UNIT 5 RECTILINEAR MOTION, PROJECTILES & RELATIVE MOTION

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5.1 INTRODUCTION

Dynamics is the study of motion of objects. It involves two aspects of study, one the geometry of the path of motion and the other involving the associated forces.

Kinematics implies the geometry of motion involving the displacement, velocity and acceleration of a particular point at a particular time or with passage of time.

The position of a point is specified by a position vector directed from the origin of reference frame to the point under consideration. The displacement of a point can be in any arbitrary direction from its original position and can be represented by corresponding components along three mutually perpendicular directions. That is it has degrees of freedom in a general motion. In this unit, the understanding of various types of displacement, velocities and acceleration of particles is developed.

Basic knowledge of algebra, geometry, trigonometry etc. will be required. Knowledge of differential and integral calculus will be advantageous. Before we develop basic concepts of kinematics it will be beneficial to introduce and define some basic associated terms.

Objectives

After studying this unit, you should be able to

- conceptualise the motion of a particle in terms of displacement, acceleration and velocity etc.,

- describe the projectile motion and develop the expression for time of flight, height, range and angle of projection, and
- explain the relative velocity of two bodies.

5.2 RECTILINEAR MOTION

5.2.1 Basic Terms

Motion

A body may be said to in a state of motion if it changes its position with respect to its surroundings. We may clearly perceive the motion of bodies such as vehicle moving on a roads, birds flying in sky. However many times we may not be able to perceive the motion but know that there must have been a motion, e.g. growth of tree, blooming of flowers etc. An observer on earth may feel he is stationary but slowly changing star positions, sun positions, indicate that he must be moving in space.

Thus a body may be stationary with reference to one frame while moving with reference to another frame like the above observer who is stationary with respect to earth, while moving with reference to sun/other stars.

Particle

A particle may be defined as a material point without any dimension. When the size of a body is very small as compared with the range to motion (i.e. earth moving in galaxial space), it can be considered as a particle. Similarly the bullet dimensions are very small as compared with its range and trajectory hence can be treated as a particle.

When a particle moves in space it describes a curve called path. The nature of path of displacement determines the type of motion. It can be rectilinear translation when it moves along a straight line, it could be curvilinear translation when the displacement path is curved or it can be rotary. Rotary or circular motion is a special case of curvilinear motion where the particle moves along concentric circles and displacements can be measured in terms of angles in radians or revolutions. Actually nothing is absolutely at rest or in motion. All rest/motion are relative.

Displacement

If a particle has motion with respect to some point which is assumed to be fixed, its displacement is its total change of position during any interval of time. The point of reference usually assumed is one which is at rest with respect to earth's surface. The displacement is a vector quantity identified by its line of action, magnitude and direction.

Consider a particle moving along the x axis as shown in Figure 5.1. The position of the particle can be defined by x coordinate measured from fixed reference point O. Let a particle moves from point
A ($x = x$, at $t = t$) to point B ($x = x + \Delta x$) at time $t = t + \Delta t$.

Then displacement of particle in time Δt is equal to AB (i.e. Δx).

Figure 5.1

Speed

Speed of a moving particle is the distance travelled by it in a given time. It is defined as rate of change of its position with respect to its surrounding irrespective of its direction.

$$\text{Mathematically, Speed } = \frac{\text{Distance covered}}{\text{Time taken}} = \frac{s}{t}$$

Speed can be uniform or variable. It is uniform when the particle would move equal distance in equal interval of time however small or large this interval may be. Variable speed means that the speed varies with distance (from point to point) or time (from instance to instance) on its path. As such it can be described only in terms of instantaneous speed at a particular point of instance.

Velocity

The velocity of a body is the rate of change of its position with respect to its surrounding in a particular direction per unit time. Since displacement has magnitude as well as direction, velocity too will have magnitude as well as direction. It can be mathematically expressed as

$$\text{Velocity} = \text{Distance covered} = \frac{\text{Displacement in a particular direction}}{\text{Time taken}}$$

or $$V = \frac{s}{t}$$

Like speed, velocity can also be uniform or variable. If the displacement of a particle is x at any time t (Figure 5.1), the motion of the particle can be completely defined by the relationship $x = f(t)$ where $f(t)$ is a function of time. The particle velocity is v.

$x = x_0 + vt$, if x_0 is initial displacement at $t = 0$ such a motion is called a uniform rectilinear motion. In case of variable velocity the variable part could be magnitude, or direction or both. In Figure 5.1, distance $AB = \Delta x$. Then Average change of displacement also defined as average velocity during time interval

$$v_{av} = \frac{\Delta x}{\Delta t}$$

If time interval, Δt, is gradually reduces to 0 the velocity is defined as instantaneous velocity or simply velocity. Thus

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \qquad \text{. . . (5.1)}$$

Velocity Parallelogram

A particle may have velocity in two or more different directions like it has displacement components in more than one direction. For example, a swimmer crossing a river may have one motion due to his own swimming efforts while another due to flow of water. Thus the actual motion of this man will be different than what it would have been in stationary water of a swimming pool. The actual motion of the man can be obtained either analytically or graphically by law of parallelogram of velocity, which can be stated as follows.

"If a moving point possess simultaneous velocities which are represented by the two sides of a parallelogram in magnitude and direction, the resultant will be equivalent to a velocity in magnitude and direction by the diagonal of the parallelogram passing through that point. The velocity which is equivalent to two or more velocities is called their Resultant. The velocities which together give a resultant are called the components of the resultant.

Let two simultaneous velocities be represented by the line AB and AC with magnitudes *u* and *v* (Figure 5.2), then the diagonal AD will represent the equivalent resultant velocity of the particle. Let *u* and *v* are velocities along AB and AC respectively, and acting at an angle α with one another.

