4.8 m, determine the location of plastic hinges and the collapse load.

Solution

From Eq. (20.3) we have

$$(M_j - M_i)x^2 + 2(M_i + M_o)Lx - (M_i + M_o)L^2 = 0$$

Putting the values and solving, we get $x = 2.25$ m

Substituting the value of $x = 2.34$ m in Eq. (20.2) and solving we get, $w = 11.07$ kN/m. The resulting BM and SF diagrams are shown in Figure 20.8.

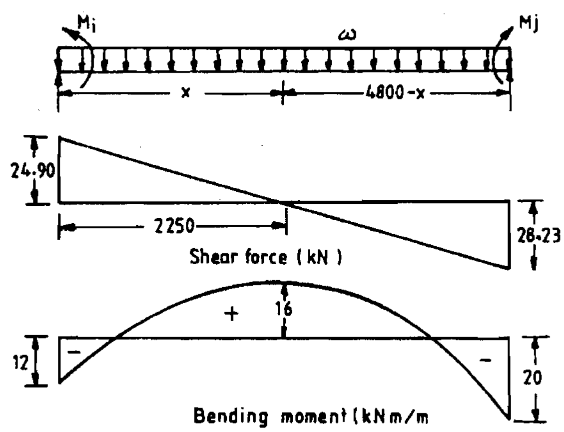


Figure 20.8 : Shear Force and Bending Moment Diagrams of One-way Continuous Slab

20.2.5 Work Done by Yield Line Moment

In the virtual work method of analysis, work done by bending and twisting moments acting along a yield line in going through rigid body rotation of the slab segment is required for which general expression has been obtained in this section.

Figure 20.9 shows moment acting along the edges of a slab segment having yield line of length L with horizontal and vertical projections of L_x and L_y respectively. The slab undergoes a rigid body rotation whose components are θ_x and θ_y. M_b and M_t are bending and torsional moment capacities acting along the yield line. These moments are shown in vector notation using right hand thumb rule. M_x and M_y are moment capacities per unit length acting along horizontal (i.e., along x-axis) and vertical (i.e., along y-axis) projections of yield line.

Considering the equilibrium of the slab segment shown in Figure 20.9, we get

$$M_b = M_x \sin^2 \theta + M_y \cos^2 \theta$$

$$\text{and, } M_t = (M_x - M_y) \sin \theta \cos \theta$$

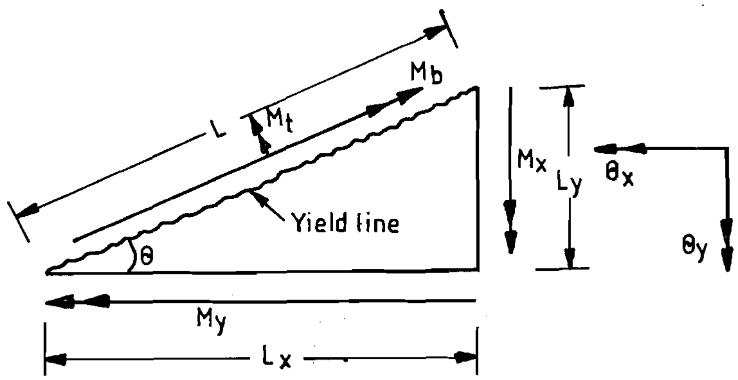


Figure 20.9 : Work Done by Yield Line Moments

Total absolute work done by M_b and M_t is equal to

$$W = M_b L (\theta_x \cos\theta + \theta_y \sin\theta) + M_t L (-\theta_x \sin\theta + \theta_y \cos\theta) \quad \ldots (20.4)$$

Putting the values of M_b and M_t and simplifying, we get

$$W = M_x L_y \theta_y + M_y L_x \theta_x \quad \ldots (20.5)$$

This is the work done by the moments $M_x L_y$ and $M_y L_x$ acting on the horizontal and the vertical projections of the yield line. The use of Eq. (20.5) will simplify the calculations by eliminating the determination of M_b, M_t and the corresponding rotations.

20.2.6 Nodal Forces

Nodal forces arise at the corners of slab segments whenever an yield line intersects another yield line, or a free edge. Johansen has given several theorems regarding nodal forces. Without going into involved mathematical proof, some of the theorems that are required in solving simple problems using equilibrium method are stated below :

(i) At the junction of any number of yield lines, irrespective of their signs, the sum of the nodal forces is equal to zero.

(ii) At the junction of yield lines each of the nodal forces is zero if the yield lines are either all negative or all positive.

According to this theorem, the nodal forces at the interior junction of yield lines are zero. This simplifies the analysis by equilibrium method.

(iii) The nodal force 'V' at the intersection of an yield line with a free edge is computed using the following equation (See Figure 20.10) :

$$V = M_y \cot\alpha \quad \ldots (20.6)$$

where, M_y is the ultimate moment capacity of slab at the node in the direction perpendicular to the free edge. The nodal force will be downward (i.e. +ve) for the segment with acute angle and upward (i.e. –ve) for the obtuse angle segment as shown in Figure 20.10.

