UNIT 10 FOURIER TRANSFORMS

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10.1 INTRODUCTION

You have studied the **Fourier series** of a periodic function in Unit 7 of the course PHE-05 entitled Mathematical Methods in Physics-II. You have learnt there that a periodic function can be expanded in a series of sine and cosine functions. Physically, we could think of the terms of the Fourier series as representing a set of harmonics. For example, the shape of the wave train of a sound wave (e.g., a musical note) is periodic in space. Such a function (extending to infinity on both sides) can be expressed as a sum of sine and cosine functions which would represent an infinite set of frequencies $n\nu$, $n = 1,2,3...$

In electricity, a periodic voltage is represented by a Fourier series. This means that the voltage is made up of an infinite but **discrete** set of frequencies $n\omega$. We may now ask a couple of related questions: Is it possible to extend this method to a function which is **not** periodic? And, can we modify it to represent functions containing a continuous set of frequencies? For example, we may like to analyse a single voltage pulse not repeated, or a flash of light, or a note of sound which is not repeated. Or else we may want to analyse a continuous spectrum of wavelength of light or a sound wave containing a continuous set of frequencies. If you recall that an integral is a limit of a sum, then you would not be surprised to know that such a method does exist. In such cases, the Fourier series (i.e., a sum of terms) is replaced by an integral known as the **Fourier transform**.

This method was in fact developed by Fourier himself to expand **aperiodic functions**. Fourier transforms are very useful in the study of waves, especially when we need to extract information about their phase. For example, the electron distribution in an atom may be obtained from the Fourier transform of the amplitude of scattered X-rays. Similarly, the output of the stellar interferometer involves a Fourier transform of the brightness across the stellar disc. Therefore, we have devoted two units of this course to Fourier transforms.

In this unit, we introduce the mathematical concepts related to Fourier transforms. In Unit 11, you will study their applications to physical problems. In order to study this unit effectively, we would advise you to revise quickly Unit 7 of PHE-05 on Fourier series, since we will be extending those very concepts here. There is another piece of advice. Always sit with a notebook and a pen when you are studying this unit. Work out all the steps given in the unit yourself. Do not read this unit passively like it were a story book. *Remember, in mathematics and physics, you must always work out the steps in a derivation yourself. Only then each step will become clear to you and will remain in your memory longer.*

Objectives

After studying this unit, you should be able to:

- determine whether the Fourier Transform of an aperiodic function exists or not;
- obtain the Fourier transforms of simple functions; and
- calculate the Fourier sine and cosine transforms of simple functions.

10.2 WHAT IS A FOURIER TRANSFORM?

To arrive at a definition of Fourier transform, we begin by rewriting the Fourier series for a periodic function using complex exponential functions rather than sine and cosine functions as we did in Unit 7 of PHE-05.

10.2.1 Complex Fourier Series

Recall that a function which has a period T and which has **a finite number of finite discontinuities** in one period, can be expanded in an infinite series of the form

Notice that in the definition of Fourier series here, we have introduced an additional concept: that of $f(x)$ having finite number of finite discontinuities in one period. You may be wondering what this means.

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nx}{T}\right) + b_n \sin\left(\frac{2\pi nx}{T}\right)\right] \tag{10.1a}$$

where $f(x+T) = f(x)$ for all values of x. The coefficients a_0, a_n and b_n are given by

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\,dx \tag{10.1b}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos\left(\frac{2\pi nx}{T}\right)dx, \tag{10.1c}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\left(\frac{2\pi nx}{T}\right)dx, \tag{10.1d}$$

The figure above shows a function $f(t)$ defined on the interval $t \in (a,b)$. The function has two discontinuities on this interval, one at t_1 and the other at t_2. Thus the function is made up of three pieces each of which is continuous. Therefore it is a *piecewise continuous function* on the given interval. The amount of discontinuity at a point, for example t_1, is the difference between its values just on the right and just on the left of that point. The function shown here has *finite* discontinuities at both the points t_1 and t_2. Since the function has two finite discontinuities on the given interval, it qualifies as having a *finite number of finite discontinuities.*