Figure 5.2

Then, $AD^2 = AB^2 + BD^2 + 2\,AB\,.\,BD\,.\cos\alpha \quad (ABD)$

or $$r^2 = u^2 + v^2 + 2uv\cos\alpha \text{ (Since } \angle ABD = \pi - \alpha) \qquad \text{. . . (5.1(a))}$$

Let angle BAD = θ, we have

$$\frac{AB}{BD} = \frac{\sin ADB}{\sin BAD} = \frac{\sin DAC}{\sin BAD}$$

or $$\frac{u}{v} = \frac{\sin(\alpha - \theta)}{\sin\theta} = \frac{\sin\alpha\cos\theta - \cos\alpha\sin\theta}{\sin\theta}$$

$$= \sin\alpha\cot\theta - \cos\alpha$$

or $$\cot\theta = \frac{u + v\cos\alpha}{v\sin\alpha}$$

or $$\tan\theta = \frac{v\sin\alpha}{u + v\cos\alpha} \qquad \text{. . . (5.1(b))}$$

Acceleration

Acceleration is the rate of change of velocity with respect to time. Like velocity, acceleration is also a vector quantity represented both by its magnitude and direction. If equal change in velocity takes place in equal interval of time, the acceleration is said to be uniform, however small this interval may be. Mathematically, it can be expressed as

$$\text{Acceleration} = \frac{\Delta v}{\Delta t}$$

where Δv is change in velocity in Δt interval of time. In limiting case :

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \qquad \ldots (5.2)$$

Since velocity is represented in terms of distance/time, the unit of acceleration would be distance/time/time, i.e. distance/time^2.

Parallelogram of Acceleration

If a moving point has two accelerations simultaneously as represented by two sides of a parallelogram in magnitude and direction, they are equivalent to an acceleration represented by the diagonal of the parallelogram passing through that point.

5.2.2 Equations of Motion in a Straight Line

Let a point move in a straight line, starting with velocity u, and moving with a constant acceleration, f, in its direction of motion; if v is its velocity at the end of time t, and s is the distance travelled by it from its starting point, then

$$v = u + ft \qquad \ldots (5.3(a))$$

$$s = ut + \frac{1}{2} ft^2 \qquad \ldots (5.3(b))$$

$$v^2 = u^2 + 2fs \qquad \ldots (5.3(c))$$

These are known as Equations of Motion in a Straight Line.

Proof

(i) Since f denotes the acceleration, i.e., the change in velocity per unit of time, ft denotes the change in the velocity in t units of time.

Since final velocity after time t = initial velocity + increase in velocity in time t.

$$v = u + ft$$

(ii) Let V be the velocity at the middle of the interval, so that

$$V = u + f\frac{t}{2}$$

Now the velocity changes uniformly throughout the interval t. Hence the velocity at any instance, preceding the middle of the interval by any time T, is as much less than V, as the velocity at the same time T after the middle of the interval is greater than V.

Since the time t could be divided into pairs of such equal moments, the space described is the same as if the point moved for time t with velocity V.

Therefore $\quad s = V \,.\, t$

$$= \left(u + f\frac{t}{2}\right) t$$

$$= ut + \frac{1}{2} ft^2$$

(iii) The third relation can be obtained by eliminating t between the first two relations.

$$v^2 = (u + ft)^2$$

$$= u^2 + 2u\,f\,t + f^2 t^2$$

$$= u^2 + 2\,f\left(u\,t + \frac{f\,t^2}{2}\right)$$

Hence $v^2 = u^2 + 2\,f\,s$

Space Described in any Particular Second

The three equations of motions given above describe the distance covered in time t. The same equations can be used to find out the distance traveled by a particle on the t^{th} second.

Distance travelled in the t^{th} second = distance travelled in t second – distance travelled in $(t – 1)$ second.

$$= \left(u\,t + \frac{1}{2}\,ft^2\right) - \left(u\,(t-1) + \frac{1}{2}\,f\,(t-1)^2\right)$$

$$= u + f\,[t^2 - (t-1)^2]$$

$$= u + f\,\frac{2t-1}{2}$$

Hence the distance travelled in the first, second, third, . . . nth second of motion are

$$u + \frac{1}{2}\,f,\; u + \frac{3}{2}\,f, \ldots, u + \frac{2n-1}{2}\,f$$

Example 5.1

A train which is moving at the rate of 60 km per hour, is brought to rest in 3 minutes with a uniform retardation, find its retardation, and also the distance that the train travels before coming to rest.

$$60 \text{ km per hour} = \frac{60 \times 1000}{60 \times 60} = \frac{50}{3} \text{ m per sec.}$$

Let f be the acceleration with which the train moves.

Since in 180 seconds a velocity of $\frac{50}{3}$ m is brought to zero using the first equation of motion, we have

$$0 = \frac{50}{3} + f\,(180)$$

$$f = -\frac{5}{54}\,\text{m/sec}^2$$

Let x be the distance described by the train before coming to a halt.

$$0 = \left(\frac{50}{3}\right)^2 + 2\left(\frac{-5}{54}\right)x$$

$$x = \left(\frac{50}{3}\right)^2 \times \frac{1}{2} \times \frac{54}{5}$$

$$= 1500 \text{ m.}$$

5.2.3 Motion Under Gravity

When a body falls towards the earth, it moves with a constant acceleration. This was shown first by Galileo by his famous experiments conducted at the leaning tower of Pisa and his experiments on inclined plane.

This acceleration is always the same at the same place but varies slightly for different places.

The value of this acceleration, which is called acceleration due to gravity, is by convention, always denoted by letter g.