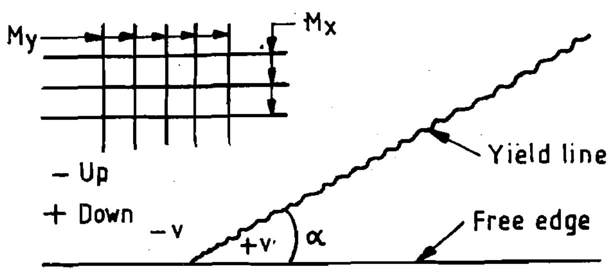


Figure 20.10 : Two-way Slab with a Free Edge

SAQ 2

Where does a nodal force arise in yield line patterns and why does it appear?

20.2.7 Analysis of Two-way Slabs

In a one-way slab, the direction of yield line is perpendicular to the direction of steel. But in a two-way rectangular slab, the direction of yield lines may not be perpendicular to the direction of reinforcement. A two-way slab can be supported on all of its edges or any three of its edges. The corner of a two-way slab may be held down or may not be held down. If the corners are held down, there may or may not be adequate corner reinforcement to prevent cracking. Typical yield line patterns in two-way slabs under different conditions are shown in Figure 20.11. A typical rectangular two-way slab has two-way reinforcement within the panel near bottom face, and one-way reinforcement across the edges near the top face. The bottom reinforcement provides ultimate moments of resistance of M_{px} and M_{py} in x- and y-directions. The subscript p refers to positive moment. Similarly, the negative reinforcement provides ultimate moments of resistance of M_{nx} and M_{ny} in the x- and y-directions. The subscript n refers to negative moment. These moments of resistance are absolute quantities per unit width of slab. The uniformly distributed collapse load w based on yield line theory may be determined in terms of the sides L_x and L_y, and moment capacities M_{px}, M_{py}, M_{nx} and M_{ny}.

In case, one or more edges of a slab are simply supported or free, the corresponding values of the negative moments of resistance at those edges will be zero. Analysis of different yield line patterns for uniformly loaded two-way slabs is given in the following sections.

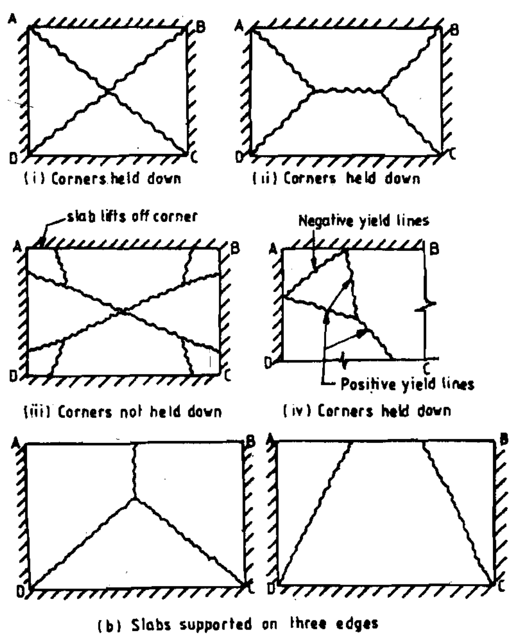


Figure 20.11 : Yield Patterns for Two-way Slabs

Analysis for Yield Line Pattern-I (Figure 20.12)

Figure 18.12 shows the yield line pattern in a two-way rectangular slab which is restrained along all its edges. Yield lines AC and BD divide the slab in four segments, 1-2-3-4. Slab segments 1 and 3 are identical. Similarly, slab segments 2 and 4 are identical. In this case, the locations of yield lines are well known. The free body diagrams of segments 3 and 4 are shown in Figure 20.12.

Let vertical deflection at the intersection of the yield lines be Δ. Therefore, the deflection at the centroid of each of the four triangles 1-2-3-4 will be $\Delta/3$.

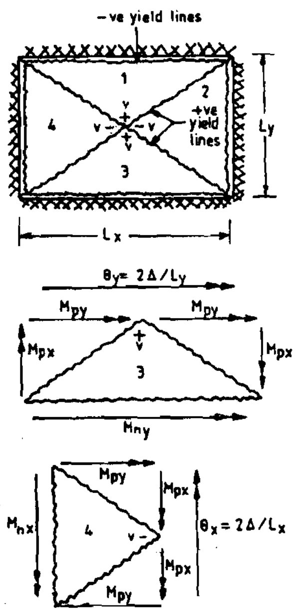


Figure 20.12 : Analysis of Yield Line Pattern-I

Analysis of Slab

The slab will now be analysed by both the methods, i.e. virtual work method and equilibrium method.

