
UNIT 16 GAMMA AND BETA DISTRIBUTIONS

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16.1 INTRODUCTION

In Unit 15, you have studied continuous uniform and exponential distributions. Here, we will discuss gamma and beta distributions. Gamma distribution reduces to exponential distribution and beta distribution reduces to uniform distribution for special cases. Gamma distribution is a generalization of exponential distribution in the same sense as the negative binomial distribution is a generalization of geometric distribution. In a sense, the geometric distribution and negative binomial distribution are the discrete analogs of the exponential and gamma distributions, respectively. The present unit discusses the gamma and beta distributions which are defined with the help of special functions known as gamma and beta functions, respectively. So, before defining these distributions, we first define gamma and beta functions in Sec. 16.2 of this unit. Then gamma distribution and beta distribution of first kind followed by beta distribution of second kind are discussed in Secs. 16.3 to 16.5.

Objectives

After studying this unit, you would be able to:

- define beta and gamma functions;
- define gamma and beta distributions;
- discuss various properties of these distributions;
- identify the situations where these distributions can be employed; and
- solve various practical problems related to these distributions.

16.2 BETA AND GAMMA FUNCTIONS

In this section, some special functions i.e. beta and gamma functions are defined with their properties and the relation between them. These will be helpful in defining beta and gamma distributions to be defined in the subsequent sections.

Beta Function

Definition: If $m > 0$, $n > 0$, the integral $\int_0^1 x^{m-1} (1-x)^{n-1} dx$ is called a beta function and is denoted by $\beta(m, n)$ e.g.

$$i) \int_0^1 \sqrt{x} (1-x)^2 dx = \int_0^1 x^{\frac{3}{2}-1} (1-x)^{3-1} dx = \beta\left(\frac{3}{2}, 3\right)$$

$$\text{or } \int_0^1 \sqrt{x} (1-x)^2 dx = \beta\left(\frac{1}{2}+1, 2+1\right) = \beta\left(\frac{3}{2}, 3\right)$$

$$ii) \int_0^1 x^{-\frac{1}{3}} (1-x)^{-\frac{1}{3}} dx = \beta\left(-\frac{1}{3}+1, -\frac{1}{3}+1\right) = \beta\left(\frac{2}{3}, \frac{2}{3}\right)$$

Properties of Beta Function

- Beta function is symmetric function i.e. $\beta(m, n) = \beta(n, m)$
- There are some other forms also of Beta function. One of these forms, which will be helpful in defining beta distribution of second kind, is

$$\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$3. (i) \frac{\beta(p, q+1)}{q} = \frac{\beta(p+1, q)}{p}$$

$$(ii) \beta(p, q) = \beta(p+q, q) \times \beta(p, q+1)$$

On the basis of the above discussion, you can try the following exercise.

E1) Express the following as a beta function:

$$i) \int_0^1 x^{-\frac{1}{3}} (1-x)^{\frac{1}{2}} dx$$

$$ii) \int_0^1 x^{-2} (1-x)^5 dx$$

$$iii) \int_0^{\infty} \frac{x^2}{(1+x)^5} dx$$

$$iv) \int_0^{\infty} \frac{x^{-\frac{1}{2}}}{(1+x)^2} dx$$

Gamma Function

Though we have defined Gamma function in Unit 13, yet we are again defining it with more properties, examples and exercises to make you clearly understand this special function.

Definition: If $n > 0$, the integral $\int_0^{\infty} x^{n-1} e^{-x} dx$ is called a gamma function and is

denoted by $\Gamma(n)$

e.g.

$$(i) \int_0^{\infty} x^2 e^{-x} dx = \Gamma(2+1) = \Gamma(3)$$

$$(ii) \int_0^{\infty} \sqrt{x} e^{-x} dx = \Gamma\left(\frac{1}{2}+1\right) = \Gamma\left(\frac{3}{2}\right)$$

Some Important Results on Gamma Function

$$1. \text{ If } n > 1, \Gamma(n) = (n-1)\Gamma(n-1)$$

$$2. \text{ If } n \text{ is a positive integer, } \Gamma n = (n-1)!$$

$$3. \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Relationship between Beta and Gamma Functions

$$\text{If } m > 0, n > 0, \text{ then } \beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

You can now try the following exercise.