Vertical Motion Under Gravity

If a body is projected vertically up from a point on the earth with a starting velocity u, it would experience an acceleration-g (the acceleration opposing the motion). The velocity of the body would continue to get less and less until it becomes zero, the body would then be for an instance at rest, but would begin to acquire a velocity in a downward direction, and retrace its steps.

Greatest Height Attained

At the highest point the velocity is just zero, hence, if x be the greatest height attained, we have

$$0 = u^2 - 2gx$$

Hence the greatest height attained = $\dfrac{u^2}{2g}$

Also the time T to attained the greatest height is given by

$$0 = u - gT$$

$$T = \frac{u}{g}$$

Velocity Due to a Given Vertical Fall from Rest

If a body be dropped from rest, its velocity after falling through a height h is obtained by substituting 0, g, and h in the equation for motion.

$$v = \sqrt{2gh}$$

Time to a Given Height

The height h at which a body has arrived in time t is given by substituting -g for f in the second equation of motion, i.e.

$$h = ut - \frac{1}{2}gt^2$$

This is a quadratic equation with both roots positive; the lesser root gives the time at which the body is at the given height on the way up, and the greater the time at which it is at the same height on the way down.

Velocity at a Given Height

The velocity *v* at a given height *h* is, using the equations of motion in a straight line is given by

$$v^2 = u^2 - 2gh$$

Hence the velocity at a given height is independent of the time from the start, and it is therefore the same at the same point whether the body be going upwards or downwards.

Motion Down a Smooth Inclined Plane

Let AB be the vertical section of a smooth inclined plane inclined at an angle α to the horizon, and let P be a body on the plane.

If there were no plane to stop its motion, the body would fall vertically with an acceleration *g*.

This acceleration *g* can be resolved in two components :

One perpendicular to the plane = $g \cos \alpha$

And another along the plane = $g \sin \alpha$

Figure 5.3

The plane prevents any motion perpendicular to itself.

The only motion possible is along the plane with an acceleration of $g \sin \alpha$.

Therefore, the velocity acquired in sliding from rest to down a length l of the plane is

$$= \sqrt{2g \sin \alpha l} = \sqrt{2gl \sin \alpha} = \sqrt{2g\,.AC}$$

This velocity is the same as that would have been acquired had the body been allowed to fall freely through the vertical height.

In other words, the velocity acquired is independent of the inclination of the plane and depends only on vertical height through which the particle has fallen.

Similarly, if the body is projected up the plane with the initial velocity *u*, greater distance attained, measured up the plane = $\dfrac{u^2}{2g \sin \alpha}$.

Time taken in traversing this distance $= \dfrac{u}{g \sin \alpha}$.

5.3 PROJECTILES

Any object that is given some initial velocity and which during the subsequent motion is subjected to only the acceleration due to gravity is termed as a projectile. For example, the motion of a bullet fired from a gun and the motion of a ball thrown horizontally are termed as projectile motion. A projectile travels in the horizontal as well as the vertical directions and traces a curvilinear path. Some related terms are explained below (Figure 5.4).

Figure 5.4

Velocity of Projection and the Angle of Projection

The velocity v_0 with which the projectile is projected and the angle α which this velocity makes with the horizontal are known as the velocity of projection and the angle of projection, respectively.

Range

Range of a projectile is the horizontal distance between the point of projection and the point where the projectile hits the ground.

Trajectory

Trajectory is the path followed by the projectile. It would be shown later that a projectile follows a parabolic path.

5.3.1 Motion of a Body Thrown Horizontally in Air

Consider an object thrown horizontally with a velocity v_0 from the point A which is at height h from the horizontal plane 0_x. Let it hit the horizontal plane at B after a time t, as shown in Figure 5.5.

Consider the motion of the projectile along Ox and Oy.

Initial velocity along $Ox = v_0$.

Initial velocity along $Oy = 0$

Acceleration along $Ox = 0$

Acceleration along $Oy = g$

Motion along $O - y$ (Vertical Direction)

Figure 5.5

In time t, the projectile shall travel a vertical distance h.

Using the relation $s = ut + \frac{1}{2} at^2$.

We have, $h = 0 + \frac{1}{2} gt^2$.

Time of flight, $t = \sqrt{\frac{2h}{g}}$

Motion along $O - x$ (Horizontal Direction). The range OB or the distance travelled by the projectile along Ox can be obtained using.

$$s = ut + 1/2\, at^2$$

$$\text{Range} = \text{OB} = ut + 0 = r$$

Substituting r for s and for $t = \sqrt{\frac{2h}{g}}$

$$r = \text{Range} = v_0 \sqrt{\frac{2h}{g}}$$

5.3.2 Equation for Path of Projectile

Consider the motion of an object projected from the origin *0* of the co-ordinate system, with the initial velocity v_o inclined at an angle α with the horizontal or the x-axis as shown in Figure 5.4

The motion of the object P can be studied by resolving its velocity v_o in the horizontal and the vertical directions as shown in Figure 5.4(b).

Horizontal component of the velocity v_o

$$v_x = v_o \cos \alpha$$

Vertical component of the velocity

$$v_y = v_o \sin \alpha$$

The horizontal velocity v_x of the object shall remain constant as no acceleration is acting in the horizontal direction ($a = 0$).

The initial velocity in the vertical direction v_y shall go on decreasing because of the constant deceleration due to gravity ($a = - g$).

The object, therefore, is having horizontal and vertical motions simultaneously. The resultant motion would be the vector sum of these two motions and the path followed would be curvilinear.

Let P be the position of the object after a time t.

