UNIT 3 ELECTROMAGNETIC WAVE PROPAGATION THEORY

Structure

3.1 INTRODUCTION

In communication systems, we use different ways to transport the electrical signal from the transmitter to the receiver. It can be sent over a metal conductor in the form of a voltage or current signal or transmitted through air in the form of electromagnetic radiation or could be converted into a light signal and sent through an optical fibre. Whatever be the mode of transport, the transmission of signal through all these media is governed by the classical theories of electromagnetic wave propagation. You have already learnt the basic electromagnetic theory in the course on Electric and Magnetic Phenomena (PHE-07). We would suggest that you go through Units 13, 14 and 15 of that course before you start studying this unit.

The physics of electromagnetic signal propagation, their radiation, absorption, etc. are contained in the four **Maxwell's equations**. You know that these equations represent four fundamental laws of electrodynamics. You have also learnt that the Coulomb's law and Ampere's law do not hold for time varying fields. This is because, the action at a distance is delayed in time from the cause producing it. For instance, a conductor carrying time varying current produces time varying magnetic field. However, the change in magnetic field due to change in current is not felt instantaneously in the entire space. In the neighbourhood of the wire it is felt early while at farther distances it is felt later. Further, the time varying magnetic fields produce electric fields. Thus, a wire carrying oscillatory (ac) current produces oscillatory electric and magnetic fields that propagate radially outward. These propagating fields are called *electromagnetic waves*. They carry momentum, energy and information from the current carrying wire called *antenna* to the receiver. You will be learning about these concepts in this unit.

In Sec. 3.2 we shall take a brief review of the basic concepts in electromagnetism, viz. Maxwell's equations. In Sec. 3.3 we shall discuss the wave equation that governs the propagation of electromagnetic waves. Transmission of signal at lower frequencies can take place by surface waves. We discuss the surface wave propagation in Sec. 3.4. At microwave frequencies, the signal is transmitted in the form of electromagnetic radiation through a hallow metallic conductor, which guides the wave to the destination. This is called waveguide. In Sec. 3.5 you will learn about the basic construction of waveguides. In all forms of transmission occurring via electromagnetic radiation, the most important component on the transmitter and receiver side is the antenna. The shape and size of antenna decides the directionality and range of signal transmission. In Sec. 3.6 we take a brief review of antennae.

Objectives

After studying this unit, you should be able to:

- state and explain Maxwell's equations of electromagnetism;
- find the plane wave solution of the wave equation;
- describe propagation of electromagnetic waves through conductor and dielectric;
- explain the concept of waveguides;
- describe the working of antenna;
- explain the radiation pattern of antenna; and
- state characteristics of antenna array.

3.2 REVIEW OF MAXWELL'S EQUATIONS

Maxwell generalised the laws of electrodynamics to time varying fields and put them in compact mathematical form, known as Maxwell's equations. These equations are relativistically covariant, i.e., they retain their form in every inertial frame of reference. In fact, it is these equations, besides Michelson Morley experiment, that led Einstein to establish that the speed of light in vacuum is the same in all inertial frames of reference. All electric and magnetic fields, irrespective of how they are produced, must satisfy Maxwell's equations. These equations represent four fundamental laws of electrodynamics. Before entering into a discussion of these equations let us recapitulate the definitions of some physical quantities.

Electric Field E: If we place a small charge q at a point and it generates a force **F**, then **F**/q, i.e., the force per unit charge is called the electric field **E** at that point. In general, it is a function of position and time. The electric field is produced by charges and also by time varying magnetic fields.

Magnetic Field B: If we bring in a moving charge in a region and it experiences a force by virtue of its charge and motion, then we say that a magnetic field **B** exists in that region. The force experienced by a particle of charge q moving with velocity **v** is related to magnetic field **B** through the relation:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \tag{3.1}$$

The force depends on the direction of motion of the particle (**v**). The force is maximum when the direction of motion and direction of magnetic field are perpendicular to each other. In such a case, the magnitude of magnetic field equals the ratio of maximum force and the product of charge and particle velocity. Magnetic field is produced by electric current and also by time varying electric field, constituting a *displacement current*.

Charge Density ρ: The electrical charge per unit volume is called charge density $\rho(\mathbf{r},t)$.

Current Density J: The charge crossing per unit area (placed perpendicular to the direction of flow) per second is called current density. If there is the flow of current I in a conductor of cross-sectional area A, then $\mathbf{J} = (I/A)\ \hat{\mathbf{i}}$, where $\hat{\mathbf{i}}$ is the unit vector in the direction of current flow.

Polarisation P: Consider an atom with many electrons revolving round the nucleus. The total negative charge of electrons equals the positive charge of the nucleus. In the absence of external electric field, the centre of negative charge cloud coincides with the nucleus. When we apply an electric field on the atom, the electron cloud gets shifted. The cloud of outermost orbiting electrons undergoes maximum displacement. Let the charge of this cloud be $-q$ and the displacement of its centre with respect to nucleus be **d**, then $-q\mathbf{d}$ is the dipole moment of the atom. If there are many atoms in the system, then the vector sum of dipole moments due to all the atoms in a small volume ΔV divided by ΔV, (i.e, the dipole moment per unit volume) is called polarisation $\mathbf{P}(\mathbf{r}, t)$.

Magnetisation M: A current carrying small loop is called a magnetic dipole. An electron revolving round a nucleus is also like a current loop, and behaves like a magnetic dipole. The product of current in the loop and the area of the loop is called the dipole moment of the loop or of an atom. Magnetic dipole moment per unit volume is called magnetisation.

Displacement Vector D: Two physical quantities described above, viz. electric field **E** and polarisation **P** combinedly define the displacement vector **D** by the relation: $\mathbf{D} = \varepsilon_0\ \mathbf{E} + \mathbf{P}$, where $\varepsilon_0 = (10^{-9}/36\pi)$ in MKS units, is the free space permittivity.

Electric Flux $\int \mathbf{D}.\mathbf{ds}$: Consider a small surface area **ds** at point **r**. On multiplying it with the component of displacement vector $\mathbf{D}(\mathbf{r})$ normal to the surface, we obtain $\mathbf{D}.\mathbf{ds}$, called electric flux linked with that surface element. Integrating it over the entire surface gives the total flux linked with the surface.

Magnetic Flux $\int \mathbf{B}.\mathbf{ds}$: $\mathbf{B}.\mathbf{ds}$ is the magnetic flux linked with surface element **ds**. When integrated over the entire area of a surface, $\int \mathbf{B}.\mathbf{ds}$ is called the magnetic flux linked with the surface.

Now we shall discuss the fundamental laws of electrodynamics.

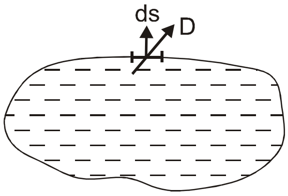

Fig. 3.1: A Gaussian surface with charges enclosed

3.2.1 Gauss's Law for Electric Field

The first law of electrodynamics states that the total outward normal electric flux through any closed surface is equal to the total charge enclosed inside the surface as shown in Fig. 3.1. Mathematically, we can write it as

$$\int_S \mathbf{D}.\mathbf{ds} = \int_V \rho\, dV \qquad (3.2)$$

With Gauss's theorem we can convert a surface integral into volume integral as

$$\int_S \mathbf{D}.\mathbf{ds} = \int_V (\nabla.\mathbf{D})\ dV$$

where $\mathbf{D} = \varepsilon_0\ \mathbf{E} + \mathbf{P}$, is the displacement vector, **E** is the electric field vector, **P** is the polarisation, ρ is the charge density and V represents the volume enclosed by the closed surface S. Using Gauss's theorem, the surface integral can be converted to volume integral, leading to

$$\nabla.\mathbf{D} = \rho \qquad (3.2a)$$

This is Maxwell's first equation. It is valid for all static and time varying fields.

3.2.2 Gauss's Law for Magnetic Field

The second law states that the total outward normal magnetic flux through any closed surface as shown in Fig. 3.2 is zero. This is because of the fact that, in magnets, there are no monopoles. We can write it as:

$$\int_s \mathbf{B.ds} = 0 \tag{3.3}$$

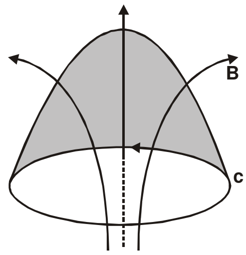

Fig. 3.2: Closed path

where **B** denotes magnetic flux density. Applying Gauss's theorem, the equation is reduced to:

$$\nabla.\mathbf{B} = 0 \tag{3.3a}$$

3.2.3 Faraday's Law of Electromagnetic Induction

Consider any closed current carrying path. The line integral $\int \mathbf{E.dl}$ along the path is known as the *electromotive force* (emf). Faraday's law states that the emf induced in any closed path equals the time rate of change of magnetic flux through the area enclosed by the path. We can consider the area to be any surface *s* having the closed path *c* as its open boundary. The law can be mathematically expressed as:

With Stoke's theorem we convert line integral into surface integral as

$$\int_c \mathbf{E.dl} = \int_s (\nabla \times \mathbf{E}).\mathbf{ds}$$

$$\int_c \mathbf{E.dl} = -\frac{\partial}{\partial t}\int_s \mathbf{B.ds} \tag{3.4}$$

The negative sign arises due to the fact that the emf induced by the change of magnetic flux is such that the current produced by it opposes the change in the magnetic flux. This is known as Lenz's law. Using Stoke's theorem, the line integral can be converted into a surface integral leading to

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3.4a}$$

3.2.4 Generalised Ampere's Law

The Ampere's Law states that for the time independent magnetic fields, the line integral of magnetic intensity **H** (called magnetomotive force, mmf) along any path equals the current passing through any area enclosed by the path, as shown in Fig. 3.3. That is

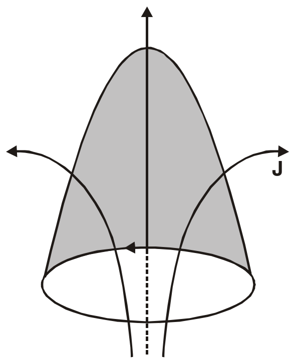

Fig. 3.3: Ampere's loop

$$\int_c \mathbf{H.dl} = \int_s \mathbf{J.ds} \tag{3.5}$$

where **J** is the electric current density due to free charge carriers, and **H** is related to magnetic field **B** as $\mathbf{B} = \mu_0\mathbf{H} + \mathbf{M}$ where **M** is the magnetisation. In free space and non-magnetic materials, $\mathbf{M} = 0$ and $\mathbf{B} = \mu_0\mathbf{H}$. Using Stoke's theorem, the line integral can be replaced by surface integral leading to Ampere's law

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{3.5a}$$

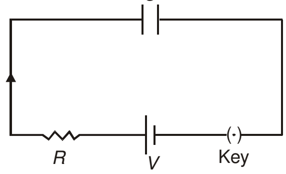

Fig.3.4: Charge accumulates on the capacitor plates during the charging of capacitor

Maxwell realized that for time varying currents, the Ampere's law is not compatible with the equation of continuity given by:

$$\frac{\partial \rho}{\partial t} + \nabla . \mathbf{J} = 0 \tag{3.6}$$

From the Ampere's Law

$$\nabla . \mathbf{J} = \nabla . (\nabla \times \mathbf{H}) = 0. \tag{3.7}$$

Hence the continuity equation would give $(\partial\rho/\partial t) = 0$, i.e., accumulation or rarefaction of charges is never possible. This is against our day-to-day experience where we observe that charge density changes with time. Charging of capacitor is the most common example of charge accumulation with time (Refer to Fig 3.4).