(a) *Virtual Work Method*

Total External Work Done (EWD) by the ultimate uniformly distributed load is given by

$$EWD = wL_x L_y \Delta/3$$

Total Internal Work Done (IWD) by the yield moments is given by

$$IWD = 2M_{ny} L_x \theta_y + 2M_{nx} L_y \theta_x + 2M_{py} L_x \theta_y + 2M_{px} L_y \theta_x$$

Equating the total EWD to the total IWD and solving for w,

$$wL_x L_y \frac{\Delta}{3} = 2M_{ny} L_x \frac{2\Delta}{L_y} + 2M_{nx} L_y \frac{2\Delta}{L_x} + 2M_{py} L_x \frac{2\Delta}{L_y} + 2M_{px} L_y \frac{2\Delta}{L_x}$$

$$\text{or,} \quad w = 21 \left(\frac{M_{nx} + M_{px}}{L_x^2} + \frac{M_{ny} + M_{py}}{L_y^2} \right) \qquad \ldots (20.7(a))$$

(b) *Equilibrium Method*

Let us consider the equilibrium of the segment 3 and take moment of all the forces about its lower edge,

$$\frac{1}{2} wL_x \frac{L_y}{2} \frac{L_y}{6} + \frac{VL_y}{2} - M_{ny} L_x - M_{py} L_x = 0$$

$$\frac{wL_x L_y}{12} + V = 2(M_{ny} + M_{py}) \frac{L_x}{L_y} \qquad \ldots (i)$$

Similarly, let us consider the equilibrium of segment 4 and take moment of all the forces about its left edge,

$$\frac{1}{2} wL_y \frac{L_x}{2} \frac{L_x}{6} - \frac{VL_x}{2} - M_{nx} L_y - M_{px} L_y = 0$$

$$\frac{wL_x L_y}{12} - V = 2(M_{nx} + M_{px}) \frac{L_y}{L_x} \qquad \ldots (ii)$$

Adding Eqs. (i) and (ii),

$$\frac{wL_x L_y}{6} = 2\,(M_{ny} + M_{py})\frac{L_x}{L_y} + 2\,(M_{nx} + M_{px})\frac{L_y}{L_x}$$

$$\text{or,} \quad w = 12\left(\frac{M_{nx} + M_{px}}{L_x^2} + \frac{M_{ny} + M_{py}}{L_y^2}\right) \qquad \ldots (20.7(b))$$

This is the same equation as Eq. (20.7(a)) obtained by the virtual work method.

Special Cases

(a) *Case I : Simply Supported Square Slab*

$$L_x = L_y = L \quad M_{nx} = M_{ny} = 0 \quad M_{px} = M_{py} = M_o \text{ (say)}$$

Eq. (20.7) gives, $$w = \frac{24M_o}{L^2} \text{ or, } M_o = \frac{wL^2}{24}$$

(b) *Case II : Square Slab with All Edges Fixed*

Support moments are equal to those at the centre of the span. The collapse load can be obtained using Eq. (20.7)

$$w = \frac{12 \times 4M_o}{L^2} \text{ or, } M_o = \frac{wL^2}{48}$$

Analysis for Yield Line Pattern-II (Figure 20.13)

Figure 20.13 shows the yield line pattern in a two-way rectangular slab, which is restrained along all its edges. Yield lines divide the slab in four segments : two triangular and two trapezoidal. Slab segments 1 and 3 are identical. Similarly, slab segments 2 and 4 are identical. The free body diagrams of slab segments 3 and 4 are shown in Figure 20.13. For convenience, slab segment 3 is sub-divided in three parts : 3A, 3B and 3C. There are two unknowns in this yield line pattern : distance x and uniformly distributed collapse load w. Let vertical deflection at the intersection of the yield line be Δ.

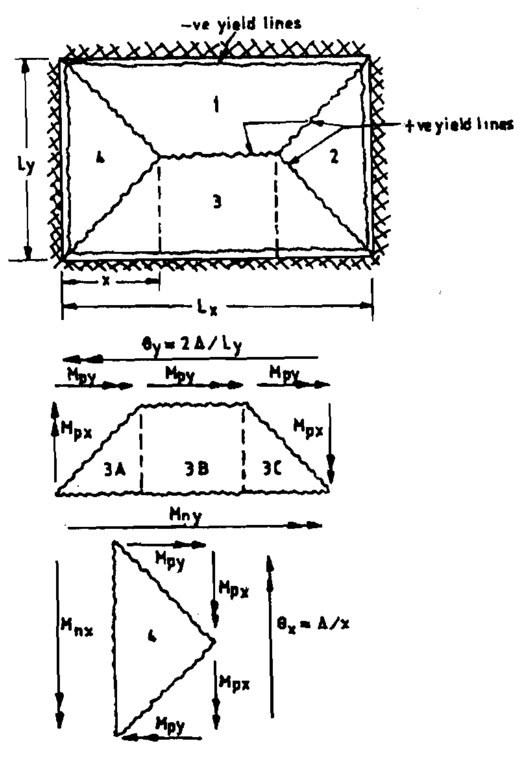


Figure 20.13 : Analysis for Yield Line Pattern - II

Analysis of Slab

The slab will now be analysed by both the methods, i.e. virtual work method and equilibrium method.