Using the relation, $s = ut + \frac{1}{2} at^2$, we can determine,

The distance travelled in the horizontal direction in time t,

$$x = (v_0 \cos \alpha)\, t \qquad \ldots (5.4(a))$$

The distance travelled in the vertical direction in time t,

$$y = (v_0 \sin \alpha)\, t - \frac{1}{2} gt^2 \qquad \ldots (5.4(b))$$

The Eqs. (5.4(a)) and (5.4(b)) are the time displacement relations for the projectile.

Eliminating the time t, we can obtain a relationship between x and y or the equation of the path of the projectile.

From Eq. 5.4(a) $\quad t = \frac{x}{v_0 \cos \alpha}$

Substituting in Eq. 5.4(b)

$$y = (v_o \sin \alpha) \left[\frac{x}{v_0 \cos \alpha}\right] - \frac{1}{2} g \left[\frac{x}{v_0 \cos \alpha}\right]^2$$

$$y = (\tan \alpha)\, x - \left[\frac{g}{2v_0^2 \cos^2 \alpha}\right] x^2 \qquad \ldots (5.4(c))$$

The above equation is of the form $y = Ax + Bx^2$ and represents a parabola. Thus the path of a projectile is a parabola.

5.3.3 Expressions for Time of Flight, Height, Range and Angle of Projection

Time of Flight

When the projectile P is at the highest point A of the path (vertex), the vertical component of the velocity becomes zero. Using the relation,

$$v = u + at$$

$$0 = v_o \sin \alpha - gt$$

$$t = \frac{v_o \sin \alpha}{g}$$

This t is half the time the projectile has been in the air (from 0 *to A*). The total time of flight is twice this value.

$$\text{Time of flight} = \frac{2v_o \sin \alpha}{g}$$

The maximum height attained by the projectile can be obtained using the relation

$v^2 - u^2 = 2as$ is given by the equation,

$$0 - (v_o \sin \alpha)^2 = -2gH$$

Maximum height attained, $H = \dfrac{(v_o \sin \alpha)^2}{2g}$ . . . (5.5(a))

Range of the Projectile

The range r is the horizontal distance travelled during the time of flight

Range = r = (horizontal velocity) × (time of flight)

$$r = (v_0 \cos \alpha) \left[\frac{2 v_0 \sin \alpha}{g} \right]$$

$$r = \frac{{v_0}^2 \sin 2\alpha}{g} = \text{Range} \qquad \text{. . . (5.5(b))}$$

The Angle of Projection for the Maximum Range of the Projectile

The ranger r for a given velocity v_0 is maximum if

$$\sin 2\alpha = 1$$

or $$2\alpha = \frac{\pi}{2}$$

$$\alpha = \frac{\pi}{4} = 45^o$$

Corresponding to this value of α, the maximum range is

$$R_{\max imum} = \frac{{v_0}^2}{g} \qquad \text{. . . (5.5(c))}$$

The Angle of Projection for a Given Range

As, $\sin 2\theta = \sin(\pi - 2\theta)$

Therefore,

$$r = \frac{{v_0}^2 \sin 2\alpha}{g}$$

$$r = \frac{{v_0}^2 \sin(\pi - 2\alpha)}{g}$$

So for a given value of the initial velocity v_0, the same range is obtained for two angles of projection; α and $\dfrac{1\,(\pi - 2\alpha)}{2}$, these two angles of projection are equally inclined to the direction corresponding to the maximum range (that is, $\dfrac{\pi}{4}$) and are as shown in Figure 5.6.

It should however be noted that the time of flight corresponding to these two angles of projections α and $\frac{\pi}{2} - \alpha$ would not be the same.

Figure 5.6

5.3.4 Velocity of Motion after a Given Time

Let v be the velocity and θ the angle which the direction of motion at the end of time t makes with the horizontal, then

$v \cos \theta$ = horizontal velocity at end of time t

$= v_0 \cos \alpha$ = the constant horizontal velocity

Also $v \sin \theta$ = the vertical velocity at the end of time t

$= v_0 \sin \alpha - gt$

By squaring and adding

$$v^2 = v_0^2 - 2\, v_0\, g\, t \sin \alpha + g^2 t^2 \qquad \ldots (5.6(a))$$

and by division

$$\tan \theta = \frac{v_0 \sin \alpha - g\, t}{v_0 \cos \alpha} \qquad \ldots (5.6(b))$$

5.3.5 Velocity and Time of a Given Height

Let v be the velocity and θ the angle which the direction of motion at the height h that the particle makes with the horizontal, then

$v \cos \theta$ = horizontal velocity at the height h

$= v_0 \cos \alpha$ = the constant horizontal velocity

Also $v \sin \theta$ = the vertical velocity at the height h

$$= \sqrt{v_0^2 \sin^2 \alpha - 2 h g} \qquad \ldots (5.7(a))$$

Squaring and adding, we get

$$v^2 = v_0^2 - 2\, g\, h$$

also by division

$$\tan \theta = \frac{\sqrt{v_0^2 \sin^2 \alpha - 2 hg}}{v_0 \cos \alpha} \qquad \ldots (5.7(b))$$

5.3.6 Motion of a Projectile on an Inclined Plane

Consider an object projected on an inclined plane *OA* which makes an angle β with the horizontal plane *O*. Let the object be projected with a velocity, v_0, making an angle, α, with the horizontal *Ox*. Let *OB* be the normal to the plane *OA* as in Figure 5.7(a).

The motion of the projectile can be considered as the vector sum of its motions along and normal to the inclined plane.

Initial velocity along the plane (along *OA*) = $v_0 \cos(\alpha - \beta)$

Initial velocity normal to the plane (along *OB*) = $v_0 \sin(\alpha - \beta)$

Acceleration due to gravity along the plane (along *OA*) = $-g \sin \beta$

Acceleration due to gravity normal to the plane (along *OB*) = $-g \cos \beta$

Let the projectile hit the inclined plane *OA* at *C* after time *t*.