Thus Ampere's Law is valid only for time independent fields. Maxwell generalised it by adding another term, $\mathbf{J_d}$ to Eq. (3.5a). Then it can be written as:

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J_d}. \quad (3.8)$$

Substituting this in Eq. (3.7), we have $\nabla.\mathbf{J} = -\nabla.\mathbf{J_d}$. Using the equation of continuity (3.6) and the Maxwell's first equation (3.2a) we get

$$\nabla.\mathbf{J_d} = \frac{\partial\rho}{\partial t} = \nabla.\left(\frac{\partial \mathbf{D}}{\partial t}\right) \quad (3.9)$$

leading to $\mathbf{J_d} = \partial\mathbf{D}/\partial t$ This is known as the **displacement current density**. The generalised Ampere's law can thus be written as

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \quad (3.10)$$

You should note that just as current density **J** is the source to produce magnetic field, the displacement current is also a source to produce magnetic field. In vacuum there are no charges, hence $\mathbf{J} = 0$, and the magnetic field of an electromagnetic wave is produced by the displacement current.

We summarise below Maxwell's four equations:

$$\nabla.\mathbf{D} = \rho \quad (3.11a)$$
$$\nabla.\mathbf{B} = 0 \quad (3.11b)$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3.11c)$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (3.11d)$$

Value of some constants:

$\varepsilon_0 = \frac{1}{4\pi \times 9 \times 10^9} s\Omega^{-1}m^{-1}$

$\mu_0 = 4\pi \times 10^{-7} \Omega s\, m^{-1}$.

$(\mu_0/\varepsilon_0)^{1/2} = 3 \times 40\pi$
$= 376.73\Omega$

The Maxwell's equations are to be supplemented by the constituent relations, relating **J**, **P** and **M** with **E** and **B**. In general **J**, **P** and **M** at time *t* depend on **E** and **B** at all times preceding *t*. However, for harmonic fields with time dependence of the form $e^{-j\omega t}$, it is always possible to write

$$\mathbf{J} = \sigma\,\mathbf{E}$$
$$\mathbf{P} = \chi\mathbf{E}$$
$$\mathbf{M} = \chi_M\,\mathbf{H}$$
$$\mathbf{D} = \varepsilon_0\,\varepsilon_r\,\mathbf{E}$$
$$\mathbf{B} = \mu_0\,\mu_r\,\mathbf{H}$$

where σ is the electrical conductivity, χ is the electric susceptibility, χ_M is the magnetic permeability. $\varepsilon_r = 1 + (\chi/\varepsilon_0)$ is the relative permittivity and $\mu_r = 1 + (\chi_M/\mu_0)$ is the relative permeability. For non-magnetic materials $\mu_r = 1$. In the following discussion we shall assume $\mu_r = 1$ unless stated otherwise.

Spend 5 Min.

SAQ 1

A sphere of radius r_0 and relative permittivity ε_r has charge distribution with charge density $\rho = \rho_0(1 - r^2/r_0^2)$. Obtain the electric field at $r < r_0$ and $r > r_0$.

After recapitulating the basic concepts let us now discuss about the wave equation.

3.3 WAVE EQUATION

Using the constituent relations $\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}$, $\mathbf{B} = \mu_0\mu_r\mathbf{H}$ and $\mathbf{J} = \sigma\mathbf{E}$, the third and fourth Maxwell's equations (Eqs. (3.11c) & (3.11d)) can be written as

$$\nabla\times\mathbf{E} = -\mu_0 \frac{\partial}{\partial t}\mathbf{H},$$

and $$\nabla\times\mathbf{H} = \sigma\,\mathbf{E} + \varepsilon_0\,\varepsilon_r \frac{\partial}{\partial t}\mathbf{E}.$$

Taking curl of the first equation and using the second, in conjunction with vector identity $\nabla\times\nabla\times\mathbf{E} = -\nabla^2\,\mathbf{E} + \nabla\left(\nabla\,.\,\mathbf{E}\right)$, we obtain the wave equation in the uniform medium

$$\nabla^2\mathbf{E} - \nabla\left(\nabla.\mathbf{E}\right) - \frac{\varepsilon_r}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} - \frac{\sigma}{\varepsilon_0\,c^2}\frac{\partial\mathbf{E}}{\partial t} = 0 \qquad (3.12)$$

where $c = (\mu_0\varepsilon_0)^{-1/2}$. For harmonic time dependent fields, of the type $e^{-j\omega t}$, we may replace $(\partial\mathbf{E}/\partial t)$ by $-j\omega\mathbf{E}$ and Eq. (3.12) can be written as

Here we have assumed ε_r to be real quantity. However, this quantity in practical dielectrics comprises real and imaginary parts–the imaginary part contributing to the dielectric losses.

$$\nabla^2\,\mathbf{E} - \nabla\left(\nabla\,.\,\mathbf{E}\right) + \frac{\omega^2}{c^2}\varepsilon_{eff}\,\mathbf{E} = 0 \qquad (3.13)$$

where $$\varepsilon_{eff} = \varepsilon_r + \frac{j\sigma}{\varepsilon_0\,\omega} \qquad (3.14)$$

ε_{eff} is the effective relative permittivity of the medium. If we multiply Eq (3.13) by (∇.), the first two terms on the left cancel each other, giving $\varepsilon_{eff}\,\nabla.\mathbf{E} = 0$. For arbitrary ε_{eff}, this gives $\nabla.\mathbf{E} = 0$. Then the wave equation takes the form

$$\nabla^2\,\mathbf{E} + \frac{\omega^2}{c^2}\varepsilon_{eff}\mathbf{E} = 0 \qquad (3.15)$$

From Eq. (3.14) you must have noted that the effective relative permittivity depends on the conductivity. For understanding the physical significance of this parameter, let us drive an expression for conductivity σ. Consider a conducting medium with free electron density n_0. When it is subjected to an electric field $\mathbf{E} = \mathbf{A}\exp(-j\omega t)$, the free electrons acquire a drift velocity **v**, governed by the momentum balance equation:

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - \nu m\mathbf{v} \qquad (3.16)$$

where $-e$ and m are the electronic charge and mass, ν is the number of collisions an electron suffers per unit time with heavy particles or phonons. In a collision an electron loses an average momentum $m\mathbf{v}$. Hence in ν collisions, it loses $m\mathbf{v}\nu$ momentum given by the second term of Eq. (3.16). In the steady state, **v** varies in the same way as **E**, i.e., we can take $\mathbf{v} = \mathbf{A_1}\exp(-j\omega t)$. Substituting this in Eq. (3.16) we get

$$\mathbf{v} = -\frac{e\mathbf{E}}{m(\nu - j\omega)} \qquad (3.17)$$

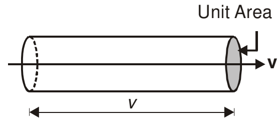

Fig.3.5: Current density calculation

Now consider a fictitious unit area placed perpendicular to **v** as shown in Fig. 3.5. We may visualise a cylinder of length v behind this area. This cylinder has volume (v × 1) and contains n_0v electrons. In one second all these electrons cross the unit area carrying a charge $-n_0$ve. Thus the current density is

$$\mathbf{J} = -n_0 e \mathbf{v} \tag{3.18}$$

$$= \frac{n_0 e^2 \mathbf{E}}{m(\nu - j\omega)} = \sigma \mathbf{E} \tag{3.19}$$

with
$$\sigma = \frac{n_0 e^2}{m(\nu - j\omega)} \tag{3.20}$$

In a conductor or semiconductor $\nu \sim 10^{12}\, s^{-1}$, hence for microwave and lower frequencies $\omega << \nu$ and σ is nearly real. At optical frequencies ($\omega >> \nu$), σ is largely imaginary, signifying that current density is $\pi/2$ radians out of phase with the electric field. Using this expression for σ in Eq. 3.14, we may write effective relative permittivity as

$$\varepsilon_{eff} = \varepsilon_r - \frac{\omega_p^2}{\omega^2 (1 + j\nu/\omega)} \tag{3.21}$$

where
$$\omega_p = (n_0 e^2 / m\varepsilon_0)^{1/2}. \tag{3.22}$$

ω_p is called the *plasma frequency*. It depends only on the total number of carriers.

In general, ε_{eff} is complex. For $\frac{\nu}{\omega} << 1$, we can write

$$\varepsilon_{eff} = \varepsilon_r - \frac{\omega_p^2}{\omega^2}. \tag{3.23}$$

You must have noticed that the value of ε_{eff} decreases with ω.

You may now like to attempt the following SAQ.

Spend 3 *Min.*

SAQ 2

In the presence of an electric field of frequency $\omega = 10^{13}$ rad s^{-1} inside a conductor, the induced current density is $\pi/3$ out of phase with electric field. Estimate the collision frequency.

Before proceeding further, let us discuss the physical nature of plane waves.

3.3.1 Plane Waves

Consider an ideal source that emits an electromagnetic wave along +x direction. Let the electric field produced by the source at $x = 0$ (in the entire yz-plane) be

$$\mathbf{E}(x = 0, t) = \mathbf{A} \cos \omega t \tag{3.24}$$

and
$$\mathbf{E}(x, t) = \mathbf{E}(x = 0, t - x/\mathrm{v}) = \mathbf{A} \cos \omega (t - x/\mathrm{v})$$

or
$$\mathbf{E}(x, t) = \mathbf{A} \cos(\omega t - kx) \tag{3.25}$$

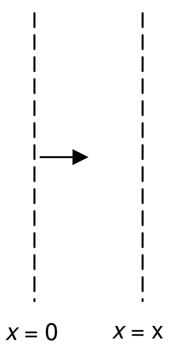

Fig. 3.6: Wave propagation in + x direction

where A is the amplitude, ω is the frequency, ωt is the phase, of the field and v is the velocity of wave propagation. The field at x at time t is the same as that it was at $x = 0$ at a previous time ($t - x$/v), since the field take x/v time to travel from $x = 0$ to x. You know that in this equation, $k = \omega$/v is called the **wave number** and the argument of cos function ($\omega t - kx$) is called the **phase of the wave**. At any time t, the phase is constant over the plane with constant x, i.e., over the entire yz-plane. The plane of constant phase is called a **wavefront**. The wave travels perpendicular to the wavefront.

For a plane wave propagating along an arbitrary direction $\hat{\mathbf{n}}$ ($\hat{\mathbf{n}}$ is an unit vector) the wavefront would be a plane perpendicular to $\hat{\mathbf{n}}$.