(a) *Virtual Work Method*

Total External Work Done (EWD) by the uniformly distributed load is given by :

$$EWD = 2W_{3A} + 2W_{3B} + 2W_{3C} + 2W_4$$

$$= 2\left(\frac{1}{2}\,wx\,\frac{L_y}{2}\right)\left(\frac{\Delta}{3}\right) + 2\left[w\,(L_x - 2x)\,\frac{L_y}{2}\right]\left(\frac{\Delta}{2}\right) + 2\left(\frac{1}{2}\,wx\,\frac{L_y}{2}\right)\left(\frac{\Delta}{3}\right) + 2\left(\frac{1}{3}\,wx\,L_y\right)\left(\frac{\Delta}{2}\right)$$

$$= \frac{w\,\Delta}{6}\,(3L_x\,L_y - 2x\,L_y)$$

Total Internal Work Done (IWD) by the yield moments is given by

$$IWD = 2M_{ny}\,L_x\,\theta_y + 2M_{nx}\,L_y\,\theta_x + 2M_{py}\,L_x\,\theta_y + 2M_{px}\,L_y\,\theta_x$$

$$= 2\,(M_{nx} + M_{px})\,\frac{\Delta\,L_y}{x} + 2\,(M_{ny} + M_{py})\,\frac{2\,\Delta\,L_x}{L_y}$$

Equating the EWD to IWD,

$$\frac{w\,\Delta}{6}\,(3\,L_x\,L_y - 2x\,L_y) = 2\,\,(M_{nx} + M_{px})\,\frac{\Delta\,L_y}{x} + 2\,\,(M_{ny} + M_{py})\,\frac{2\,\Delta\,L_x}{L_y}$$

$$w = \frac{\left[12\left(M_{px} + M_{nx}\right)L_y^2 + 24x\left(M_{py} + M_{ny}\right)L_x\right]}{L_y^2\left(3x\,L_x - 2x^2\right)} \qquad \ldots (20.8(a))$$

Setting to zero the derivative of w to get the minimum collapse load, i.e., $\frac{dw}{dx} = 0$.

or, $$24L_x\left(M_{py} + M_{ny}\right)L_y^2\left(3x\,L_x - 2x^2\right) - 12\left[\left(M_{px} + M_{nx}\right)L_y^2 + 2x\,L_x\left(M_{py} + M_{ny}\right)\right]\left(3\,L_x - 4\,x\right)L_y^2 = 0$$

or, $$4\left(M_{py} + M_{ny}\right)L_x\,x^2 + 4\left(M_{px} + M_{nx}\right)L_y^2\,x - 3\left(M_{px} + M_{nx}\right)L_x\,L_y^2 = 0 \qquad \ldots (20.9(a))$$

This is a quadratic equation in x.

(b) *Equilibrium Method*

Let us consider the equilibrium of the slab segment 3 and taking moment of all forces about its lower edge,

$$2\left(\frac{1}{2}\,wx\,\frac{L_y}{2}\,\frac{L_y}{6}\right) + w\left(L_x - 2x\right)\frac{L_y}{2}\,\frac{L_y}{4} - M_{ny}\,L_x - M_{py}\,L_x = 0$$

or, $$w = \frac{24(M_{py} + M_{ny})\,L_x}{2x\,L_y^2 + 3L_y^2\,(L_x - 2x)} \qquad \ldots (20.8(b))$$

Similarly, considering the equilibrium of the slab segment 4 and taking moment of all forces about its left edge,

$$\frac{1}{2} w x L_y \frac{x}{3} - M_{nx} L_y - M_{px} L_y = 0$$

$$\text{or,} \quad w = \frac{6\,(M_{px} + M_{nx})}{x^2} \qquad \text{. . . (20.8(c))}$$

Equating the two expressions of w, we get

$$4\,(M_{py} + M_{ny})\, L_x x^2 + 4\,(M_{px} + M_{nx})\, L_y^2\, x - 3\,(M_{px} + M_{nx})\, L_x L_y^2 = 0 \qquad \text{. . . (20.9(b))}$$

This is the same expression as Eq. (20.9(a))

For $x = L_x/2$, Eq. (20.9(a)) gives,

$$\frac{M_{px} + M_{nx}}{M_{py} + M_{ny}} = \frac{L_x^2}{L_y^2} \qquad \text{. . . (20.10(a))}$$

If the above condition is satisfied, the slab will exhibit yield pattern-I (Figure 20.12)

For $x < \frac{L_x}{2}$, it can be shown that

$$\frac{M_{px} + M_{nx}}{M_{py} + M_{ny}} < \frac{L_x^2}{L_y^2} \qquad \text{. . . (20.10(a))}$$

Example 20.2

Find the collapse load for a 6.0 m × 4.5 m rectangular slab fixed at all edges for which the mid span moment is 80% of the support moment in the corresponding direction. The moment in the longer direction is 40% of that in the shorter direction.

Solution

$M_{py} = 0.8\, M_{ny} = M$ (say) $\quad M_{px} = 0.8\, M_{nx} = 0.4\, M$

$L_x = 6.0$ m and $L_y = 4.5$ m

Let us first determine the yield line pattern by using Eq. (20.10)

For yield pattern-I $\qquad \frac{M_{px} + M_{nx}}{M_{py} + M_{ny}} = \frac{L_x^2}{L_y^2}$

And, for yield pattern-II $\qquad \frac{M_{px} + M_{nx}}{M_{py} + M_{ny}} < \frac{L_x^2}{L_y^2}$

Since, $\frac{M_{px} + M_{nx}}{M_{py} + M_{ny}} = \frac{0.90M}{2.25M} = 0.4$ and $\frac{L_x^2}{L_y^2} = \frac{6.0^2}{4.5^2} = 1.78$

Hence, yield pattern-II prevails.