Figure 5.7(a)

Figure 5.7(b)

Motion Normal to the Inclined Plane (Along OB)

Let the projectile hit the inclined plane after a time *t*. The distance travelled by the projectile normal to the inclined plane in this time is zero.

Using the relation, $$s = ut - \frac{1}{2} at^2$$

$$0 = v_0 \sin(\alpha - \beta)\, t - \frac{1}{2} (g \cos \beta)\, t^2$$

Time of flight, $t = \dfrac{2v_0 \sin(\alpha - \beta)}{g \cos \beta}$ . . . (5.8(a))

Motion Along the Inclined Plane (Along OA)

Range of the projectile is the distance OC travelled by the projectile along the inclined plane.

Using the relation, $s = ut + \dfrac{1}{2} at^2$

Range along the inclined plane

$$= (v_0 \cos(\alpha - \beta))\, t - (\frac{1}{2} g \sin \beta)\, t^2$$

Substituting for t from the Eq. (5.7(a))

Range along the plane

$$= v_0 \cos(\alpha - \beta)\left[\frac{2v_0 \sin(\alpha - \beta)}{g \cos\beta}\right] - \frac{1}{2} g \sin\beta \left[\frac{2v_0 \sin(\alpha - \beta)}{g \cos\beta}\right]^2$$

$$= \frac{2v_0^2 \sin(\alpha - \beta)}{g \cos\beta}\left[\cos(\alpha - \beta) - \frac{\sin(\alpha - \beta)\sin\beta}{\cos\beta}\right]$$

$$= \frac{2v_0^2}{g\cos\beta} \sin(\alpha - \beta)\left[\frac{\cos(\alpha - \beta)\cos\beta - \sin(\alpha - \beta)\sin\beta}{\cos\beta}\right]$$

$$= \frac{2v_0^2}{g\cos\beta} \sin(\alpha - \beta)\frac{\cos(\alpha - \beta + \beta)}{\cos\beta}$$

$$= \frac{2v_0^2}{g\cos^2\beta} \sin(\alpha - \beta)\cos\alpha$$

Range along the inclined plane

$$= \frac{2v_0^2}{g\cos^2\beta} \sin(\alpha - \beta)\cos\alpha \quad \text{. . . (5.8(b))}$$

Above relation can also be expressed as Range along the inclined plane

$$= \frac{2v_0^2 \cos^2\alpha}{g\cos\beta}(\tan\alpha - \tan\beta) \quad \text{. . . (5.8(c))}$$

Maximum Range

$$R = \frac{2v_0^2}{g\cos^2\beta}\sin(\alpha - \beta)\cos\alpha$$

Using, $\sin(A + B) - \sin(A - B) = 2\cos A \sin B$

$$R = \frac{v_0^2}{g\cos^2\beta}\left[\sin(2\alpha - \beta) - \sin\beta\right] \quad \text{. . . (5.8(d))}$$

As the angle of the inclined plane is constant, the maximum value of the range, for a given initial velocity, is obtained when $\sin(2\alpha - \beta)$ is maximum.

or $\sin(2\alpha - \beta) = 1 = \sin\dfrac{\pi}{2}$

$$2\alpha - \beta = \frac{\pi}{2}$$

$$\alpha = \frac{\pi}{4} + \frac{\beta}{2} \qquad \text{. . . (5.8(e))}$$

For determining the value of the maximum range, the above value can be substituted in the expression for range to get

Maximum range along the inclined plane $\dfrac{v_0^{\,2}}{g\cos^2\beta}(1 - \sin\beta)$

$$R_{max} = \frac{v_0^{\,2}}{g\cos^2\beta}(1 - \sin\beta)$$

$$R_{max} = \frac{v_0^{\,2}}{g(1 + \sin\beta)} \qquad \text{. . . (5.8(f))}$$

Motion Down the Inclined Plane

The motion down the inclined plane can be studied by replacing β by – β in the expressions obtained above.

Example 5.2

A projectile is fired with a velocity of 500 m/s at an elevation of 30°, find the velocity and the direction of the projectile after 30 second of its firing.

Solution

$v_0 = 500$ m/s $\quad \alpha = 30^o, \quad t = 30$ s

Initial velocity in the horizontal direction $v_x = 500 \cos 30^o = 433$ m/s

Initial velocity in vertical direction $v_y = 500 \sin 30^o = 250$ m/s

Motion in the horizontal direction :

Horizontal velocity remains constant = 433 m/s

So, horizontal velocity after 30 s = $v_x' = 433$ m/s

Motion in the Vertical Direction

Let the vertical velocity after a time $t = 30$ s be v_y^1.

Using the relation $v = u + at$.

$$v_y^1 = v_y + g\,t = 250 - 9.81 \times 30$$

$$= 44.3 \text{ m/s, downwards.}$$

$$v_0^1 = \sqrt{{v_x^1}^2 + {v_y^1}^2}$$

$$= \sqrt{(433)^2 + (44.3)^2}$$

$$= 435.26 \text{ m/s}$$

$$\tan\theta = \frac{v_y^1}{v_x^1} = \frac{44.3}{433.0} = \frac{1}{10}$$

$\therefore \qquad \theta = 5.84^o$

Example 5.3

Body A is thrown with a velocity of 10 m/s at an angle of 60^o to the horizontal. If another body B is thrown at an angle of 45^o to the horizontal find its velocity if it has the same (a) horizontal range (b) maximum height (c) time of flight, as the body A.