Now from your school maths course, recall the relations

$$e^{ix} = \cos x + i\sin x, \quad e^{-ix} = \cos x - i\sin x, \tag{10.2a}$$

which are true for any x. Add and subtract the above two equations and you will obtain the inverse relations

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{10.2b}$$

Now use Eq. (10.2b) in the expressions for Fourier series of Eqs. (10.1 to d) (of course, replacing x in Eq. (10.2b) by $2\pi nx/T$) and you will get this equation:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \frac{e^{2n\pi ix/T} + e^{-2n\pi ix/T}}{2} + b_n \frac{e^{2n\pi ix/T} - e^{-2n\pi ix/T}}{2i}\right] \tag{10.3}$$

Next combine the terms with the positive exponent and those with the negative exponent. Also remember that $1/i$ equals $-i$, so that you can write b_n/i as $-ib_n$ in the last term in Eq. (10.3). You will then obtain

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n - ib_n}{2} e^{2n\pi ix/T} + \frac{a_n + ib_n}{2} e^{-2n\pi ix/T}\right] \tag{10.4}$$

We now define new coefficients through the equations

$$c_0 = a_0, \quad c_n = \frac{a_n - ib_n}{2}, \quad d_n = \frac{a_n + ib_n}{2}, \quad n = 1,2,3,\ldots \tag{10.5}$$

Then we can write the function $f(x)$ of Eq. (10.4) in terms of these coefficients as

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{2n\pi ix/T} + \sum_{n=1}^{\infty} d_n e^{-2n\pi ix/T} \quad (10.6)$$

We once again define coefficients by

$$c_{-n} = d_n, \quad n \geq 1, \quad (10.7)$$

that is, we call d_1 as c_{-1}, d_2 as c_{-2}, etc. Now substitute for d_n from Eq. (10.7) in the last term of Eq. (10.6) and make a change of the summation index by defining $n = -m$ (only in this term). Note that as n takes values from 1 to ∞, m will take values between -1 and $-\infty$. Then you can rewrite this term as follows:

$$\sum_{n=1}^{\infty} d_n e^{-2n\pi ix/T} = \sum_{n=1}^{\infty} c_{-n} e^{-2n\pi ix/T} = \sum_{m=-1}^{-\infty} c_m e^{2m\pi ix/T}$$

$$= \sum_{n=-1}^{-\infty} c_n e^{2n\pi ix/T}, \quad (10.8)$$

where in the last part of the derivation, we have simply put $m = n$. If we rewrite the term like this, we find that the right hand side of Eq. (10.6) contains the coefficients c_n for all values of n: positive, negative and zero. We can thus write the right hand side of Eq. (10.6) as a single summation over n from $-\infty$ to $+\infty$, in a compact form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2n\pi ix/T} \quad (10.9)$$

What are the expressions for the new coefficients c_n? These are related to the earlier coefficients a_0, a_n and b_n through Eqs. (10.5) and (10.7) which in turn are related to the given function $f(x)$ through Eqs. (10.1b to d). In order to obtain c_n, start with the second of the Eqs. (10.5) and make the substitutions for a_n and b_n. You will obtain

$$c_n = \frac{a_n - ib_n}{2} = \frac{2}{2T} \int_{-T/2}^{T/2} f(x)[\cos(2n\pi x/T) - i\sin(2n\pi x/T)]\, dx$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} f(x) e^{-2n\pi ix/T} dx, \quad n > 0 \quad (10.10a)$$

The coefficient c_0 is just equal to a_0, which is given by Eq. (10.1b). You will see that it agrees with the form of c_n which we have found above in Eq. (10.10a) if we put n=0. Therefore, you can now say that Eq. (10.10a) is true for $n \geq 0$. Finally, you can see from Eq. (10.5) that the coefficients d_n can be obtained from c_n by replacing the imaginary quantity i by $-i$ in Eq. (10.10a). But d_n is just equal to c_{-n}, as defined in Eq. (10.7). You will thus get

$$d_n = c_{-n} = \frac{1}{T} \int_{-T/2}^{T/2} f(x) e^{2n\pi ix/T} dx, \quad n \geq 1 \quad (10.10b)$$

This is of the same form as Eq. (10.10a) with n replaced by $-n$. Eqs. (10.10a and b) together tell us that the form of Eq. (10.10a) holds good for all values of n: negative, zero and positive.