Determining the value of x by using Eq. (20.9)

$$4\,(M_{py} + M_{ny})\, L_x x^2 + 4\,(M_{px} + M_{nx})\, L_y^2\, x - 3\,(M_{px} + M_{nx})\, L_x L_y^2 = 0$$

Putting the values and solving, we get $x = 1.88$ m

We can use any of the three Eqs. (20.8(a)), (20.8(b)) or (20.8(c)) to determine the collapse load, therefore, from Eq. (20.8(a))

$$w = \frac{\left[12\left(M_{px} + M_{nx}\right) L_y^2 + 24x\left(M_{py} + M_{ny}\right) L_x\right]}{L_y^2\left(3x\, L_x - 2x^2\right)} = 1.527\, M$$

20.2.8 Rectangular Slab Simply Supported at Three Edges and Free at the Other

Analysis of Yield Line Pattern-I (Figure 20.14)

Figure 20.14 shows the yield line pattern in a two-way rectangular slab which is slab supported at three edges and free at the other. Yield lines divide the slab in three segments : 1-2-3. Slab segments 1 and 3 are identical. The free body diagrams of slab segments 2 and 3 are shown in Figure 20.14. For convenience, the slab segment 3 is sub-divided in two parts : 3A and 3B. There are two unknowns in this pattern : distance y and uniformly distributed collapse load w. Let vertical deflection at the intersection of yield lines be Δ. Let us determine the collapse load by both methods.

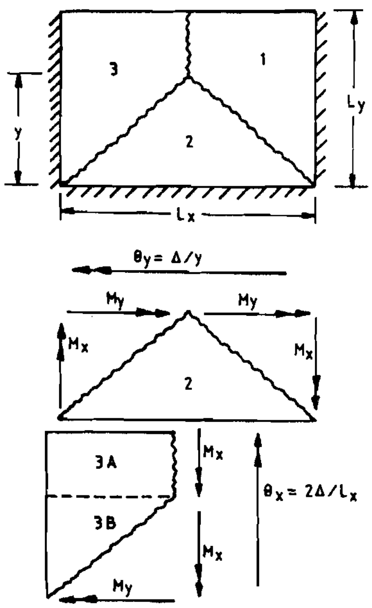


Figure 20.14 : Analysis for Yield Pattern-I

Analysis of Slab

The slab will now be analysed by both the methods, i.e. virtual work method and equilibrium method.

(a) *Virtual Work Method*

Total External Work Done (EWD) by the uniformly distributed load is given by :

$$EWD = W_2 + 2W_{3A} + 2W_{3B}$$

$$= \frac{1}{2} wL_x y \frac{\Delta}{3} + 2w \frac{L_x}{2} (L_y - y) \frac{\Delta}{2} + 2w \frac{1}{2} \frac{L_x}{2} y \frac{\Delta}{3}$$

$$= wL_x \frac{\Delta}{6} (3L_y - y)$$

The total Internal Work Done (IWD) by the yield moments is given by

$$IWD = 2M_x L_y \theta_x + M_y L_x \theta_y$$

$$= 2M_x L_y \frac{2\Delta}{L_x} + M_y L_x \frac{\Delta}{y}$$

Equating the total EWD to the total IWD,

$$wL_x \frac{\Delta}{6} (3L_y - y) = 4M_x L_y \frac{\Delta}{x} + M_y L_x \frac{\Delta}{y}$$

$$\text{or,} \quad w = \frac{24M_x L_y y + 6M_y L_x^2}{L_x^2 (3y L_y - y^{2)}} \qquad \ldots (20.11(a))$$

Setting to zero the derivative of w with respect to y in order to get the minimum value of w, i.e., $\frac{dw}{dy} = 0$.

$$\text{or,} \quad 24M_x L_y L_x^2 (3y L_y - y^2) - (24M_x yL_y + 6M_y L_x^2) L_x^2 (3L_y - 2y) = 0$$

$$\text{or,} \quad 4M_x L_y y^2 + 2M_y L_x^2 y - 3M_y L_x^2 L_y = 0 \qquad \ldots (20.12(a))$$

This quadratic equation can be solved to get the value of y as

$$y = \frac{-2\,r L_x^2 + \sqrt{4r^2 L_x^4 + 48r L_x^2 L_y^2}}{8\,L_y} \qquad \ldots (20.12(b))$$

where, $r = \frac{M_y}{M_x}$, but y should be less than L_y, that is

$$= \frac{-2\,r L_x^2 + \sqrt{4r^2 L_x^4 + 48r L_x^2 L_y^2}}{8\,L_y} < L_y$$

$$\text{or,} \quad \sqrt{4r^2 L_x^4 + 48r L_x^2 L_y^2} < \left(8L_y^2 + 2r L_x^2\right)$$

$$\text{or,} \quad r < 4\frac{L_y^2}{L_x^2} \qquad \ldots (20.13)$$

(b) *Equilibrium Method*

Since the moment capacities under three intersecting yield lines are identical, the nodal forces are zero. Let us consider the equilibrium of segment 3 and taking moments of all the forces about its left edge,

$$\frac{wL_x}{2}(L_y - y)\frac{L_x}{4} + \frac{wL_x}{4} y \frac{L_x}{6} - M_x L_y = 0$$

$$\text{or,} \quad w = \frac{24M_x L_y}{3L_x^2 (L_y - y) + L_x^2 y}$$

$$\text{or,} \quad w = \frac{24M_x L_y}{L_x^2 (3L_y - 2y)} \qquad \ldots (20.11(b))$$