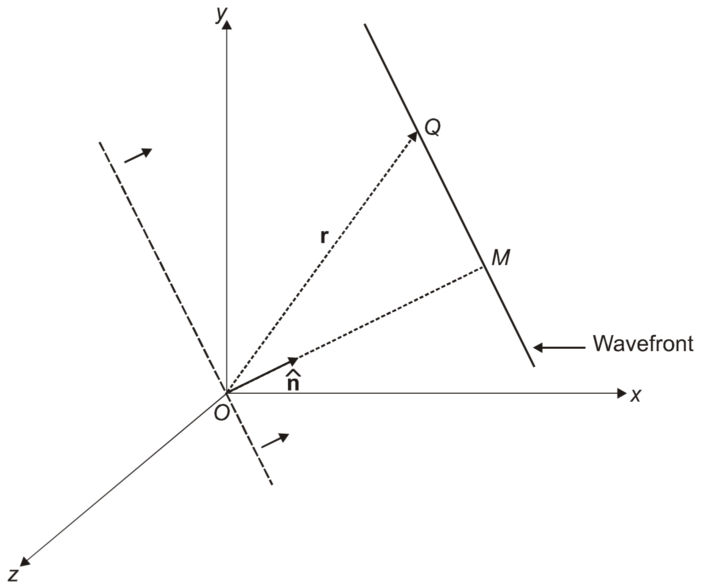

Fig. 3.7: Plane wavefront

Refer to Fig.3.7. The field at **Q** (**r**, *t*) is same as at point *M* at time *t* (since both *Q* and *M* are on the same wavefront). This field is same as that was at the origin *O* at time $\left(t - \frac{OM}{\text{v}}\right)$.

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{A}\cos\omega\left(t - \frac{OM}{\text{v}}\right)$$
$$= \mathbf{A}\cos(\omega t - \mathbf{k.r}) \qquad (3.26)$$

A more convenient representation of an electromagnetic wave is

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{A}\, e^{-j(\omega t - \mathbf{k.r})} \qquad (3.27)$$

where real part of the right hand side is implied. The amplitude and frequency of the waves are determined by the sources producing them, whereas velocity of wave propagation is determined by the properties of the medium through which they propagate.

To understand the effects of material properties on wave propagation let us now consider the propagation of waves in dielectric and conducting materials.

3.3.2 Plane Waves in Dielectrics and Conductors

The wave field is given by Eq. (3.27). By partially differentiating this equation we get:

$$\frac{\partial \mathbf{E}}{\partial x} = jk_x \mathbf{E} \text{ and, } \frac{\partial^2 \mathbf{E}}{\partial x^2} = -k_x^2\ \mathbf{E}$$

Similarly working out the partial derivatives for *y* and *z* components and using Eq. (3.15) we get:

$$k^2 = \frac{\omega^2}{c^2}\varepsilon_{eff}. \qquad (3.28)$$

This is known as the *dispersion relation*.

The phase velocity of the wave is

$$v_{ph} = \frac{\omega}{k} = \frac{c}{\eta} \tag{3.29}$$

where

$\eta = \sqrt{\varepsilon_{eff}}$ is the refractive index of the medium.

Now we can obtain more insight into the nature of electromagnetic waves from Maxwell's equations. Since **B**, **H**, **D** and **J** in the steady state have same (**r,** t) dependence as **E**, we can replace the $\partial / \partial t$ and ∇ operators operating on these quantities by $-j\omega$ and $j\mathbf{k}$ respectively. Then from Eq (3.11c) and (3.11d) we get

$$\mathbf{k} \times \mathbf{E} = \omega\mu_0 \mathbf{H},$$
$$\mathbf{k} \times \mathbf{H} = -\omega\varepsilon_0\, \varepsilon_{eff}\, \mathbf{E}. \tag{3.30}$$

Pre-multiplying these equations by **k** we get

$$\mathbf{k.E} = 0,$$
and $$\mathbf{k.H} = 0$$

These relations imply that the electric and magnetic fields are transverse to the direction of wave propagation **k**. Further, from Eq (3.30), we can conclude that **E** is perpendicular to **H**. Thus **E**, **H** and **k** are mutually orthogonal.

The ratio Z = (**E/H**) is known as the *impedance of the medium*. From Eq. (3.30) we get that, $Z = Z_0/\eta$, where $Z_0 = (\mu_0/\varepsilon_0)^{1/2}$ = 377 ohm is the free space impedance.

To apply the concepts learnt so for, you may now like to answer the following SAQ.

Spend 4 *Min.*

SAQ 3

An electromagnetic wave has electric field $\mathbf{E} = \mathbf{A}\; e^{-j\left(\omega t - 2\frac{\omega}{c}x - \frac{\omega}{c}y\right)}$. Obtain the propagation vector, wavelength and phase velocity of the wave.

After considering the basics of propagation, let us now discuss the wave propagation in dielectric medium.

Wave Propagation in Dielectrics

$k^2 = \frac{\omega^2}{c^2}\varepsilon_{eff}$

$k_r^2 - k_i^2 + j2k_r k_i = \frac{\omega^2}{c^2}(\varepsilon' + j\varepsilon'')$

$k_r^2 - k_i^2 = \frac{\omega^2}{c^2}\varepsilon'$

and $2k_r k_i = \frac{\omega^2}{c^2}\varepsilon''$

For $\varepsilon'' << \varepsilon', k_i << k_r$

hence, $k_r = \frac{\omega}{c}\varepsilon'^{1/2}$

and $k_i = \frac{\omega^2}{c^2}\frac{\varepsilon''}{2k_r} = \frac{\omega}{c}\frac{\varepsilon''}{2\varepsilon'^{1/2}}$

In a pure dielectric, the electric conductivity $\sigma = 0$ and $\eta = \varepsilon_r^{1/2}$. Since the refractive index is greater than 1 (ε_r >1), the wave travels with a lower velocity than the velocity of light in vacuum. At radio wave and microwave frequencies $(\omega / 2\pi \le 30\,\text{GHz})$, ε_r in most dielectrics, is independent of wave frequency, i.e., there is no dispersion in them.

However, ε_r has finite imaginary part $\varepsilon = \varepsilon' + j\varepsilon''$ in lossy dielectrics. It simply implies that the displacement vector **D** is not in phase with electric field **E**. In this case, k is complex and wave suffers damping as it propagates through the dielectrics. If we substitute $k = k_r + jk_i$ in Eq. (3.28) and separate real and imaginary parts, then in the limit of $\varepsilon'' << \varepsilon'$ we get

$$k_r = \frac{\omega}{c}\varepsilon'^{1/2}, \text{and } k_i = \frac{\omega}{c}\frac{\varepsilon''}{2\varepsilon'^{1/2}}. \tag{3.31}$$

k_i^{-1} is known as the *attenuation length*.

At optical frequencies, ε_r depends on the frequency of the wave, hence dielectric behaves as a dispersive medium. Glass, for instance, has a different refractive index for different colours of light in the visible region and is a dispersive medium. The propagation of signals of finite frequency width in such media leads to distortion of signals as different frequency components travel with different velocities.

Wave Propagation in Conductors

In a conductor, electrical conductivity, σ given by Eq.(3.20), plays an important role in wave propagation.

$$\sigma = \frac{ne^2(\nu + j\omega)}{m\,(\nu^2 + \omega^2)} \tag{3.32}$$

Typically $n \sim 10^{28}\ \text{m}^{-3}$ and $\nu \sim 10^{12}\ \text{s}^{-1}$ in a conductor.

Using this expression for σ in Eq (3.14), and substituting in Eq (3.28) we get

$$k^2 = \frac{\omega^2}{c^2}\left(\varepsilon_r - \frac{\omega_p^2(1 - j\nu/\omega)}{\omega^2 + \nu^2}\right) \tag{3.33}$$

where $\omega_p = (ne^2/m\varepsilon_0)^{1/2}$ is the plasma frequency.

At microwave and radio wave frequencies (i.e. $\omega/2\pi \leq 30$ GHz)

$$\omega^2 << \nu^2, \text{ and } \omega_p^2 >> \omega\nu\varepsilon_r$$

Hence, the imaginary and real parts of k are equal and the wave is strongly damped. This can be expressed as:

$$k = k_r + jk_i, \text{with}\, k_r = k_i \cong \frac{\omega_p}{c}\left(\frac{\omega}{2\nu}\right)^{1/2}. \tag{3.34}$$

k_i^{-1} is known as the **skin depth** and is denoted by a symbol δ. It scales as $\omega^{-1/2}$, i.e., as the frequency increases, the waves tend to travel only in the area near to the surface of the conductor. Typically, for copper $\delta \sim 0.85$ cm for 50 Hz while for 100 MHz, it reduces to 0.7×10^{-3} cm.

At optical frequencies, $\omega >> \nu$

Hence, $$k \sim \frac{\omega}{c}\left(\varepsilon_r - \frac{\omega_p^2}{\omega^2}(1 - j\nu/\omega)\right)^{\frac{1}{2}} \tag{3.35}$$

In the first approximation we can take $\nu/\omega = 0$. Then the wave propagates (i.e. k is real) only when $\omega > (\omega_p/\varepsilon_r^{1/2})$.

For $\omega < (\omega_p/\varepsilon_r^{1/2})$, k is imaginary and there is no propagation of wave. This limiting frequency, $(\omega_p/\varepsilon_r^{1/2})$ is called the **plasma edge**. For good conductors plasma edge frequency falls in the ultra-violet range. This is the reason why visible and shorter frequencies do not propagate through the conductors.

For $\omega > \omega_p/\varepsilon_r^{1/2}$, k can be written as $k = k_r + jk_i$, with

$$k_r \approx \frac{\omega}{c}\left(\varepsilon_r - \frac{\omega_p^2}{\omega^2}\right)^{1/2},$$

and $$k_i \approx \frac{\omega_p^2 \nu}{2c^2 \omega k_r}. \qquad (3.36)$$

This treatment is also valid in ionised media (knows as plasma) with $\varepsilon_r = 1$.

You may now like to solve the following SAQ.

Spend 4 *Min.*

SAQ 4

In the ionosphere (earth's atmosphere above 90 km from the earth's surface) electron density has a peak value of $n = 10^{12}$ m^{-3}. Estimate the lowest radio wave frequency that can propagate through it. Ignore collisions.

3.3.3 Boundary Conditions

At a boundary between the two media, μ, ε and σ change suddenly, hence it is important to find relation between the fields on the two sides of the boundary. This can be done by determining the boundary conditions from Maxwell's equations.

a. Continuity of tangential component of E

Consider the boundary between two media, I and II at $y = 0$ (Fig.3.8). Choose a rectangular path *ABCDA* of length l and vanishing width b ($b \to 0$). Now let us evaluate the line integral $\int_c \mathbf{E.dl}$.

$$\int_c \mathbf{E.dl} = E_{xI}\, l - E_{xII}\, l.$$

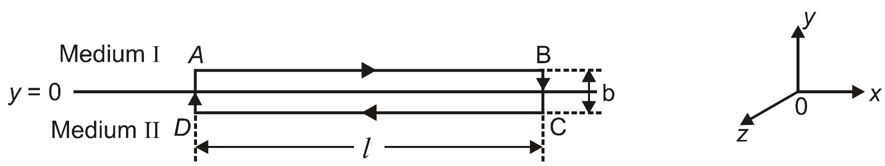

Fig. 3.8: Path of the line integral across the interface $y = 0$

This must equate the time derivative of the magnetic flux linked with the loop. As the area of the loop is reduced to zero, the flux linked with it also vanishes, resulting in,

$$\int_c \mathbf{E.dl} = 0$$

or $$E_{xI} = E_{xII} \qquad (3.37)$$

Similarly $E_{zI} = E_{zII}$ at the interface. Thus the tangential component of the electric field is continuous across the boundary.