Solution

Body A

Initial velocity, $v_a = 10$ m/s

Angle of projection, $\alpha_A = 60^o$

Body B

Initial velocity be v_b

Angle of projection $\alpha_b = 45^o$

Velocity v_b for the same range can be obtained by using relation

$$r_a = r_b$$

$$\therefore \qquad \frac{v_a^2 \sin 2\alpha_A}{2g} = \frac{v_b^2 \sin 2\alpha_B}{2g}$$

$$\therefore \qquad v_b^2 = \frac{{v_a}^2 \sin 2\alpha_A}{\sin 2\alpha_B} = \frac{(10)^2 \sin (2 \times 60^o)}{\sin (2 \times 45^o)} = 100\ (0.866)$$

$$\therefore \qquad v_a = 9.3 \text{ m/s}$$

Velocity v_b for max height $H_A = H_B$

$$\text{or} \qquad \frac{v_a^2 \sin^2 \alpha_A}{2g} = \frac{v_b^2 \sin^2 \alpha_B}{2g}$$

$$\text{or} \qquad v_b^2 = \frac{v_a^2 \sin^2 \alpha_A}{\sin^2 \alpha_B} = \frac{10^2 \sin^2 60^o}{\sin^2 45^o} = 100 \times \left(\frac{0.866}{0.707}\right)^2 = 150.03$$

$$\text{or} \qquad v_b = 12.25 \text{ m/s}$$

Velocity $\underline{v_b}$ for the equal time of flight

$$t_A = t_B$$

$$\frac{2v_a \sin \alpha_A}{g} = \frac{2v_b \sin \alpha_B}{g}$$

$$v_b = \frac{v_a \sin \alpha_A}{\sin \alpha_B} = \frac{10 \times \sin 60^o}{\sin 45^o} = \frac{10 \times 0.866}{0.707}$$

$$v = 12.25 \text{ m/s}$$

SAQ 1

A boy throws a ball so that it may just clear a wall 3.6 m high. The boy is at distance of 4.8 m from the wall. The ball was found to hit the ground at a distance of 3.6 m on the other side of the wall (Figure 5.8).

Figure 5.8

Find the angle at which the ball should be thrown.

SAQ 2

Two adjacent guns, having the same muzzle velocity of 400 m/s, fire simultaneously at angles of θ_1 and θ_2 for the same target situated at a range of 5000 m.

Figure 5.9

Find the time difference between the two hits.

SAQ 3

The pilot of an aeroplane flying horizontally at a height of 2000 m with a constant speed of 540 km/hour wishes to hit a target on the ground. At what distance from the target should he release the bomb in order to hit the target? At what angle would the target appear to him from that distance?

Figure 5.10

SAQ 4

A soldier positioned on hill fires a bullet at an angle of 30^{o} upwards from the horizontal (Figure 5.11). The target lies 50 m below him and the bullet is fired with a velocity of 100 m/s.

Figure 5.11

Determine

(a) the maximum height to which the bullet will rise above the position of the soldier,

(b) the velocity with which the bullet will hit the target, and

(c) the time required to hit the target.

5.4 RELATIVE MOTION

As discussed earlier, no motion in universe can be considered as absolute. The motion of a moving object on earth is always considered relative to earth which is assumed to be fixed. As we know earth rotates about its axis while it moves in a curvilinear path about the sun which itself is not stationary. Thus the absolute motion of the referred moving object is quite complex. It becomes much more simpler without much loss of practical accuracy if we consider its motion relative to earth. In a similar way the motion of any object (say B) can be analysed with respect to another object A which may be assumed to be fixed. Thus the motion of B (say velocity) can be considered to be consisting of two parts (i) the velocity of B with respect to A, and (ii) the self velocity of object A.

5.4.1 Relative Velocity of Man and Rain

Let us consider the motion of a person in rain, as in Figure 5.12(a), moving in direction OA while rain is falling vertically in direction OC. The relative velocity diagram of the person with respect to rain will be as shown in Figure 5.12(b). Assuming man to be stationary, the rain will have a velocity of its own OC and the superimposed velocity of person (OB in opposite direction of OA). The resultant velocity CB will represent the relative velocity of rain and person both in magnitude and direction.

(a) **(b)**

Figure 5.12

Example 5.4

A passenger sitting in a train at 40 km/h feels rain coming down at 45^o to the vertical. However another person standing on side of railway track, feels the rain to be vertical. Find the actual velocity of the rain.

Solution

Velocity of train = 40 km/h

To find the actual velocity of the rain let us proceed as follows (Refer Figure 5.12(b)).

- Draw a line OA representing the actual direction motion of train moving at 40 km/h.
- Cut off OB equal to 40 km/h to some suitable scale in the opposite direction of the actual motion of train.

- At O, draw a perpendicular line which represents the *actual* direction of the train.
- From C, draw a line making an angle 45^{o} with CO (i.e. vertical) which represents the relative velocity of the rain.
- By measurement, we find that actual velocity of the rain = CO = 40 km/h

Hence actual velocity of rain = 40 km/h.

Analytical Method

In right angled ΔOBC

$$\frac{OB}{OC} = \tan 45^{o} = 1$$

i.e. $$\frac{40}{OC} = \text{ or } OC = 40 \text{ km/h}$$

Example 5.5

When a motor cyclist is riding west at 40 km/h, he finds the rain meeting him at angle of 45^{o} with vertical. When he rides at 24 km/h, he finds the rain at an angle of 30^{o} with the vertical. What is the actual velocity (magnitude and direction) of the rain?

Figure 5.13

Solution

When velocity = 40 km/h, apparent direction of the rain = 45^{o} with the vertical. When velocity = 24 km/h, apparent direction of the rain = 30^{o} with the vertical. The procedure for drawing the velocity diagram is as follows (Refer Figure 5.13).