Similarly, considering the equilibrium of segment 2,

$$\frac{1}{2} wL_x y \frac{y}{3} = M_y L_x$$

$$\text{or,} \quad w = \frac{6M_y}{y^2} \qquad \ldots (20.11(c))$$

Equating w from Eqs. (20.22) and (20.23)

$$4M_x L_y y^2 + 2M_y L_x^2 y - 3M_y L_x^2 L_y = 0 \qquad \ldots (20.12(c))$$

This equation is the same as Eq. (20.18)

Analysis of Yield Line Pattern-II (Figure 20.15)

Figure 20.15 shows the yield line pattern in a two-way rectangular slab, which is simply supported at three edges and free at the remaining edge. Yield lines divide the slab in three segments : 1, 2, 3. Slab segments 1 and 3 are identical. The free body diagrams of slab segments 1 and 2 are shown in Figure 20.15. For convenience, the slab segment 2 is subdivided in three parts : 2A, 2B, and 2C. There are two unknowns in this pattern : distance x and uniformly distributed collapse load w. Let vertical deflection at the point where yield lines meet the free edge be Δ.

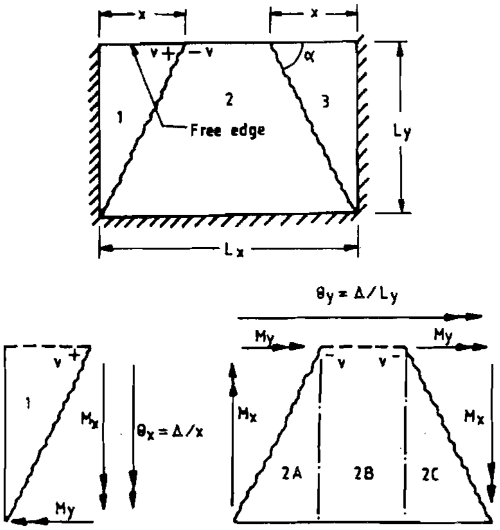


Figure 20.15 : Analysis for Yield Line Pattern-II

Analysis of Slab

The slab will now be analysed by both the methods i.e., virtual work method and equilibrium method.

(a) *Virtual Work Method*

Total External Work Done (EWD) by the uniformly distributed load is given by :

$$EWD = W_1 + W_{2A} + W_{2B} + W_{2C} + W_3$$

$$= 2\left(\frac{1}{2}\, wxL_y\, \frac{\Delta}{3}\right) + \frac{1}{2}\, wxL_y\, \frac{\Delta}{3} + w\left(L_x - 2x\right) L_y\, \frac{\Delta}{2}$$

$$= wL_y\, (3L_x - 2x)\, \frac{\Delta}{6}$$

Total Internal Work Done (IWD) by the yield moments is given by

$$IWD = 2M_x\, L_y\, \theta_x + 2M_y\, x\, \theta_y$$

$$= 2M_x\, L_y\, \frac{\Delta}{x} + 2M_y\, x\, \frac{\Delta}{L_y}$$

Equating the total EWD to the total IWD,

$$wL_y\, (3L_x - 2x)\, \frac{\Delta}{6} = 2M_x\, L_y\, \frac{\Delta}{x} + 2M_y\, x\, \frac{\Delta}{L_y}$$

or, $$w = \frac{12M_x\, L_y^2 + 12M_y\, x^2}{L_y^2\, (3x\, L_x - 2x^2)} \qquad \ldots (20.14(a))$$

Setting to zero the derivative of w with respect to x, i.e., $\frac{dw}{dx} = 0$.

$$\text{or,} \quad 24M_y x L_y^2 (3x L_x - 2x^2) - (12M_x L_y^2 + 12M_y x^2) L_y^2 (3L_x - 4x) = 0$$

$$\text{or,} \quad 3r L_x x^2 + 4L_y^2 x - 3L_x L_y^2 = 0 \quad \text{. . . (20.15(a))}$$

where, $r = \frac{M_y}{M_x}$, solving for x,

$$x = \frac{-4L_y^2 \pm \sqrt{16L_y^4 + 36r L_x^2 L_y^2}}{6r L_x} \quad \text{. . . (20.15(b))}$$

if x is less than $L_x / 2$,

$$= \frac{-4L_y^2 + \sqrt{16L_y^4 + 36r L_x^2 L_y^2}}{6r L_x} < \frac{L_x}{2}$$

$$16L_y^4 + 36r L_x^2 L_y^2 < \left[3r L_x^2 + 4L_y^2\right]^2$$

$$\text{or,} \quad r > \frac{4 L_y^2}{3L_x^2} \quad \text{. . . (20.16)}$$

(b) *Equilibrium Method*

The nodal force can be determined by using Eq. (20.6)

$$V = M_y \cot \alpha \quad \Rightarrow \quad V = M_y \frac{x}{L_y}$$

Let us consider the equilibrium of slab segment 1 and taking moment of all force about its left edge,

$$\frac{1}{2} wx L_y \frac{x}{3} + V_x - M_x L_y = 0$$

$$\text{or,} \quad w = \frac{6M_x L_y^2 - 6M_y x^2}{x^2 L_y^2} \quad \text{. . . (20.14(b))}$$