In a special case when medium II is a perfect conductor ($\sigma = \infty$), the electric field inside it is zero (otherwise infinite current will flow), hence, the continuity condition demands that the tangential component of the electric field, $E_{\|}$, in the first medium must also vanish at the boundary.

b. Continuity of tangential component of H

Consider the boundary between two media I and II at $y = 0$ and choose the path *ABCDA* across it as shown in Fig.3.8. We evaluate $\int_c \mathbf{H.dl}$ along this path.

According to generalised Ampere's law

$$\int_c \mathbf{H.dl} = \int_s \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) .\mathbf{ds}$$

In the limit, where width of the path $b \to 0$, the surface integral on the right varnishes, hence

$$H_{xI} = H_{xII} \tag{3.38}$$

Similarly we can show that $H_{zI} = H_{zII}$. Thus the tangential component of **H** is continuous across the boundary.

c. Continuity of normal component of D

Consider a cylinder of cross-section A and vanishing height b ($b \to 0$) across the boundary between two media as shown in Fig.3.9.

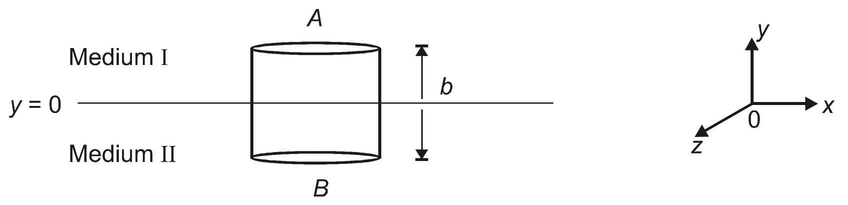

Fig. 3.9: A cylindrical Gaussian surface across the boundary between two media

We evaluate the surface integral $\int_s \mathbf{D.ds}$ over that closed surface

$$\int_s \mathbf{D.ds} = D_{yI} A - D_{yII} A$$

According to Gauess's law this must be equal to the charge enclosed within the cylinder. In the case of a dielectric, there is no charge enclosed, hence

$$D_{yI} = D_{yII} \tag{3.39}$$

That is, the normal component of the displacement vector is continuous across the boundary. In a special case when medium II is a conductor, finite surface charge density ρ_s is possible. In that case, the charge enclosed inside the Gaussian surface is $\rho_s A$, hence

$$D_{y\mathrm{I}} - D_{y\mathrm{II}} = \rho_s. \tag{3.40}$$

d. Continuity of normal component of B

Applying Gauss's law for the magnetic field, $\int_s \mathbf{B.ds} = 0$, over the surface of Fig.3.9,

we obtain

$$B_{yI} = B_{yII}. \tag{3.41}$$

This means that, the normal component of magnetic field is continuous across the boundary.

In essence, the boundary conditions imply the continuity of the components of electric and magnetic field parallel to the interface ($E_{\parallel}$and $H_{\parallel}$) and the components of **D** and **B** normal to the interface ($D_{\perp}$ and $B_{\perp}$).

3.3.4 Poynting's Theorem

The electric and magnetic fields are produced by the charge and current distributions. In order to establish the fields, it is necessary to establish certain charge distributions. The work done in doing so can be viewed as the energy stored in the electric and magnetic fields. In the Course on Electric and Magnetic Phenomena, (PHE-07) you have learnt that the energy stored in the electric field per unit volume is

$$W_E = \frac{\mathbf{E}\,.\,\mathbf{D}}{2} \qquad (3.42)$$

while that in the magnetic field per unit volume is

$$W_B = \frac{\mathbf{B}\,.\,\mathbf{H}}{2} \qquad (3.43)$$

Using the Maxwell's equations, we can investigate how the energy flows from one region of space to another when fields vary with time.
From the generalised Ampere's law we have

$$\mathbf{J} = \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t}\ .$$

$$\begin{aligned}\mathbf{H}.\nabla\times\mathbf{E} &= -\mathbf{H}.\frac{\partial \mathbf{B}}{\partial t}\\ &= \mathbf{H}.\frac{\partial}{\partial t}\mu_0\mathbf{H}\\ &= -\mu_0\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{H}.\mathbf{H})\\ &= -\frac{\partial}{\partial t}\frac{\mathbf{B}.\mathbf{H}}{2}\\ &= -\frac{\partial}{\partial t}W_{\mathbf{B}}\end{aligned}$$

Hence $$\mathbf{J}\,.\,\mathbf{E} = \mathbf{E}\,.\,\nabla \times \mathbf{H} - \mathbf{E}\,.\frac{\partial \mathbf{D}}{\partial t}\ . \qquad (3.44)$$

Using vector identity and third Maxwell's equation we can write

$$\mathbf{E}\,.\,\nabla\times\mathbf{H} = \nabla\,.\,(\mathbf{E}\times\mathbf{H}) + \mathbf{H}\,.\,\nabla\times\mathbf{E} = \nabla\,.\,(\mathbf{E}\times\mathbf{H}) - \frac{1}{2}\,\frac{\partial}{\partial t}(\mathbf{B}\,.\,\mathbf{H})$$

$$\mathbf{J}\,.\,\mathbf{E} = \nabla\,.\,(\mathbf{E}\times\mathbf{H}) - \frac{\partial}{\partial t}(W_E + W_B)$$

Hence, $$\frac{\partial W_{EB}}{\partial t} + \mathbf{J}\,.\,\mathbf{E} = \nabla.\,(\mathbf{E}\times\mathbf{H}) \qquad (3.45)$$

where $W_{EB} = (W_E\ +\ W_B)$. We can interpret the left hand side as the sum of the rate of increase of field energy density and the energy absorbed per unit volume per second. Hence the right hand side must represent energy entering in the unit volume per second. Thus, from the definition of divergence, $\mathbf{E}\times\mathbf{H}$ represents the energy flow density, i.e., the amount of electromagnetic energy flowing per unit area per unit time. We can define this by a vector known as the **Poynting's vector**.

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} \qquad (3.46)$$

In case of plane monochromatic waves, **E** and **H** are written in complex representations as:

$$\mathbf{E} = \mathbf{A_0}\exp[-\,j(\omega t - \mathbf{k.r})]$$

and $$\mathbf{H} = \mathbf{k}\times\mathbf{E}/\omega\mu \qquad (3.47)$$

Here, you must remember that **S** is the vector product of real part of **E** (that is the actual value of **E**) and the real part of **H** since it is a product of real energy transfers.

i.e. $$\mathbf{S} = \operatorname{Re}\mathbf{E} \times \operatorname{Re}\mathbf{H}$$
$$= \frac{1}{2}\operatorname{Re}\left[\mathbf{E}\times\mathbf{H} + \mathbf{E}^* \times \mathbf{H}\right] \qquad (3.48)$$

You know the vector identity:

$$\operatorname{Re}\mathbf{A} \times \operatorname{Re}\mathbf{B} = \frac{1}{2}\operatorname{Re}\left[\mathbf{A}\times\mathbf{B} + \mathbf{A}^* \times \mathbf{B}\right]$$

where * denotes complex conjugate.

$$\mathbf{S} = \frac{1}{2\omega\mu}\operatorname{Re}\left[\mathbf{E}\times(\mathbf{k}\times\mathbf{E}) + \mathbf{E}^* \times (\mathbf{k}\times\mathbf{E})\right]$$

Using vector triple product identity we get

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A.C}) - \mathbf{C}(\mathbf{A.B})$$

$$\mathbf{S} = \frac{1}{2\omega\mu}\operatorname{Re}\left[\mathbf{k}\,(\mathbf{E.E}) + \mathbf{k}.(\mathbf{E.E}^*)\right]$$

where Re denotes the real part of the quantity.

Substituting values of **E** and **H** and remembering the fact that for electromagnetic waves **k**.**E** = 0 we get

$$\mathbf{S} = \frac{1}{2\mu\omega}\operatorname{Re}\left[\mathbf{k}\,A_0^2\, e^{-2j(\omega t - j\mathbf{k}.\mathbf{r})} + \mathbf{k}\,A_0^2\, e^{-j(\mathbf{k}-\mathbf{k}^*)\,.\,\mathbf{r}}\right] \qquad (3.49)$$

The time average of the first term over a wave period is zero. Further $A = A_0 e^{-k_r x}$ is the amplitude of the wave, hence

$$\mathbf{S_{av}} = \frac{A^2}{2\mu\omega}\mathbf{k_r} \qquad (3.50)$$

where $\mathbf{k_r}$ is the real part of **k**.

Now you may attempt the following SAQ.

SAQ 5

Spend 3 Min.

An electromagnetic wave is expressed as $\mathbf{E} = \hat{\mathbf{y}}\, A\, e^{-j\left(\omega t - 2\frac{\omega}{c}x + \frac{\omega}{c}z\right)}$. Obtain the time average Poynting's vector.

An important mode of electromagnetic wave transmission is along the boundary of two media. These are called *surface waves*. You will learn about these waves in the next section.

3.4 SURFACE WAVES

A surface wave is a guided electromagnetic wave that propagates along the boundary between a conductor and a dielectric (or conductor and free space). Its amplitude is maximum on the boundary and falls off away from it in either medium. Medium frequency radio waves (300 kHz < $\omega/2\pi$< 3 MHz) propagate as surface waves along the surface of the earth with earth acting as a conducting medium. In the course of their propagation, surface waves suffer attenuation. For short waves and shorter wavelengths, the attenuation rate is larger.

Consider a conductor–free space interface at $y = 0$ as shown in Fig. 3.10. The conductor (y<0) is characterized by effective permittivity ε_{eff}, whereas free space

($y > 0$) has effective permittivity of unity. A surface wave of frequency ω propagates along x axis. The x and t dependence of the fields can be written as

$$\mathbf{E} = \mathbf{A}(y)e^{-j(\omega t - k_x x)} \quad (3.51)$$

Here we assume that there is no variation along z.

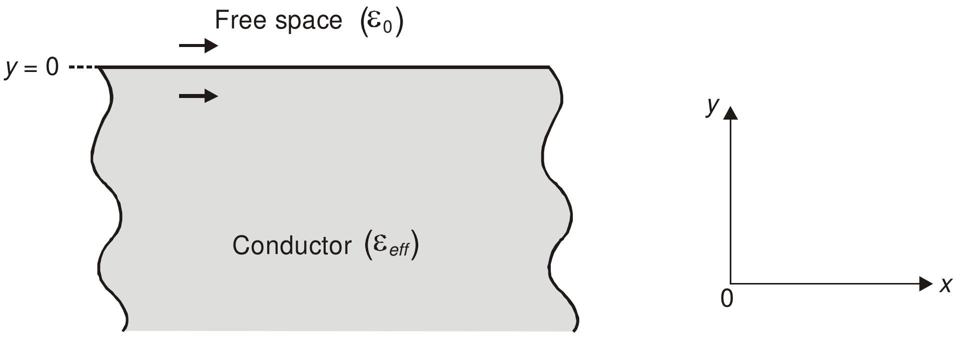

Fig. 3.10: Surface wave propagation

We consider $\mathbf{E} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}}$. We solve the wave equation for E_x in the two media, and apply the continuity of E_x and H_z across $y = 0$ to obtain the dispersion relation.