E2) Evaluate:

$$(i) \int_0^{\infty} e^{-x} x^{\frac{5}{2}} dx$$

$$(ii) \int_0^{\infty} (1-x)^{10} dx$$

$$(iii) \int_0^{\infty} x^{\frac{1}{2}} e^{-x} dx$$

16.3 GAMMA DISTRIBUTION

Gamma distribution is a generalisation of exponential distribution. Both the distributions are good models for waiting times. For exponential distribution, the length of time interval between successive happenings is considered i.e. the time is considered till one happening occurs whereas for gamma distribution, the length of time between 0 and the instant when r^{th} happening

occurs is considered. So, if $r = 1$, then the situation becomes the exponential situation. Let us now define gamma distribution:

Definition: A random variable X is said to follow gamma distribution with parameters $r > 0$ and $\lambda > 0$ if its probability density function is given by

$$f(x) = \begin{cases} \frac{\lambda^r e^{-\lambda x} x^{r-1}}{\Gamma(r)}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Remark 1:

(i) It can be verified that

$$\int_0^{\infty} f(x) dx = 1$$

Verification:

$$\begin{aligned} \int_0^{\infty} f(x) dx &= \int_0^{\infty} \frac{\lambda^r e^{-\lambda x} x^{r-1}}{\Gamma(r)} dx \\ &= \int_0^{\infty} \frac{\lambda e^{-\lambda x} (\lambda x)^{r-1}}{\Gamma(r)} dx \end{aligned}$$

Putting $\lambda x = y \Rightarrow \lambda dx = dy$

Also, when $x = 0, y = 0$ and when $x \rightarrow \infty, y \rightarrow \infty$

$$\begin{aligned} &= \frac{1}{\Gamma(r)} \int_0^{\infty} e^{-y} y^{r-1} dy \\ &= \frac{1}{\Gamma(r)} \Gamma(r) \quad [\text{Using gamma function defined in Sec. 16.2}] \\ &= 1 \end{aligned}$$

(ii) If X is a gamma variate with two parameters $r > 0$ and $\lambda > 0$, it is expressed as $X \sim \gamma(\lambda, r)$.

(iii) If we put $r = 1$, we have

$$\begin{aligned} f(x) &= \frac{\lambda e^{-\lambda x} x^0}{\Gamma(1)}, x > 0 \\ &= \lambda e^{-\lambda x}, x > 0 \end{aligned}$$

which is probability density function of exponential distribution.

Hence, exponential distribution is a particular case of gamma distribution.

(iv) If we put $\lambda = 1$, we have

$$f(x) = \frac{e^{-x} \cdot x^{r-1}}{\Gamma(r)}, x > 0, r > 0$$

It is known as gamma distribution with single parameter r . This form of the gamma distribution is also widely used. If X follows gamma distribution with single parameter $r > 0$, it is expressed as $X \sim \gamma(r)$.

Mean and Variance of Gamma Distribution

If X has a gamma distribution with parameters $r > 0$ and $\lambda > 0$, then its

$$\text{Mean} = \frac{r}{\lambda}, \text{ Variance} = \frac{r}{\lambda^2}.$$

If X has a gamma distribution with single parameter $r > 0$, then its

$$\text{Mean} = \text{Variance} = r.$$

Additive Property of Gamma Distribution

1. If X_1, X_2, \dots, X_k are independent gamma variates with parameters $(\lambda, r_1), (\lambda, r_2), \dots, (\lambda, r_k)$ respectively, then $X_1 + X_2 + \dots + X_k$ is also a gamma variate with parameter $(\lambda, r_1 + r_2 + \dots + r_k)$.
2. If X_1, X_2, \dots, X_k are independent gamma variates with single parameters r_1, r_2, \dots, r_k respectively, then $X_1 + X_2 + \dots + X_k$ is also a gamma variate with parameter $r_1 + r_2 + \dots + r_k$.

Example 1: Suppose that on an average 1 customer per minute arrive at a shop. What is the probability that the shopkeeper will wait more than 5 minutes before

- (i) both of the first two customers arrive, and
- (ii) the first customer arrive?

Assume that waiting times follows gamma distribution.