Similarly, considering the equilibrium of slab segment 2 and taking moment of all forces about its bottom edge,

$$w = \frac{24x M_y}{L_y^2 (3L_x - 4x)} \quad \text{. . . (20.14(c))}$$

Equating w from Eqs. (20.25) and (20.26)

$$3L_x \left(\frac{M_y}{M_x}\right) x^2 + 4L_y^2 x - 3L_x L_y^2 = 0 \quad \text{. . . (20.15(c))}$$

This equation is same as Eq. (20.15(a)).

Example 20.3

A square slab is simply supported on three sides and is free on the fourth side. If the moment capacities are equal in both directions, calculate the collapse load.

Solution

Given $L_x = L_y = L, \; M_x = M_y = M$

Let us first determine the yield line pattern in which the slab is likely to collapse.

For yield pattern-I $\quad \frac{M_y}{M_x} < \frac{4 L_y^2}{L_x^2}$

and, for yield pattern-II $$\frac{M_y}{M_x} > \frac{4\,L_y^2}{3\,L_x^2}$$

Since, $$r = \frac{M_y}{M_x} = 1\,, \quad \frac{L_y}{L_x} = 1$$

Therefore, slab will collapse in yield pattern-I. Determining the value of y by using Eq. (19.19)

$$y = \frac{-2r\,L_x^2 + \sqrt{4r^2\,L_x^4 + 48r\,L_x^2\,L_y^2}}{8\,L_y}$$

$$y = \frac{-2\,L^2 + \sqrt{4L^4 + 48L^4}}{8L} = 0.65L$$

Collapse load can be determined by using Eqs. (20.11(a)), (20.11(b)) or (20.11(c)). From Eq. (20.11(a)),

$$w = \frac{6M_y}{y^2} = \frac{6M}{(0.65L)^2} = 14.2\,\frac{M}{L^2}$$

Example 20.4

A slab whose length is 1.5 times its breadth is simply supported on three edges. It is free on one of its longer sides. If the moment carrying capacity along longer span is 60% that of along shorter span, calculate the collapse load.

Solution

Given that $L_x = 1.5L_y = 1.5L$ (say), $M_x = 0.6M_y = 0.6M$ (say)

Determining the yield line pattern in which the slab is likely to collapse,

For yield pattern-I : $\dfrac{M_y}{M_x} < 4\,\dfrac{L_y^2}{L_x^2}$

For yield pattern-II : $$\frac{M_y}{M_x} > \frac{4\,L_y^2}{3L_x^2}$$

But, $$r = \frac{M_y}{M_x} = \frac{1}{0.6} = \frac{5}{3}\,, \quad \frac{L_y^2}{L_x^2} = \frac{1}{2.25} = \frac{4}{9}$$

For given data, both the above conditions are satisfied. So we will calculate the collapse load for both the yield line patterns and the one giving minimum value will be the correct collapse load and the correct yield pattern.

For yield pattern-I, the value of y from Eq. (20.12(b))

$$y = \frac{-2r\,L_x^2 + \sqrt{4r^2\,L_x^4 + 48r\,L_x^2\,L_y^2}}{8L_y} = 0.98L$$

Collapse load can be determined by using any of the Eqs. (20.11(a)), (20.11(b)) or (20.11(c)). From Eq. (20.11(c)) we have

$$w = \frac{6M_y}{y^2} = \frac{6M}{(0.98L)^2} = 6.25\,\frac{M}{L^2}$$

For pattern II, the value of x by using Eq. (20.15(b))

$$x = \frac{-4L_y^2 + \sqrt{16L_y^4 + 36r\,L_x^2\,L_y^2}}{6r\,L_x} = 0.75L$$

Collapse load can be determined by using any of the Eqs. (20.14(a)), (20.14(b))

$$w = \frac{24x M_y}{L_y^2 (3L_x - 4x)}$$

$$= \frac{12M}{L^2}$$

Since the first pattern yields the minimum value of collapse load, hence the correct value of collapse load is 6.25 M / L^2 and the slab will fail in pattern I.