The wave equation governing the fields in two media can be written as (cf. Eq. 3.15)

$$\nabla^2\mathbf{E} + \frac{\omega^2}{c^2}\varepsilon'_{eff}\mathbf{E} = 0$$

where $\varepsilon'_{eff} = \varepsilon_{eff}$ inside conductor and $\varepsilon'_{eff} = 1$ in free space. Using $\frac{\partial}{\partial x} = jk_x$ to write $\nabla^2 = \frac{\partial^2}{\partial y^2} - k_x^2$, we obtain for E_x

$$\frac{\partial^2 E_x}{\partial y^2} - \alpha^2 E_x = 0 \quad (3.52)$$

where $\alpha^2 = \alpha_1^2 \equiv k_x^2 - \frac{\omega^2}{c^2}$ for $y > 0$ (3.53a)

and $\alpha^2 = \alpha_2^2 \equiv k_x^2 - \frac{\omega^2\varepsilon_{eff}}{c^2}$ for $y < 0$. (3.53b)

The well behaved solutions of Eq. (3.52) that vanish at $y \to -\infty$ and $y \to \infty$ are

$$E_x = Ae^{-\alpha_1 y}e^{-j(\omega t - k_x x)} \text{ for } x > 0 \quad (3.54a)$$

and $E_x = A'e^{\alpha_2 y}e^{-j(\omega t - k_x x)}$ for $x < 0$ (3.54b)

Demanding the continuity of E_x at $y = 0$ we get $A' = A$.

Using these equations, we obtain

$$E_y = \left(\frac{jk_x}{\alpha_1}\right)Ae^{-\alpha_1 y}e^{-j(\omega t - k_x x)} \text{ for } y > 0 \quad (3.55a)$$

and $E_y = -\left(\frac{jk_x}{\alpha_2}\right)Ae^{\alpha_2 y}e^{-j(\omega t - k_x x)}$ for $y < 0$ (3.55b)

Substituting for E_x and E_y we get the magnetic field of the wave

$$H_z = j\left(\frac{\omega}{c^2\mu_0\alpha_1}\right)Ae^{-\alpha_1 y}e^{-j(\omega t - k_x x)} \text{ for } y > 0 \tag{3.56a}$$

and
$$H_z = -j\left(\frac{\omega\varepsilon_{eff}}{c^2\mu_0\alpha_2}\right)Ae^{-\alpha_2 y}e^{-j(\omega t - k_x x)} \text{ for } y < 0 \tag{3.56b}$$

Applying the continuity of H_z at $y = 0$ we get, $1/\alpha_1 = -\varepsilon_{eff}/\alpha_2$ or

$$k_x^2 = \frac{\omega^2}{c^2}\frac{\varepsilon_{eff}}{1+\varepsilon_{eff}} \tag{3.57}$$

This is the dispersion relation for surface wave. At radio wave frequencies σ is real ($\approx n_0 e^2/m\nu$), $\sigma/\omega\varepsilon_0 >> 1$ and $\varepsilon_{eff} \approx j\frac{\sigma}{\omega}\varepsilon_0$.

Hence
$$k_x^2 \approx \frac{\omega^2}{c^2}(1 + j\omega\varepsilon_0/\sigma)$$

or
$$k_x = \frac{\omega}{c} + j\frac{\omega^2\varepsilon_0}{2\sigma c} \tag{3.58}$$

k_x has a large real part, giving phase velocity of the surface wave $v_{ph} = \omega/k_x \sim c$.

The imaginary part of k_x gives the attenuation rate of the wave in course of its propagation. It scales as ω^2, i.e., the higher frequency waves are attenuated more. The quantities characterizing the decay rates of wave amplitudes in the two media, α_1, α_2, are

$$\alpha_1 = \alpha_{1r} + j\alpha_{1i}, \tag{3.59a}$$

$$\alpha_2 = \alpha_{2r} - j\alpha_{2i}, \tag{3.59b}$$

where
$$\alpha_{1r} = \alpha_{1i} = \frac{\omega}{c}\left(\frac{\omega\varepsilon_0}{2\sigma}\right)^{1/2} \tag{3.59c}$$

and
$$\alpha_{2r} = \alpha_{2i} = \frac{\omega}{c}\left(\frac{\sigma}{2\omega\varepsilon_0}\right)^{1/2} \tag{3.59d}$$

The characteristic distances, over which field amplitudes fall off from the interface in the free space and in the conductor are α_{1r}^{-1} and α_{2r}^{-1} respectively. Since $\alpha_{2r} >> \alpha_{1r}$, the surface wave field is strongly localized in the conductor within a skin layer of width ~ α_{2r}^{-1}; whereas it extends to a larger width (~ α_{1r}^{-1}) in free space as shown in Fig. 3.11. α_{2r} is the same as k_i in Eq 3.34 at $\omega << \nu$.

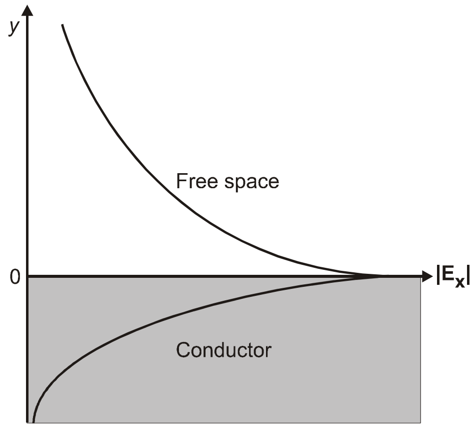

Fig. 3.11: Surface wave attenuation

You may use the above relations for solving the following SAQ.

Spend 5 Min.

SAQ 6

A surface wave of $\omega = 2 \times 10^{15}$ rad s^{-1} propagates over a metal-free space interface. The metal has $\varepsilon_r = 10$, $\omega_p = 8 \times 10^{15}$ radians. Estimate the wavelength of the surface wave. If the wave amplitude on the surface is 10^4 Vm^{-1}, estimate its value at a depth of 0.1 μm inside the metal. Ignore collisions.

After learning about the propagation of surface wave at the boundary of the two media, let us discuss about a very important mode of guided wave transmission at microwave frequencies. Here, the hollow metallic conductors called *waveguides* are used.

3.5 WAVEGUIDES

A waveguide is a hollow metallic pipe through which an electromagnetic wave can propagate without spreading, i.e,. without diffraction divergence. The waveguide could have rectangular cross-section or a circular cross-section, accordingly it is called rectangular waveguide or a cylindrical waveguide. These are shown in Fig. 3.12. The axis of a waveguide is usually denoted by z-axis. The reflecting metallic walls of the waveguide provide confinement to electromagnetic waves in x-y or r-θ directions. This may be understood as follows.

An electromagnetic wave propagating through a waveguide must have electric and magnetic field distribution such that the tangential components of electric field at all the metallic walls vanish. This is true if we assume that the walls are ideal, having infinite conductivity. Consider, for instance, the propagation of an electromagnetic wave of frequency ω in a rectangular waveguide of inner widths a and b along $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions respectively and axis along $\hat{\mathbf{z}}$. Let the electric field of the wave be parallel to y-axis ($\mathbf{E} \parallel \hat{\mathbf{y}}$). Since it is a propagating disturbance along $\hat{\mathbf{z}}$, it may be written as

$$E_y = A\, e^{-j(\omega t - kz)}. \qquad (3.60)$$

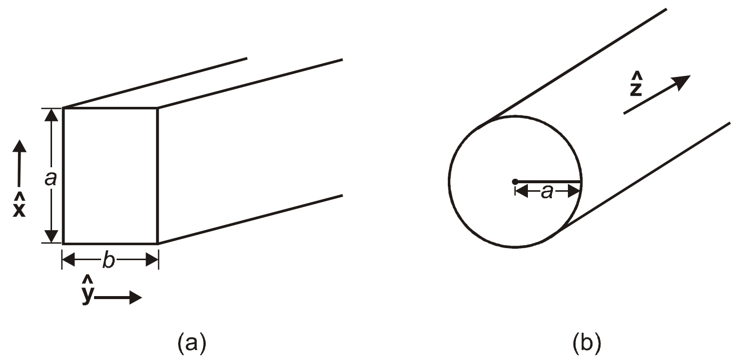

Fig. 3.12: a) A rectangular waveguide of dimensions *a* and *b* in *x* and *y* directions; and b) A cylindrical waveguide of radius *a*

Since E_y the tangential component of electric field at metallic walls, it is expected to vanish at wall boundaries $x = 0$ and $x = a$. But if A, the amplitude of E_y, were independent of x, then E_y will not vanish. Hence A must be a function of x. It can be a function of y as well but that is not essential or mandatory. In order to deduce the x dependence of A, let us go over to the wave equation, which for E_y takes the form:

$$\nabla^2 E_y + \frac{\omega^2}{c^2} E_y = 0 \qquad (3.61)$$

Substituting the expression for E_y from Eq. (3.60) and assuming that *A* is a function of *x* but it is independent of *y* and *z*, we get

$$\frac{\partial^2 A}{\partial x^2} + \alpha^2 A = 0 \qquad (3.62)$$

where $$\alpha^2 = \frac{\omega^2}{c^2} - k^2 \qquad (3.63)$$

The general solution of Eq.(6.60) is

$$A = A_1 \sin \alpha x + A_2 \cos \alpha x$$

The boundary condition at $x = 0$, $A = 0$ demands $A_2 = 0$. Another boundary condition at $x = a$, $A = 0$ demand $\sin(\alpha a) = 0$ or

$$\alpha = n\pi/a \qquad (3.64)$$

where *n* is an integer.

The wave electric field thus can be written as

$$E_y = A_1 \sin\left(\frac{n\pi}{a} x\right) e^{-j(\omega t - kz)}. \qquad (3.65)$$

The propagation constant *k* of the wave, on using the value of α from Eq. (3.64) in Eq. (3.63) can be written as

$$k^2 = \frac{\omega^2}{c^2} - \frac{n^2\pi^2}{a^2}. \qquad (3.66)$$

This equation is the *dispersion relation for the waveguide*.

You must have noted that the boundary conditions have quantised α, the quantity that characterises the *x* variation of the field. For a given value of *n*, the field has a specific *x* variation and a specific *k*. For another value of *n*, *x* variation is different and so is *k*. The field structure with a particular value of *n* is called a **mode**. Thus the waveguide has many modes of propagation. The modes that have $E_z = 0$ are called *TE* (Transverse Electric) modes. Two integers subscripts are affixed to *TE*. For example, TE_{nm}, *n* characterising the *x* variation and *m* the *y* variation. In this case there is no *y* variation, hence a mode is denoted as TE_{n0}.

For TE_{n0} mode, the dispersion relation can be written as

$$\omega^2 = \frac{n^2\pi^2}{a^2} c^2 + k^2 c^2. \qquad (3.67)$$

The minimum value of k^2 for a propagating wave is zero, hence the minimum frequency that can propagate in the TE_{n0} mode is

$$\omega_n = \frac{n\pi c}{a} \qquad (3.68)$$

This is called the *cut-off frequency*. For TE_{10} mode the cut-off frequency is ($\pi\, c/a$), for TE_{20} mode it is ($2\pi\, c/a$) and so on. The cut-off frequency increases with the mode number.

Note that at a given frequency, different modes have different k, hence different phase velocities $v_{ph} = \omega/k$. The phase velocity can be written as:

$$v_{ph} = \frac{c}{\left(1 - \frac{\omega_n^2}{\omega^2}\right)^{1/2}} \tag{3.69}$$

For a particular mode ω_n is fixed but v_{ph} is a function of ω. So a waveguide acts as a dispersive medium. If we transmit an amplitude modulated wave or a pulse, comprising many frequency components, in a particular mode through the waveguide, different frequencies will travel with different phase velocities leading to distortion the signal at the receiver end. For $\omega >> \omega_n$, the phase velocity approaches c and dispersion becomes negligible. In Fig. 3.13 we have plotted dispersion relation for some typical TE modes. The cut off frequencies for higher order modes are higher. As ω increases, all the modes asymptotically approach the $\omega = kc$ line.