- Draw North, East, West and South lined meeting at 0.
- Cut off OA equal to 40 km to some suitable scale towards East (opposite to west), i.e. actual direction of the motor cyclist.
- At A, draw an angle of 45^{o} with OA which represents the relative direction of the rain.
- In the second case, since the motor cyclist is riding at 24 km/h therefore cutoff OB equal to 24 km to the scale towards East.

- At B, draw an angle of 60^o (because rain meets the man at an angle of 30^o with the vertical) with OA which represents the direction of relative velocity of the rain. Let the two lines, meet at C.
- Join CO. This gives the actual direction and velocity of the rain.

By Measurement

CO = 38 km/h

$\alpha = 3^o\,16'$

Analytical Method

From C, draw a perpendicular CD to the line OA.

Let $OD = x,$

and $CD = y$

From ΔDCA, we have

$$\frac{DA}{DC} = \tan 45^o$$

or $$\frac{40 - x}{y} = 1 \qquad (\because \ \tan 45^o = 1)$$

i.e. $$y = 40 - x \qquad \ldots \text{(i)}$$

Similarly, in ΔDCB

$$\frac{DB}{DC} = \tan 30^o$$

$$\frac{24 - x}{y} = 0.577$$

or $$y = \frac{24 - x}{0.577} \qquad \ldots \text{(ii)}$$

Eqs. (i) and (ii), we get

$$40 - x = \frac{24 - x}{0.577}$$

$$0.577\,(40 - x) = 24 - x$$

$$23.08 - 0.577\,x = 24 - x$$

$$0.423\,x = 0.92 \quad \text{or} \quad x = 2.17 \text{ m}$$

Substituting the value of x in (i), $y = 40 - 2.17 = 37.83$ m

and $$\tan\alpha = \frac{x}{y} = \frac{2.17}{37.83} = 0.0574 \ : \ \alpha = 3^o - 16'$$

from Δ OCD

$$OC = \sqrt{OD^2 + CD^2} = \sqrt{\{(2.17)^2 + (37.83)^2\}} = 37.9 \text{ km/h}$$

5.4.2 Relative Velocity of Two Bodies

Let us consider two bodies A and B moving relative to each other with velocities V_A and V_B, respectively. In general let the direction of motion of two bodies be inclined to an angle α with each other (Figure 5.14).

Figure 5.14

To obtain the relative velocity of A with respect to B, we construct $OA = V_A$ in the direction of velocity of body A and $OA_1 = V_A$ in the opposite direction. Next we plot $OB = V_B$ in the actual direction of motion of body B and construct the parallelogram $OBCA_1$. The diagonal OC will represent V_{AB} in magnitude and direction the relative velocity between A and B. It may be noted that either A is considered stationary (direction OA reversed) or B is considered stationary in relation to other body.

Example 5.6

Two ships, A and B, are travelling in the same path towards each other. A with a velocity of 70 km/hr while B with 105 km/hr. When they are 1 km apart, ship B turns through 30^{o} to avoid collision. Calculate the nearest distance apart and how long before the distance is reached after B takes the avoiding action.

(a) **(b)**

Figure 5.15

Solution

Draw to scale the paths of two ships A and B. Motion of ship B is 105 km/hr at 30^{o} to the line AB while AB = 1 km. Draw velocity diagram to scale as in Figure 5.15(b) and draw *Oa* parallel to AB = 70 km/hr. Also, draw *Ob* 30^{o} to AB; *b* represents velocity B which is 105 km/hr. Then *ab* represents in magnitude and direction the velocity of B as seen from A.

Draw BC parallel to *ab* to represent path of B relative to the A. The shortest distance between A and B is obtained by drawing perpendicular AC from A on BC.

Then by measurement nearest approach AC = 311 m.

Distance travelled by ship B on path BC = 950 m.

Velocity of B relative to A on this path = *ab* = 135.7 km/hr.

5.5 SUMMARY

In this unit, we have studied the rectilinear motion of a particle by describing equation of motion. In this section, we have also understood the motion under gravity as well as motion of a body along an inclined plane.

Any object projected into space or air, such that it moves under the effect of gravity alone, is called a projectile. The path followed by the particle is called its trajectory. When a projectile is fired parallel to horizontal or making an angle with horizontal, it moves along a parabolic path. In this unit, we have understood projectile motion in detail. At the end of this unit, we have also highlighted the relative motion. With the preparation of this unit, on the way to describe motion, we can now go on to study the causes and laws of motion in the next unit.

5.6 ANSWERS TO SAQs

SAQ 1

Let, the velocity of projection be = v_0

The angle of projection be = α

Range = 4.8 + 3.6 = 8.4 m

The top of the wall AB must lie on the path of the projectile.

The equation of the path with the point of projection as origin is,

$$y = x \tan \alpha - \frac{g x^2}{2 v_0^2 \cos^2 \alpha} \quad \ldots \text{(i)}$$

Point B (4.8, 3.6) lies on this path or curve.