Solution:

- i) Let X denotes the waiting time in minutes until the second customer arrives, then X has gamma distribution with $r = 2$ (as the waiting time is to be considered up to 2nd customer)

$\lambda = 1$ customer per minute.

$$\begin{aligned} \therefore P[X > 5] &= \int_5^{\infty} f(x) dx = \int_5^{\infty} \frac{\lambda^r e^{-\lambda x} x^{r-1}}{\Gamma(r)} dx \\ &= \int_5^{\infty} \frac{(1)^2 e^{-x} x^{2-1}}{\Gamma(2)} dx = \int_5^{\infty} \frac{e^{-x} x^1}{1} dx = \int_5^{\infty} x^1 e^{-x} dx \\ &= \left[\left\{ x \frac{e^{-x}}{-1} \right\}_5^{\infty} - \int_5^{\infty} (1) \frac{e^{-x}}{-1} dx \right] \quad \text{[Integrating by parts]} \\ &= (0 + 5e^{-5}) + \int_5^{\infty} e^{-x} dx = 5e^{-5} + \left[\frac{e^{-x}}{-1} \right]_5^{\infty} \\ &= 5e^{-5} - (0 - e^{-5}) \\ &= 6e^{-5} \end{aligned}$$

$$= 6 \times 0.0070 \quad [\text{See the table given at the end of Unit 10}]$$

$$= 0.042$$

ii) In this case $r = 1$, $\lambda = 1$ and hence

$$P[X > 5] = \int_5^{\infty} \frac{\lambda^r e^{-\lambda x} \cdot x^{r-1}}{\Gamma(r)} dx$$

$$= \int_5^{\infty} \frac{(1)^1 e^{-x} x^0}{\Gamma(1)} dx = \int_5^{\infty} e^{-x} dx = \left[\frac{e^{-x}}{-1} \right]_5^{\infty} = 0 + e^{-5} = 0.0070$$

Alternatively,

As $r = 1$, so it is a case of exponential distribution for which

$$f(x) = \lambda e^{-\lambda x}, x > 0$$

$$\therefore P[X > 5] = \int_5^{\infty} \lambda e^{-\lambda x} dx = \int_5^{\infty} (1) e^{-x} dx = \left[\frac{e^{-x}}{-1} \right]_5^{\infty} = 0 + e^{-5} = 0.0070$$

Here is an exercise for you.

E3) Telephone calls arrive at a switchboard at an average rate of 2 per minute. Let X denotes the waiting time in minutes until the 4th call arrives and follows gamma distribution. Write the probability density function of X . Also find its mean and variance.

Let us now discuss the beta distributions in the next two sections:

16.4 BETA DISTRIBUTION OF FIRST KIND

You have studied in Sec. 16.3 that beta function is related to gamma function in the following manner:

$$\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

Now, we are in a position to define beta distribution which is defined with the help of beta function. There are two kinds of beta distribution – beta distribution of first kind and beta distribution of second kind. Beta distribution of second kind is defined in next section of the unit whereas beta distribution of first kind is defined as follows:

Definition: A random variable X is said to follow beta distribution of first kind with parameters $m > 0$ and $n > 0$, if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{\beta(m, n)} x^{m-1} (1-x)^{n-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

The random variable X is known as beta variate of first kind and can be expressed as $X \sim \beta_1(m, n)$

Remark 5: If $m = 1$ and $n = 1$, then the beta distribution reduces to

$$f(x) = \frac{1}{\beta(1,1)} x^{1-1} (1-x)^{1-1}, 0 < x < 1$$

$$= \frac{x^0 (1-x)^0}{\beta(1,1)}, 0 < x < 1$$

$$= \frac{1}{\beta(1,1)}, 0 < x < 1$$

But $\beta(1,1) = \frac{\Gamma(1)\Gamma(1)}{\Gamma(2)} = \frac{1 \cdot 1}{1}$

Therefore, $f(x) = \frac{(1)(1)}{(1)} = 1$

$$\therefore f(x) = 1, 0 < x < 1$$

$$= \frac{1}{1-0}, 0 < x < 1$$

which is uniform distribution on $(0, 1)$.

[\therefore p.d.f. of uniform distribution on (a, b) is $f(x) = \frac{1}{b-a}, a < x < b$]

So, continuous uniform distribution is a particular case of beta distribution.

Mean and variance of Beta Distribution of First Kind

Mean and Variance of this distribution are given as

$$\text{Mean} = \frac{m}{m+n}$$

$$\text{Variance} = \frac{mn}{(m+n)^2(m+n+1)}$$

Example 4: Determine the constant C such that the function

$f(x) = Cx^3(1-x)^6, 0 < x < 1$ is a beta distribution of first kind. Also, find its mean and variance.