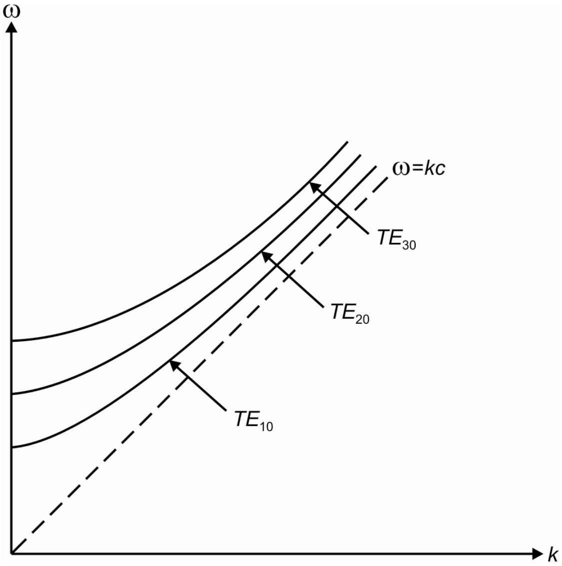

Fig. 3.13: Dispersion relations for TE modes

In Fig.3.14 we show the x variation of field for different modes. For the TE_{10} mode, A is maximum on waveguide axis and E_y is in same phase over the entire cross-section. For TE_{20} mode the field amplitude is zero on waveguide axis and has two maxima at $x = a/4$ and $3a/4$. The fields for $x < a/2$ and $x > a/2$ have a phase difference of π. For the TE_{30} mode, the field amplitude has three maxima.

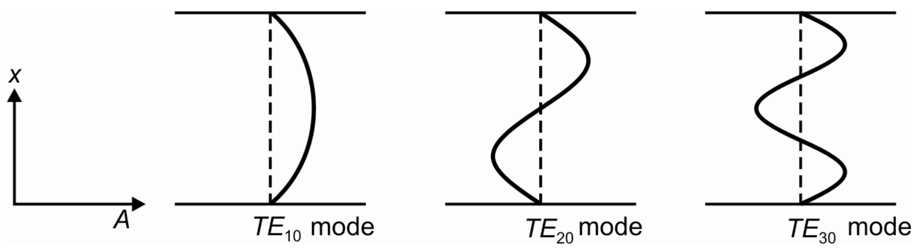

Fig.3.14: Field structure of TE_{10}, TE_{20} and TE_{30} modes in a rectangular waveguide

Spend 4 Min.

SAQ 7

The electric field of a *TE* mode in a rectangular waveguide is

$\mathbf{E} = \hat{\mathbf{y}} A_1 \sin\left(\frac{\pi}{a}x\right) e^{-j(\omega t - kz)}$. Obtain the magnetic field.

You have found out that for the given *TE* mode, $H_z \neq 0$, i.e., the wave has a magnetic field component along z. Similarly, there are other kind of modes for which $H_z = 0$ but $E_Z \neq 0$. These are called *transverse magnetic* or TM_{nm} modes.

The above discussion presumed metal walls with infinite conductivity. In real situation, waveguide wall conductivity is large but finite. Consequently, electromagnetic fields penetrate within the skin layer causing power dissipation. This leads to attenuation of the wave as it propagates. The attenuation-constant increases with frequency, so these waveguides are not suitable for optical communication. We have to use optical fibres that employ the phenomenon of critical reflection to guide electromagnetic waves.

An optical fibre comprises a core of refractive index η (~1.5) surrounded by a cladding of slightly smaller refractive index (η–Δη) as shown in Fig. 3.15. A ray propagating through the core when hits the core cladding boundary at an angle θ greater than the critical angle defined by

$$\theta_c = \sin^{-1} \frac{\eta - \Delta\eta}{\eta}, \qquad (3.70)$$

it suffers total internal reflection and remains confined in the core. The wave theory of propagation in optical fibres reveals that a fibres also support many modes of propagation and each mode suffers from dispersion effects.

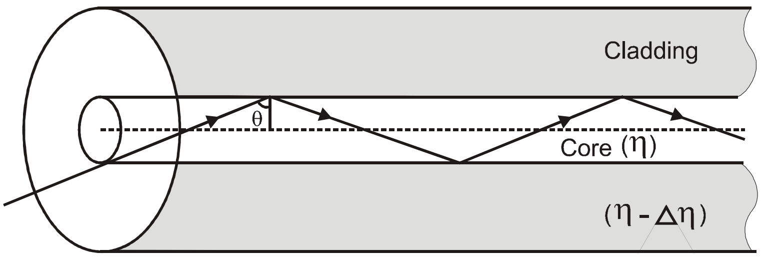

Fig. 3.15: Schematic of an optical fibre with a core and cladding. A ray incident on the core-cladding interface suffers total internal reflection when angle of incidence exceeds the critical angle

After learning about the guided wave propagation confined by conductor boundaries, let us now briefly discuss about the transmission through free space with the use of *antennae*.

3.6 REVIEW OF ANTENNAE

A current carrying wire produces a magnetic field **B** in the region surrounding it. If the current is periodic in time, it will produce a time periodic magnetic field. According to Faraday's law of electromagnetic induction (Eq.3.11c), this magnetic field is accompanied by a time varying electric field **E**. The Poynting's vector due to these **E** and **B** fields turns out to be finite, pointing away from the wire. Thus a wire carrying periodic current is a source of electromagnetic radiation. This is shown in Fig. 3.16a. If the wire were surrounded by a conducting cylinder carrying an equal and opposite current, like in case of a coaxial cable, there would be no magnetic field outside the cylinder as shown in Fig.3.16b. Only the exposed portion of the wire not surrounded by a conducting cylinder is the source of electromagnetic waves. This exposed portion of the wire is called an **antenna**.

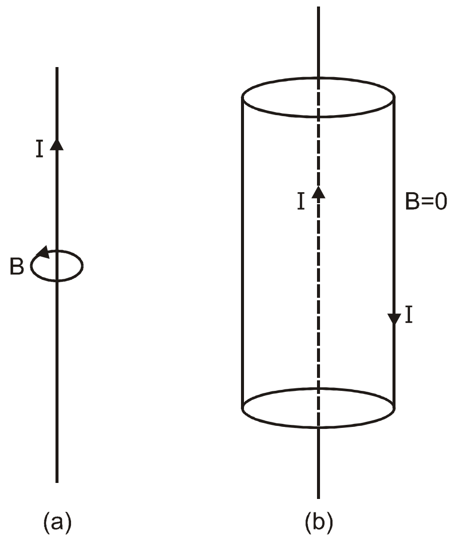

Fig. 3.16: Radiation through current carrier

A simplest antenna is a small length of straight wire with a cut in the middle. In Fig. 3.17 wire *AB* is an antenna. It is cut in the middle and points *C* and *D* are connected to inner and outer conductors of a co-axial cable respectively to feed current from a current source. Such a wire is known as *centre-fed dipole antenna*. In this case the circuit is completed through the free space impedance.

An important aspect of the fields produced by an antenna is that the fields produced at a distance *d* away from an antenna element at time *t* depend on the current in the

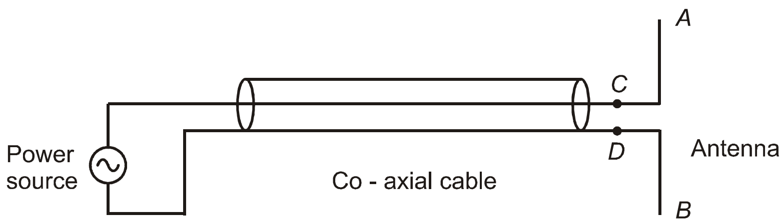

Fig. 3.17: Antenna

element at time $t - (d/c)$, preceded by the time of propagation of the signal (fields). Here *c* is the velocity of light in vacuum.

In the antenna theory, magnetic field can be expressed in terms of a vector potential as follows.

From Maxwell's second equation, $\nabla.\mathbf{B} = 0$ we may write **B** as a curl of a vector since divergence of curl of a vector always vanishes.

$$\mathbf{B} = \nabla \times \mathbf{A}\,. \qquad (3.71)$$

In this equation, **B** is a physical quantity, which could be measured. While $\mathbf{A}(\mathbf{r}, t)$ is a suitable mathematical function, to be chosen such that its curl produces the field **B**. **A** is called the *vector potential* which you have already studied in the PHE-07 Course.

The vector potential $\mathbf{A}(\mathbf{r},t)$ due to a static current element (called a dipole) of length **dl** and current *I* located at $r = 0$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi r} I\,\mathbf{dl} \qquad (3.72)$$

3.6.1 Vector Potential and Radiation Fields due to a Small Dipole

Consider a small dipole antenna of length $\hat{\mathbf{z}}\, dl$, shown in Fig. 3.18, carrying a time dependent current $I = I_0\, e^{-j\omega t}$ and placed at $r = 0$. The vector potential at ($\mathbf{r}$, t) due to this antenna is similar to the expression given by Eq. (3.72). However **A** at **r** at time t depends on current in the antenna at an earlier time $t - r/c$, preceded by the time of flight r/c. Hence, the vector potential takes the following form:

$$\mathbf{A}(\mathbf{r},t) = \hat{\mathbf{z}}\frac{\mu_0 I_0\, dl}{4\pi r}\, e^{-j\left(\omega t - \frac{\omega}{c} r\right)}. \qquad (3.73)$$

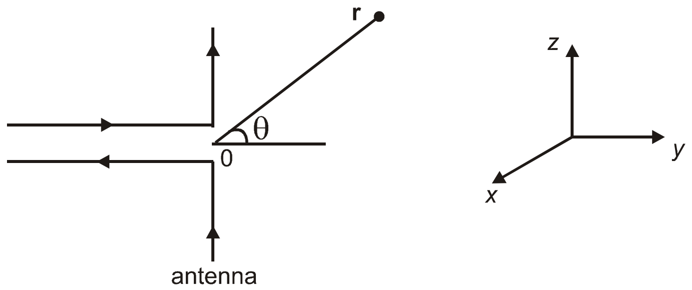

Fig. 3.18: Schematic of dipole antenna of length *dl* parallel to $\hat{\mathbf{z}}$

The magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$ is

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0 I_0\, dl}{4\pi r}\left(\frac{j\omega}{c} - \frac{1}{r}\right)\hat{\mathbf{r}}\times\hat{\mathbf{z}}\, e^{-j\left(\omega t - \frac{\omega}{c} r\right)} \qquad (3.74)$$

where **r** refers to spherical polar coordinates. **B** has two terms: the first one, whose amplitude goes as $1/r$ is called *radiation field* which is significant for electric field. The other with $1/r^2$ dependence is called the *induction field*; it is significant for magnetic field. At $r = c/\omega \approx \lambda/6$ (λ being the wavelength of the wave), the two terms are of equal magnitude. For $r >> c/\omega$ the induction field can be neglected, and we can write

$$\mathbf{B} = \frac{\mu_0 I_0\, dl}{4\pi r}\frac{j\omega}{c}(\hat{\mathbf{r}}\times\hat{\mathbf{z}})\, e^{-j\left(\omega t - \frac{\omega}{c} r\right)}. \qquad (3.75)$$

Using the fourth Maxwell's equation, the electric field can be written as

$$\mathbf{E} = \frac{j}{\omega\varepsilon_0}\nabla\times\mathbf{H}. \qquad (3.76)$$

At $r>>\lambda$ we can replace ∇ by $(j\omega/c)\hat{\mathbf{r}}$, assuming the r-variation of phase much stronger than that of the amplitude. Then we get

$$\mathbf{E} = -\frac{I_0\, dl}{4\pi r}\frac{j\omega}{\varepsilon_0 c^2}(\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\hat{\mathbf{z}}))\, e^{-j\left(\omega t - \frac{\omega}{c} r\right)}. \qquad (3.77)$$

The time average Poynting's vector, also know as *intensity* is

$$\mathbf{S} = \frac{1}{2}\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*) = \left(\frac{I_0\, dl}{4\pi r}\right)^2 \frac{\omega^2}{2c^3\varepsilon_0}\sin^2\theta\,\hat{\mathbf{r}}. \qquad (3.78)$$

Here, you should note that the intensity scales as the square of wave frequency, inversely as the square of distance and goes as $\sin^2\theta$ where θ is the angle between the direction of observation and the direction of current flow in the antenna. It is maximum transverse to the direction of current flow and vanishes in the end-on direction of the antenna.