20.2.9 Circular Slabs

A circular slab can be considered as a polygonal slab with sides tending to infinity. Therefore, consider an isotropically reinforced regular n sided polygonal slab continuous over the edges (Figure 20.16) and loaded by uniformly distributed load of intensity w. Let M_b and M_t be the positive (for bottom reinforcement) and negative (for top reinforcement at supports) ultimate moment capacities per m length of slab. Let L be the length of each side of polygon and r be the radius of the inscribed circle. The postulated yield line pattern of the slab is shown in Figure 20.16. Since the yield lines meeting at the centre of the slab are all positive and are governed by the same mesh, the nodal forces at this point are zero. Considering the equilibrium of one of the segment AOB, we get

$$\left(M_b + M_t\right) L = w \left(\frac{L r}{2}\right) \left(\frac{r}{3}\right) = \frac{w L r^2}{6} \quad \ldots (20.16)$$

Therefore,

$$w = \frac{6}{r^2} \left(M_b + M_t\right) \quad \ldots (20.17)$$

This equation can be applied to a wide range of cases of regular sided polygonal slabs by substituting the suitable values of parameters, e.g.

for an equilibrium triangular slab ($n = 3$) : $r = \frac{L}{2\sqrt{3}}$,

for square slab ($n = 4$) : $r = L/2$,

for hexagonal slab ($n = 6$) : $r = \frac{\sqrt{3} L}{2}$.

For circular slab $n \to \infty$, $L \to 0$; and r becomes the radius of the circle,

Therefore

$$w = \frac{6}{r^2} \left(M_b + M_t\right) \quad \ldots (20.18)$$

Though infinite number of positive yield lines have been considered by taking $n \to \infty$, and $L \to 0$, but practically, a circular slab will collapse with finite number of positive yield lines and hence the negative yield line as shown in Figure 20.17 will not be exactly circular and instead it will be polygonal. Therefore, the load carrying capacity of the circular slab will be slightly more than that given by Eq. (20.18) because the value of r in this equation will be slightly less than the radius of the circular slab. For the case of a simply supported slab, M_t will be zero.

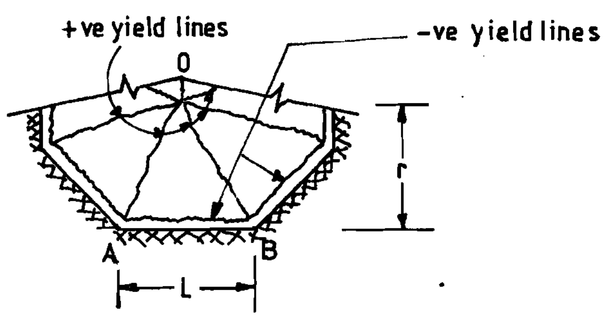


Figure 20.16 : Yield Line Pattern in a Polygonal Slab Carrying Uniformly Distributed Load

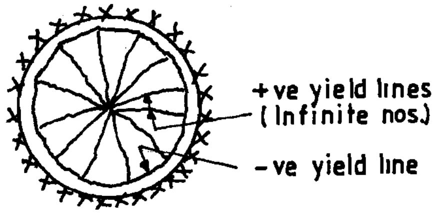


Figure 20.17 : Yield Line Pattern in a Circular Slab Carrying Uniformly Distributed Load

Example 20.5

Determine ultimate moment capacity of a simply supported circular slab of 4.5 m diameter. The slab is supported on 230 mm thick masonry walls and has to carry a live load of 3 kN / sq. m and finish load of 1 kN / sq. m.

Solution

Assuming the slab to be 120 mm thick and isotropically reinforced. The effective radius of the slab, $a = (4.5 + 0.23)/2 = 2.365$ m

Loads

Self weight of slab (120 mm) = 0.12×25		= 2.00
Finish load		= 1.00
Live load		= 3.00
Total load,	w	= 7.00
Total factored load,	w_u	= 10.50

From Eq. (20.18), putting $M_t = 0$ because the slab is simply supported, maximum ultimate moment capacity required for the slab,

$$M = \frac{w_u a^2}{6} = 9.79 \text{ kN m / m}$$

Therefore, orthogonal mesh required in the slab shall correspond to a factored moment of 8.86 kN m / m.

From elastic analysis (discussed in Section 3.2), radial and circumferential BM at the centre of slab $\frac{3 w_u a^2}{16} = 11.01$ kN m / m.

A comparison of the results of yield line analysis with the results of elastic analysis shows that the BM obtained from yield line analysis is 11.1% less than the elastic BM, if one designs the slab on the basis of yield line analysis then proper checking of deflections and cracking will be mandatory. As the normal design procedure only involves the control deflection indirectly, therefore, the use of the results of yield line analysis for design purposes should be avoided.

20.3 SUMMARY

Reinforced concrete slabs when loaded to failure develop a characteristics pattern of cracks depending upon the shape of the slab, support conditions and the distribution of reinforcement in the slab. This crack pattern is called as yield line pattern. With the increase of the load on the slab, the region of highest moment will yield first and the yield lines are propagated until they reach the boundaries of the slab. The final failure will take place by the rotation of the slab elements about the axes of rotation which are usually the support edges of the slab. It is important that for complete development of yield line pattern the slab must be under reinforced so that sufficient rotation capacity is available for the initiation and propagation of the yield lines. Yield line pattern developed in slabs depends on the shape of slab and its edge conditions.

Ultimate load capacity of a slab can be determined either by virtual work method or by equilibrium method.

20.4 ANSWERS TO SAQs

Refer the relevant preceding text in the unit or other useful books on the topic listed in the section "Further Reading" to get the answers of the SAQs.