Substituting in Eq. (i)

$$3.6 = 4.8 \tan \alpha - \frac{g \times (4.8)^2}{2 {v_0}^2 \cos^2 \alpha} \quad \ldots \text{(ii)}$$

The range of projectile,

$$r = \frac{v_0^2 \sin 2\alpha}{g} = 8.4 \quad \ldots \text{(iii)}$$

Eqs. (ii) and (iii) involve unknowns α and v

From Eq. (iii)

$$\frac{{v_0}^2}{g} = \frac{8.4}{\sin 2\alpha}$$

Substituting for $\frac{v_0{}^2}{g}$ in Eq. (ii)

$$3.6 = 4.8 \tan\alpha - \frac{(4.8)^2}{2\cos^2\alpha}\frac{\sin 2\alpha}{8.4}$$

$$= 4.8 \tan\alpha - \frac{(4.8)^2}{2\times 8.4}\frac{2\sin\alpha\cos\alpha}{\cos^2\alpha}$$

$$= 4.8 \tan\alpha - 2.74 \tan\alpha = 2.05 \tan\alpha$$

$$\therefore \qquad \tan\alpha = \frac{3.6}{2.057} = 1.75$$

$$\therefore \qquad \alpha = 60.2^{o}$$

SAQ 2

Initial velocity of each gun = 400 m/s

Range of each gun = 5000 m

Range $r = \frac{v_0^2}{g}\sin 2\alpha$

1st gun $\alpha = \theta_1$

$$r = 5000 = \frac{(400)^2}{9.81}\sin 2\theta_1 \qquad \ldots (I)$$

2nd gun $\alpha = \theta_2$

$$r = 5000 = \frac{(400)^2}{9.81}\sin 2\theta_2 \qquad \ldots (II)$$

From Eqs. (I) and (II)

$$\sin 2\theta_1 = \sin 2\theta_2$$

Therefore, either $2\theta_2 = 2\theta_1$

or $2\theta_2 = (\pi - 2\theta_1)$ . . . (III)

From Eq. (I) $\sin\ 2\theta_1 = \frac{(5000\times 9.8)}{(400)^2} = 0.3065$

$$2\theta_1 = 17.8^{o}$$

From Eq. (III) $\theta_1 = 8.9^{o}$

$$2\theta_2 = \pi - 2\theta_1 = 180^{o} - 17.8^{o}$$

$$\theta_2 = 81.1^{o}$$

(To determine the time of flight, consider the horizontal motion of the projectiles)

1st gun time = t_1

range = r = 5000 m

$$\alpha = \theta_1 = 8.9^o$$

$$v_x = v_0 \cos \theta_1 = 400 \cos 8.9^o$$

using the relation $s = ut + \frac{1}{2} at^2$

$$5000 = (400 \times \cos 8.9^o)\, t_1 + 0$$

$$t_1 = \frac{5000}{400 \cos 8.9^o} = 12.65$$

$$t_1 = 12.65 \text{ sec.}$$

Second gun time $= t_2$

$$v_x = 400 \cos 81.1^o$$

$$v = 5000 \text{ m}$$

$$5000 = (400 \cos 81.1^o)\, t_2$$

$$t_2 = \frac{5000}{400 \cos 81.1^o} = 80.79 \text{ sec.}$$

Time difference between the two hits

$= 80.79 - 12.65$ sec

$= 68.14$ sec.

SAQ 3

Initial velocity of the bomb is the same as that of the aeroplane that is, 540 km/hour.

Initial velocity of the bomb in the horizontal direction = 540 km/hour = 150 m/sec

Initial velocity of the bomb in the vertical direction = 0.

Motion in the Vertical Direction

Initial velocity = 0

Distance $h = 2000$ m

Acceleration $g = 9.81$ m/s^2

Using $s = ut + \frac{1}{2} gt^2$

$$2000 = 0 + \frac{1}{2} \times 9.81 \times t^2$$

$$t = \sqrt{\frac{400}{9.81}} = 20.2 \text{ sec}$$

where, t is the time required to travel down a height 2000 m.

Motion in the Horizontal Direction

Initial velocity = 150 m/s

Acceleration = 0.0

Using, $s = ut + \frac{1}{2}at^2$, the horizontal distance travelled by the bomb in the time t = 20.2 s is

$$s = 150 \times 20.2$$

$$s = 3030 \text{ m}$$

So, the bomb should be released when the aeroplane is at a distance of 3030 m from the target and the angle is

$$\tan\theta = \frac{2000}{3030} = 0.660$$

$$\therefore \quad \theta = 33.4^o$$

SAQ 4

Initial velocity in the horizontal direction v_x = 100 cos 30^o = 86.6 m/s

Initial velocity in the vertical direction = v_y = 100 sin 30^o = 50 m/s

(a) To find the maximum height attained by the bullet consider the vertical motion of the bullet. Using,

$$v^2 - u^2 = 2as$$

$$u = v_y = 50 \text{ m/s}, v = 0, a = -9.81 \text{ m/s}^2$$

$$-(50)^2 = 2(-9.81)\,h$$

$$h = \frac{(50)^2}{2 \times 9.81}$$

$$h = 127.4 \text{ m}$$

(b) The horizontal component of the velocity v of the bullet remains constant.

The horizontal velocity of the bullet when it hits the target v_x' = 86.6 m/s

The vertical velocity of the bullet when it hits the target can be found using

$$v^2 - u^2 = 2as$$

$$u = v_y = 50 \text{ m/s}, v' = v'_y, a = -9.81 \text{ m/s}^2, \; s = -50 \text{ m}$$

$$(v'_y)^2 - (50)^2 = -2 \times 9.81 \times 50$$

$$v'_y = \sqrt{50^2 + 981}$$

$$= \pm 59 \text{ m/sec}$$

The velocity v' with which the bullet hits the target

$$v' = \sqrt{(v'_x)^2 + (v'_y)^2}$$

$$= \sqrt{(86.6)^2 + (59)^2}$$

$$= 104.8 \text{ m/s}$$

(c) The initial upward velocity of the bullet $v_y = 50$ m/s changes to $v'_y = -59$ m/s, when it hits the target (under the acceleration due to gravity in time t).

Using, $v = u + at$

Time required to hit the target

$$t = \frac{v - u}{a}$$

$$= \frac{50 + 59}{9.81}$$

$$= 11.1 \text{ sec}$$