3.6.2 Radiation Pattern

It is useful to plot polar lines from the centre of antenna in different directions such that the length of a line is proportional to intensity S_r of radiation in that direction. The locus of tips of these lines gives the **radiation pattern** of the antenna. In the case of a dipole antenna S_r is proportional to $\sin^2\theta$. The radiation pattern in the *xz*- (or *yz*-) plane is as shown in Fig.3.19. It has maxima at $\theta = \pi/2$. In the *xy* - plane ($\theta = \pi/2$ plane) the radiation pattern is a circle since there is no ϕ dependence.

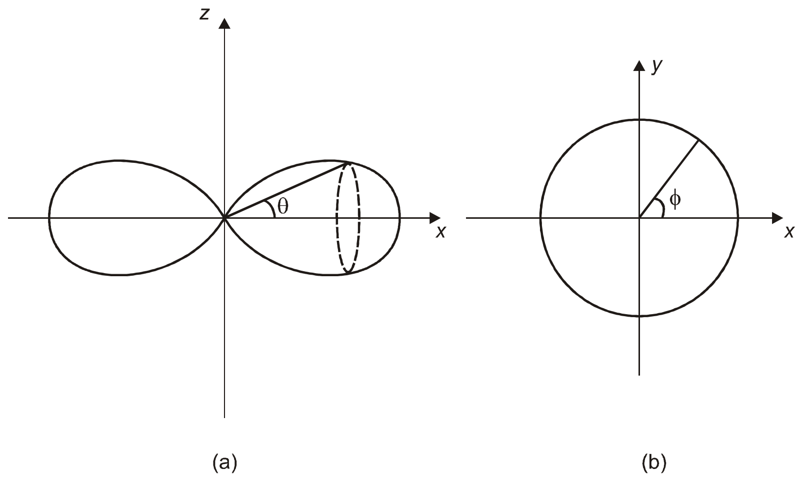

Fig.3.19: Radiation pattern of a short dipole antenna (a) in the *xz*-plane ($\phi = 0$); and (b) in the *xy*-plane ($\theta = \pi/2$)

Total power radiated by the antenna is

$$W = \int_0^{2\pi}\int_0^{\pi} P_r \; r^2 \sin\theta \, d\theta \, d\phi = \frac{Z_0\,\omega^2 I_0^2\, dl^2}{12\,\pi\, c^2} \qquad (3.79)$$

where $Z_0 = (\mu_0/\varepsilon_0)^{1/2} \approx 377$ Ohm for free space. Expressing W as $I_0^2 R_{rad}/2$, we obtain the radiation resistance of the dipole antenna

$$R_{rad} = 80\,\pi^2 \left(\frac{dl}{\lambda}\right)^2 . \qquad (3.80)$$

In the above treatment, we have assumed that the current distribution along the length of the antenna is uniform. However, in actual centre-fed dipole antennae the current is maximum at the centre and vanishes at the ends.

You may now like to solve one numerical problem.

Spend 3 *Min.*

SAQ 8

An antenna of length dl = 1 m carries current $I = I_0\, e^{-j\omega t}$ with I_0 = 10 A and $\omega/2\pi$ = 100 MHz. Estimate the radiation intensity in the side-on direction at a distance of 20 km. How does it scale with ω?

One of the important parameters of antennae is the **antenna gain**. It characterises the directivity of the radiation. It is the ratio of maximum value of S_r to the average value of S_r, or the ratio of maximum intensity obtained in certain direction by the antenna to the intensity obtained from an isotropic radiator of the same power.

For any practical antenna, the power per unit solid angle will vary depending on the direction in which it is measured, and therefore it may be written generally as a function of the angular coordinates θ and ϕ as $W(\theta, \phi)$. The power gain of the antenna is then defined as the ratio of $W(\theta, \phi)$ to the power per unit solid angle radiated by a loss-less isotropic radiator. The gain function, denoted by $G(\theta, \phi)$, is

Isotropic radiator radiates in all directions uniformly. Then power per unit solid angle is $W_i = \frac{W_{source}}{4\pi}\,\text{Wsr}^{-1}$

$$G(\theta,\phi) = \frac{W(\theta,\phi)}{W_i} = \frac{4\pi W(\theta,\phi)}{W_{source}} \tag{3.81}$$

The gain function is a very important antenna characteristic that can be measured, or in some cases, calculated.

For most antennae, the gain function shows a well-defined maximum, which is denoted by G_M, and the radiation pattern of the antenna is

$$g(\theta,\phi) = \frac{G(\theta,\phi)}{G_M} \tag{3.82}$$

The radiation pattern is seen to be simply the gain function normalised to its maximum value. The maximum value G_M is referred to as the *gain* of the antenna, but this is only a gain in the sense that the antenna concentrates or focuses the power in the direction corresponding to the maxima. It does not increase the total power radiated.

Closely associated with the power gain is the *directive gain* of the antenna. This is the ratio of $W(\theta, \phi)$ to the average power per unit solid angle radiated by the *actual* antenna and is denoted by $D(\theta, \phi)$. The average power per unit solid angle is $\eta_A W_{source}/4\pi$, where η_A is the antenna efficiency and W_{source} is the power input, as before. Thus the average is seen to be equal to $\eta_A W_{source}$ and therefore the directivity is related to power gain by

$$D(\theta,\phi) = \frac{G(\theta,\phi)}{\eta_A} \tag{3.83}$$

In particular, the maximum value of $D(\theta, \phi)$ is termed the *directivity* or *directive gain* given by

$$D_M = \frac{G_M}{\eta_A} \tag{3.84}$$

A transmitting antenna can also be employed as a receiving antenna. An electromagnetic wave incident on the antenna induces an electrical current that can be amplified and processed to retrieve relevant information from the wave. It can be shown that the directional pattern of a receiving antenna is identical to its directional (radiation) pattern as a transmitting antenna.

3.6.3 Antenna Array

The directivity of electromagnetic radiation from an antenna (antenna gain) can be increased by placing a large number of identical antennae at regular intervals from each other to form an antenna array. This is shown in Fig. 3.20. Each antenna radiates in all directions. If the currents in all the antennae are in same phase then the fields due to all the antennae at a long distance in normal direction from the array meet in the same phase giving electric field N times the field due to a single antenna, where N is the number of antennae in an array. At an angle θ to this normal direction, as shown in Fig.3.20 the radiation signals from successive antenna have path difference of (d sin θ), i.e. a phase difference of $\left(\frac{\omega}{c}d\right)\sin\theta$. If the phase difference is an integer multiple of 2π, these signals add up. Any other path difference gives smaller resultant signal. Such an array has strong directivity in the direction θ = 0. If we want to direct the radiation at angle θ, then we can introduce a phase difference in currents in successive antennae by an amount $\phi = -\left(\frac{\omega}{c}\right) d \sin\theta$, so that it nullifies the phase difference introduced by the difference in path lengths. This scheme of increasing antenna gain and directing the beam in desired direction is useful in satellite communication and radar.

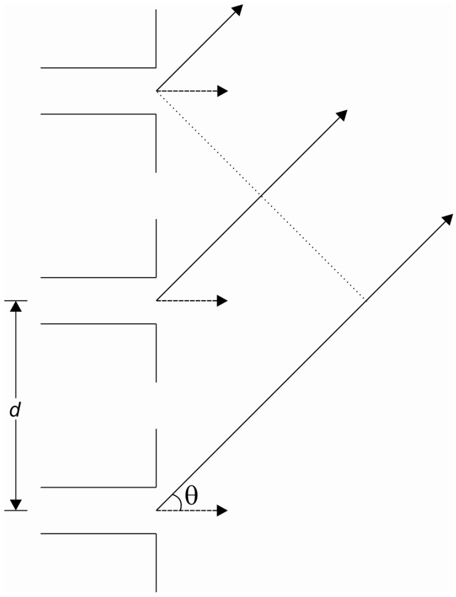

Fig. 3.20: Antenna array with separation between successive antennae = *d*

Let us now summarise the points you have learnt in this unit.

3.7 SUMMARY

- Electromagnetic fields in any medium satisfy four Maxwell's equations:
  i) $\nabla . \mathbf{D} = \rho$,
  ii) $\nabla . \mathbf{B} = 0$,
  iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, and
  iv) $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$.
- A plane electromagnetic wave is expressed as $\mathbf{E} = \mathbf{A} e^{-j(\omega t - \mathbf{k.r})}$, $\mathbf{B} = \mathbf{k} \times \mathbf{E} / \omega$ where $\mathbf{k.A} = 0$, $k = \frac{\omega}{c} \varepsilon_{eff}^{1/2}$.
- The effective relative permittivity of a medium is $\varepsilon_{eff} = \varepsilon_r + (j\sigma/\omega)\varepsilon_0$ where ε_r is the relative permittivity and σ is the conductivity.
- Refractive index of a medium is $\eta = \varepsilon_{eff}^{1/2}$.

- The plasma frequency is defined as $\omega_p = \left(\frac{\eta e^2}{m\varepsilon_0}\right)^{1/2}$
- As frequency increases, the wave tends to travel through the area near the surface of the conductor. This is characterized by the skin depth

$$\delta = \frac{\omega_p}{c}\left(\frac{\omega}{2\nu}\right)^2 = \frac{\omega}{c}\left(\frac{\sigma}{2\omega\varepsilon_0}\right)^{1/2}$$

- The wave propagates through a conductor only for the frequency $\omega > \omega_p / \varepsilon_r^{1/2}$
- The dispersion relation for n^{th} mode in wave guide is

$$k^2 = \frac{\omega^2}{c^2} - \frac{n^2\pi^2}{a^2}.$$

- Any unshielded conductor carrying current can act as an antenna.
- The intensity of antenna radiation patterns is expressed by

$$\mathbf{S} = \left(\frac{I_0 dl}{4\pi r}\right)\frac{\omega^2}{2c^3\varepsilon_0}\sin^2\theta\,\hat{\mathbf{r}}$$

- By using antenna array with an appropriate phase difference, the gain of the antenna can be increased.