Solution: As $f(x)$ is a beta distribution of first kind.

$$\therefore \int_0^1 f(x) dx = 1$$

$$\Rightarrow \int_0^1 Cx^3(1-x)^6 dx = 1$$

$$\Rightarrow C \int_0^1 x^3(1-x)^6 dx = 1$$

Continuous Probability Distributions

$$\Rightarrow C\beta(3+1, 6+1) = 1 \text{ [By definition of Beta distribution of first kind]}$$

$$\Rightarrow C = \frac{1}{\beta(4, 7)}$$

$$= \frac{|4+7}{|4|7} \quad \left[\because \beta(m, n) = \frac{|m| |n|}{|(m+n)|} \right]$$

$$= \frac{|11}{|4|7} = \frac{|10}{|3|6}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times |6}{3 \times 2 \times |6} = 840$$

$$\text{Thus, } f(x) = 840x^3(1-x)^6$$

$$= 840x^{4-1}(1-x)^{7-1}$$

$$= \frac{x^{4-1}(1-x)^{7-1}}{\beta(4, 7)}$$

$$\left[\because \frac{1}{\beta(4, 7)} = 840 \text{ just obtained above in this example} \right]$$

$$\therefore m = 4, n = 7$$

$$\Rightarrow \text{Mean} = \frac{m}{m+n} = \frac{4}{4+7} = \frac{4}{11},$$

$$\text{and Variance} = \frac{mn}{(m+n)^2(m+n+1)}$$

$$= \frac{4 \times 7}{(4+7)^2(4+7+1)}$$

$$= \frac{28}{(121)(12)} = \frac{7}{(121)(3)} = \frac{7}{363}$$

Now, you can try the following exercises.

E4) Using beta function, prove that

$$\int_0^1 60x^2(1-x)^3 dx = 1$$

E5) Determine the constant k such that the function

$$f(x) = kx^{\frac{1}{2}}(1-x)^{\frac{1}{2}}, 0 < x < 1, \text{ is a beta distribution of first kind. Also find its mean and variance.}$$

16.5 BETA DISTRIBUTION OF SECOND KIND

Let us now define beta distribution of second kind.

Definition: A random variable X is said to follow beta distribution of second kind with parameters $m > 0$, $n > 0$ if its probability density function is given by

$$f(x) = \begin{cases} \frac{x^{m-1}}{\beta(m, n)(1+x)^{m+n}}, & 0 < x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Remark 6: It can be verified that $\int_0^{\infty} \frac{x^{m-1}}{\beta(m, n)(1+x)^{m+n}} dx = 1$

Verification:

$$\int_0^{\infty} \frac{x^{m-1}}{\beta(m, n)(1+x)^{m+n}} dx = \frac{1}{\beta(m, n)} \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \frac{1}{\beta(m, n)} \left[\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx \text{ is another form of beta function. (see Sec. 16.2 of this Unit)} \right]$$

$$= 1$$

Remark 7: If X is a beta variate of second kind with parameters $m > 0$, $n > 0$, then it is expressed as $X \sim \beta_2(m, n)$

Mean and Variance of beta Distribution of second kind

$$\text{Mean} = \frac{m}{n-1}, n > 1;$$

$$\text{Variance} = \frac{m(m+n-1)}{(n-1)^2(n-2)}, n > 2$$

Example 5: Determine the constant k such that the function

$$f(x) = \frac{kx^3}{(1+x)^7}, 0 < x < \infty,$$

is the p.d.f of beta distribution of second kind. Also find its mean and variance.

Solution: As $f(x)$ is a beta distribution of second kind,

$$\therefore \int_0^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_0^{\infty} \frac{kx^3}{(1+x)^7} dx = 1$$

$$\Rightarrow k \int_0^{\infty} \frac{x^{4-1}}{(1+x)^{4+3}} dx = 1$$

$$\Rightarrow k\beta(4,3) = 1$$

$$\Rightarrow k = \frac{1}{\beta(4,3)} = \frac{1}{\frac{1}{4} \frac{1}{3}} = \frac{1}{\frac{1}{12}} = \frac{12}{1} = 12$$

Here $m = 4, n = 3$

$$\therefore \text{Mean} = \frac{m}{n-1} = \frac{4}{3-1} = \frac{4}{2} = 2$$

$$\text{Variance} = \frac{m(m+n-1)}{(n-1)^2(n-2)} = \frac{4(4+3-1)}{(3-1)^2(3-2)} = \frac{4 \times 6}{4 \times 1} = 6$$

Now, you can try the following exercises.