3.8 TERMINAL QUESTIONS

Spend 25 *Minutes*

1. An infinitely long cylinder of radius r_0 and relative permittivity ε_r has charge density $\rho = \rho_0\, r/r_0$, where r is the distance from the axis of the cylinder. Obtain the electric field at $r < r_0$, and $r > r_0$.

2. In a plasma the electric field of an electromagnetic wave is $\mathbf{E} = \mathbf{A}\; e^{-j\left(\omega t - \frac{\omega}{4c}z\right)}$. Obtain the electron density of the plasma in terms of ω. Take $\varepsilon_r = 1$.

3. A metal has ε_r =10, $\omega_p = 8 \times 10^{15}$ rad s^{-1} and $\nu = 10^{12}$ s^{-1}. Estimate the skin depth at $\omega \cong 6 \times 10^{10}$ rad s^{-1} (microwave of 3 cm vacuum wavelength).

4. Estimate the phase velocity of a TE_{10} mode at twice the cut off frequency for this mode. What will be the phase velocity of TE_{30} mode at this frequency?

3.9 SOLUTIONS AND ANSWERS

Self Assessment Questions

1. Consider a point P inside the sphere at a distance r from the centre. Choose a Gaussian surface in the form of a sphere of radius r with centre at O refer to Fig. 4.21. At each point on the surface of that sphere, due to symmetry, the electric field would be radial and have the same magnitude E_r. Hence, the total outward normal flux through this surface is

$$\int \mathbf{D.ds} = \varepsilon_0\, \varepsilon_r E_r 4\pi r^2 .$$

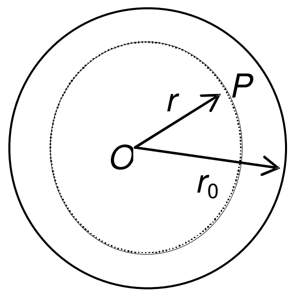

Fig. 4.21: Gaussian surface

To obtain the charge enclosed within the Gaussian surface, consider two concentric spheres of radii r' and $r' + dr'$. The surface area of the first sphere is $4\pi r'^2$ and the volume between the two spheres is $4\pi r'^2\, dr'$. The charge contained within this volume element is $\rho(r')\, 4\pi r'^2\, dr'$. Hence the charge enclosed within the Gaussian surface of radius $r < r_0$ is

$$\int \rho\, dV = \int \rho_0\left(1 - r'^2 / r_0^2\right)4\pi r'^2 dr' = 4\pi\rho_0\left(\frac{r^3}{3} - \frac{r^5}{5r_0^2}\right).$$

Equating the flux to charge enclosed we get

$$E_r = \frac{\rho_0}{\varepsilon_0 \, \varepsilon_r}\left(\frac{r}{3} - \frac{r^3}{5r_0^2}\right).$$

For an exterior point, $r > r_0$, the Gaussian surface lies outside the charged sphere, hence the charge enclosed within it is merely the charge contained within radius r_0

$$\int \rho \, dV = \int_o^{r_0} \rho_0 \left(1 - r'^2 / r_0^2\right) 4\pi r'^2 dr' = 8\pi\rho_0 r_0^3 / 15.$$

Equating this to the flux we get

$$E_r = \frac{2\rho_0 r_0^3}{15\varepsilon_0 \, \varepsilon_r \, r^2}.$$

2. From Eq. (3.20) we have

$$\sigma = \frac{n_0 e^2}{m(\nu - j\omega)}$$

$$= \frac{n_0 e^2}{m\sqrt{\nu^2 + \omega^2}} e^{j\phi}$$

where $\phi = \tan^{-1} \omega/\nu$. The current density due to the electric field $\mathbf{E} = \mathbf{A}\, e^{-j\omega t}$ is

$$\mathbf{J} = \frac{n_0 e^2 \mathbf{A}}{m\sqrt{\nu^2 + \omega^2}} e^{-j(\omega t - \phi)}.$$

The phase difference between **J** and **E** is $\phi = \pi/3$. Hence $\tan \phi = \sqrt{3}$ and collision frequency $\nu = \dfrac{\omega}{\tan\phi} = \dfrac{\omega}{\sqrt{3}} \approx 5.77 \times 10^{12} s^{-1}$.

3. In general we may write $\mathbf{E}(\mathbf{r}, t)$ as $\mathbf{E}(\mathbf{r}, t) = \mathbf{A}e^{-j\omega t - k_x x - k_y y - k_z z}$ comparing the given field with this expression we get

$$k_x = 2\omega/c,\ k_y = \omega/c \text{ and } k_z = 0$$

hence $\mathbf{k} = (2\omega/c)\,\hat{\mathbf{x}} + (\omega/c)\,\hat{\mathbf{y}}$ and

$$k = (k_x^2 + k_y^2)^{1/2} = \sqrt{5}\,\omega / c$$

Wavelength $\lambda = 2\pi/k = 2\pi c / \omega\sqrt{5}$

and Phase Velocity, $v_{ph} = \omega / k = c / \sqrt{5}$.

4. For an electromagnetic wave in plasma

$$k^2 = \frac{\omega^2}{c^2}(1 - \omega_p^2 / \omega^2)$$

Wave propagates when $\omega > \omega_p = (ne^2 / m\varepsilon_0)^{1/2} \approx 5.6 \times 10^7 \text{ rad}\, s^{-1}$.

5. Here $\mathbf{k} = \dfrac{2\omega}{c}\hat{\mathbf{x}} - \dfrac{\omega}{c}\hat{\mathbf{z}}$

Hence $\mathbf{S}_{\mathbf{av}} = \dfrac{A^2}{2\mu\omega}\dfrac{\omega}{c}(2\hat{\mathbf{x}} - \hat{\mathbf{z}}) = \dfrac{A^2}{2\mu c}(2\hat{\mathbf{x}} - \hat{\mathbf{z}})$

Magnitude of $\mathbf{S}_{\mathbf{av}}$ is called *intensity*.

6. Surface wave dispersion relation is

$$k^2 = \frac{\omega^2}{c^2}\frac{\varepsilon_{eff}}{\varepsilon_{eff} + 1}$$

For the given parameters

$$\varepsilon_{eff} = \varepsilon_r - \frac{\omega_p^2}{\omega^2} = -6$$

$$k = \sqrt{\frac{6}{5}}\frac{\omega}{c}$$

$$\lambda = \sqrt{\frac{5}{6}}\frac{c}{\omega} \approx 0.14\mu\text{m}$$

Field amplitude decays in the metal as

$$A = A_0 e^{-\alpha_2 d}$$

where d is the depth

$$\alpha_2^2 = k^2 - \frac{\omega^2}{c^2}\varepsilon_{eff} = -\frac{\omega^2}{c^2}\frac{\varepsilon_{eff}^2}{1+\varepsilon_{eff}}$$

$$\alpha_2 = \frac{\omega}{c}\frac{6}{\sqrt{5}} = \sqrt{6}/\lambda$$

At $d = 0.1$ μm

$$\alpha_2 d = \sqrt{6}d/\lambda = \frac{\sqrt{6}\times 0.1\mu\text{m}}{0.14} = 1.75$$

$$A = A_0\, e^{-1.75} = 10^4 \times 0.17 = 1.7 \times 10^3 \text{ V m}^{-1}$$

7. From third Maxwell's equation, $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$, you can note that in the steady state the t, z dependence of **B** should be the same as that of **E**. Hence we can write $\partial\mathbf{B}/\partial t$ as $-j\omega\mathbf{B}$ and the Maxwell's equation

$$\mathbf{B} = \frac{1}{j\omega}\nabla\times\mathbf{E} = \left[-\hat{\mathbf{x}}\frac{k}{\omega}A_1 \sin\left(\frac{\pi}{a}x\right) + \hat{\mathbf{z}}\frac{A_1}{j\omega}\frac{\pi}{a}\cos\left(\frac{\pi}{a}x\right)\right] e^{-j(\omega t - kz)}$$

8. The intensity at $\theta = \pi/2$ is

$$S = \left(\frac{I_0 dl}{4\pi r}\right)^2 \frac{\omega^2}{2c^3\varepsilon_0} = 1.3\times 10^{10} \text{ Wm}^{-2}.$$

S scales as ω^2.

Terminal Questions

1. To find the electric field at a point r distance away from the cylinder axis, consider a cylindrical Gaussian surface of radius r and length unity. By symmetry the electric field is radial. Hence the total outward flux through the Gaussian surface is

$$\int \mathbf{D}.\mathbf{ds} = \varepsilon_0\, \varepsilon_r\, E_r 2\pi r.$$

There is no contribution from the ends of the cylinder as electric field is tangential to the surface. The charge enclosed within the Gaussian surface is

$$\int \rho\, dV = \int_0^r \rho(r')\, 2\pi r' dr' = \frac{2\pi\rho_0 r^3}{3r_0}.$$

Equating the electric charge enclosed, with the flux for ($r < r_0$), we get

$$E_r = \frac{\rho_0 r^2}{3\varepsilon_0\varepsilon_r r_0}$$

For external point ($r > r_0$), the Gaussian surface is outside the conducting cylinder. Hence the change will be confined only between 0 & r_0.

$$\therefore \int \rho\, dV = \int_0^{r_0} \rho(r')\, 2\pi r' dr' = \frac{2\pi\rho_0 r^3}{3}.$$

Equating it to the flux we get

$$E_r = \frac{\rho r_0^2}{3\varepsilon_0\varepsilon_r r}.$$

2. In a plasma

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

Here $k = \omega/4c$, hence $\omega_p = \sqrt{15}\omega/4$

Since $\omega_p = (ne^2/m\varepsilon_0)^{1/2}$ we get

$$n = \frac{15}{16}\frac{\omega^2}{e^2}m\varepsilon_0.$$

3. For an e.m. wave in a conductor

$$k^2 = \frac{\omega^2}{c^2}\left(\varepsilon_r - \frac{\omega_p^2}{\omega^2 + \nu^2}\left(1 - j\frac{\nu}{\omega}\right)\right)$$

For $\omega << \nu$, $k^2 = \frac{\omega^2}{c^2}\frac{\omega_p^2}{\omega\nu} j$

Skin depth $\delta = \frac{1}{k_i} = \sqrt{2}\frac{c}{\omega_p}\left(\frac{\nu}{\omega}\right)^{\frac{1}{2}} \approx 0.2\,\mu\text{m}.$

4. For TE_{n0} mode using the expression $\omega_n = n\pi\, c/a$ the dispersion relation can be written as

$$\omega^2 = \omega_n^2 + k^2c^2$$

When $\omega = 2\omega_1$ for TE_{10} mode we get $k = \sqrt{3}\frac{\omega_1}{c}$, and $v_{ph} = \frac{\omega}{k} = \frac{2}{\sqrt{3}}c.$

For TE_{30} mode $\omega^2 = \omega_3^2 + k^2c^2$

Since $\omega_3 = 3\omega_1$ and $\omega = 2\omega_1$, k^2 is negative, hence the wave does not propagate.

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2. *Communication Systems* by Haykin, Simon; (III Edition) John Wiley & Sons
3. *Modern Digital and Analog Communication Systems* by Lathi, B.P.; (III Edition) (Oxford University Press)