E6) Using beta function, prove that

$$\int_0^{\infty} \frac{x^3}{(1+x)^{\frac{13}{2}}} dx = \frac{64}{15015}$$

E7) Obtain mean and variance for the beta distribution whose density is given by

$$f(x) = \frac{60x^2}{(1+x)^7}, 0 < x < \infty$$

16.6 SUMMARY

The following main points have been covered in this unit:

1) A random variable X is said to follow **gamma distribution with parameters $r > 0$ and $\lambda > 0$** if its probability density function is given by

$$f(x) = \begin{cases} \frac{\lambda^r e^{-\lambda x} x^{r-1}}{\Gamma(r)}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

2) **Gamma distribution** of random variable X with **single parameter $r > 0$** is

$$\text{defined as } f(x) = \frac{e^{-x} x^{r-1}}{\Gamma(r)}, x > 0, r > 0$$

3) For **gamma distribution** with two parameters λ and r , **Mean** = $\frac{r}{\lambda}$ and

$$\text{Variance} = \frac{r}{\lambda^2}.$$

- 4) A random variable X is said to follow **beta distribution of first kind** with parameters $m > 0$ and $n > 0$, if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{\beta(m, n)} x^{m-1} (1-x)^{n-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Its **mean and variance** are $\frac{m}{m+n}$ and $\frac{mn}{(m+n)^2(m+n+1)}$, respectively.

- 5) A random variable X is said to follow **beta distribution of second kind** with parameters $m > 0$, $n > 0$ if its probability density function is given by:

$$f(x) = \begin{cases} \frac{x^{m-1}}{\beta(m, n)(1+x)^{m+n}}, & 0 < x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Its **Mean and Variance** are $\frac{m}{n-1}$, $n > 1$; and $\frac{m(m+n-1)}{(n-1)^2(n-2)}$, $n > 2$ respectively.

- 6) Exponential distribution is a particular case of gamma distribution and continuous uniform distribution is a particular case of beta distribution.

16.7 SOLUTIONS/ANSWERS

$$\mathbf{E1} \text{ (i)} \int_0^1 x^{-\frac{1}{3}} (1-x)^{\frac{1}{2}} dx = B\left(-\frac{1}{3}+1, \frac{1}{2}+1\right) = B\left(\frac{2}{3}, \frac{3}{2}\right)$$

$$\text{(ii)} \int_0^1 x^{-2} (1-x)^5 dx = \int_0^1 x^{-1-1} (1-x)^{6-1} dx$$

is not a beta function, since $m = -1 < 0$, but m and n both should be positive.

$$\text{(iii)} \int_0^{\infty} \frac{x^2}{(1+x)^5} dx = \int_0^{\infty} \frac{x^{3-1}}{(1+x)^{3+2}} dx = \beta(3, 2)$$

[$\because m = 3, n = 2$ (see Property 2 of Beta function Sec. 16.2)]

$$\text{(iv)} \int_0^{\infty} \frac{x^{-\frac{1}{2}}}{(1+x)^2} dx = \int_0^{\infty} \frac{x^{\frac{1}{2}-1}}{(1+x)^{\frac{1}{2}+3/2}} dx = \beta\left(\frac{1}{2}, \frac{3}{2}\right)$$

$$\mathbf{E2} \int_0^{\infty} e^{-x} \cdot x^{5/2} dx = \left(\frac{5}{2} + 1\right)$$

$$= \left(\frac{7}{2}\right)$$

$$= \left(\frac{5}{2}\right) \sqrt{\frac{5}{2}} = \left(\frac{5}{2}\right) \left(\frac{3}{2}\right) \sqrt{\frac{3}{2}} = \left(\frac{5}{2}\right) \left(\frac{3}{2}\right) \left(\frac{1}{2}\right) \sqrt{\frac{1}{2}}$$

[Result 1 on gamma function (See Sec. 16.2)]

$$= \left(\frac{5}{2}\right) \left(\frac{3}{2}\right) \left(\frac{1}{2}\right) \sqrt{\pi} \quad [\text{Result 3 on gamma function}]$$

$$= \left(\frac{15}{8}\right) \sqrt{\pi}$$

$$(ii) \int_0^{\infty} x^1 (1-x)^{10} dx = \beta(1+1, 10+1)$$

$$= \beta(2, 11)$$

$$= \frac{\sqrt{2} \sqrt{11}}{(13)}$$

[∵ see relation between beta and gamma function]

$$= \frac{(1)!(10)!}{(12)!} \quad [\text{Result 2 on gamma function}]$$

$$= \frac{(10)!}{(12)(11)(10)!} = \frac{1}{12 \times 11} = \frac{1}{132}$$

$$(iii) \int_0^{\infty} x^{-\frac{1}{2}} e^{-x} dx = \left[-\frac{1}{2} + 1 \right] = \left[\frac{1}{2} \right] = \sqrt{\pi}$$

E3) Here $\lambda = 2, r = 4$.

$$\therefore f(x) = \frac{\lambda^r e^{-\lambda x} \cdot x^{r-1}}{\Gamma(r)}, x > 0$$

$$= \frac{2^4 \cdot e^{-2x} \cdot x^3}{\Gamma(4)}, x > 0$$

$$= \frac{16e^{-2x} \cdot x^3}{|3|}, x > 0$$

$$= \frac{8}{3} x^3 e^{-2x}, x > 0$$

$$\text{Mean} = \frac{r}{\lambda} = \frac{4}{2} = 2,$$

$$\text{Variance} = \frac{r}{\lambda^2} = \frac{4}{2^2} = 1$$

$$\mathbf{E4)} \int_0^1 60x^2 (1-x)^3 dx = 60 \int_0^1 x^{3-1} (1-x)^{4-1} dx = 60\beta(3, 4)$$

$$= 60 \frac{\sqrt{3} \sqrt{4}}{\sqrt{7}} = 60 \times \frac{\sqrt{2} \sqrt{3}}{\sqrt{6}} = \frac{60 \times 2 \times 3 \times 2}{6 \times 5 \times 4 \times 3 \times 2} = 1$$

$$\mathbf{E5)} \int_0^1 kx^{-\frac{1}{2}}(1-x)^{\frac{1}{2}} dx = 1$$

$$\Rightarrow k\beta\left(-\frac{1}{2}+1, \frac{1}{2}+1\right) = 1$$

$$\Rightarrow k = \frac{1}{\beta\left(\frac{1}{2}, \frac{3}{2}\right)} = \frac{\sqrt{2}}{\left|\frac{1}{2}\right| \left|\frac{3}{2}\right| \sqrt{\pi} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)} = \frac{2}{\sqrt{\pi} \sqrt{\pi}} = \frac{2}{\pi}$$

Now, as the given p.d.f. of beta distribution of first kind is

$$f(x) = \frac{2}{\pi} x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}, 0 < x < 1$$

$$= \frac{x^{\frac{1}{2}-1}(1-x)^{\frac{3}{2}-1}}{\beta\left(\frac{1}{2}, \frac{3}{2}\right)}, 0 < x < 1$$

$$\therefore m = \frac{1}{2}, n = \frac{3}{2}$$

$$\text{and hence mean} = \frac{m}{m+n} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{3}{2}} = \frac{1}{4}$$

$$\text{Variance} = \frac{mn}{(m+n)^2(m+n+1)}$$

$$= \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{\left(\frac{1}{2} + \frac{3}{2}\right)^2 \left(\frac{1}{2} + \frac{3}{2} + 1\right)} = \frac{\frac{3}{4}}{(2)^2(3)} = \frac{3}{4 \times 4 \times 3} = \frac{1}{16}$$

$$\mathbf{E6)} \int_0^{\infty} \frac{x^3}{(1+x)^{13/2}} dx = \int_0^{\infty} \frac{x^{4-1}}{(1+x)^{4+\frac{5}{2}}} dx$$

$$= \beta\left(4, \frac{5}{2}\right) = \frac{\sqrt{4} \sqrt{\frac{5}{2}}}{\left(4 + \frac{5}{2}\right) \sqrt{\frac{13}{2}}}$$

$$= \frac{6 \times \sqrt{\frac{5}{2}}}{\frac{13}{2} \cdot \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \sqrt{\frac{5}{2}}} = \frac{6 \times 32}{13 \times 11 \times 9 \times 7 \times 5} = \frac{64}{15015}$$

Continuous Probability
Distributions

$$\begin{aligned} \text{E7) } f(x) &= \frac{60x^2}{(1+x)^7}, 0 < x < \infty \\ &= \frac{60x^{3-1}}{(1+x)^{3+4}}, 0 < x < \infty \end{aligned}$$

$$= \frac{x^{3-1}}{\beta(3,4)(1+x)^{3+4}}, 0 < x < \infty \quad \left[\because \beta(3,4) = \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = \frac{2!3!}{6!} = \frac{1}{60} \right]$$

$$m=3, n=4$$

$$\text{Hence, mean} = \frac{m}{n-1} = \frac{3}{4-1} = 1$$

$$\text{Variance} = \frac{m(m+n-1)}{(n-1)^2(n-2)} = \frac{3(3+4-1)}{(4-1)^2(4-2)} = \frac{3 \times 6}{9 \times 2} = 1.$$