We can now put all these results together and define the complex Fourier series of a periodic function.

Complex Fourier series of a periodic function

A periodic function $f(x)$ with a period T and having a finite number of finite discontinuities in one period, can be expanded in a complex Fourier series of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi n i x/T} \qquad (10.11a)$$

where the coefficients are given by

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\, e^{-2\pi n i x/T} dx, \quad n = \text{any integer}. \qquad (10.11b)$$

The coefficients c_n are called the **complex Fourier coefficients** of $f(x)$.

Given this background we are ready to define the Fourier transform.

10.2.2 Defining Fourier Transforms

As we have said in the introduction to this unit, the powerful method of Fourier series is restricted to periodic functions. Very often we come across occasions where we have to deal with aperiodic functions (that is, functions which are not periodic). Let us now extend this method to such functions.

Consider a periodic function having a period T. If we now go on increasing the period so that it tends to infinity, the function will not repeat itself and will be aperiodic. Let us illustrate this point with the help of a couple of examples. Consider first the function shown in Fig. 10.1(a) and mathematically defined by

$$f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } 1 < |x| < T/2 \text{ with } T > 2, \end{cases} \qquad (10.12a)$$

and

$$f_T(x+T) = f_T(x)$$

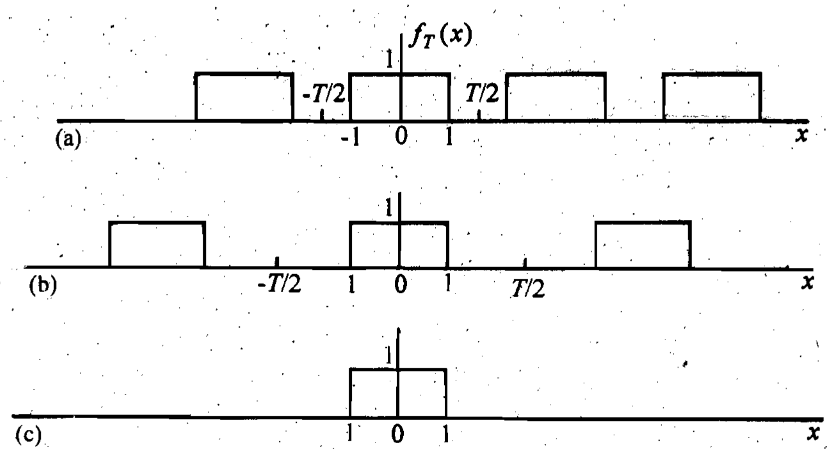

Fig.10.1: (a) A rectangular periodic pulse with width 2 and period T; (b) the same pulse with a larger T; (c) as T is made infinitely large, there is only one pulse, giving an aperiodic function.

The pulse of Fig. 10.1a is a periodic pulse of fixed width 2 and unit height repeating with a period T. Now we make the period T larger and larger (such as 10, 100, ...) as in. (Fig. 10.1b). Finally when T tends to infinity, there is a single square pulse of unit height and width 2. This is an aperiodic function shown in Fig. 10.1c and defined by

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1. \end{cases} \qquad (10.12b)$$

Similarly, consider another function (Fig. 10.2a) mathematically defined by

$$g_T(x) = e^{-|x|} \quad \text{for } |x| < T/2$$

and

$$g_T(x+T) = g_T(x). \qquad (10.13a)$$

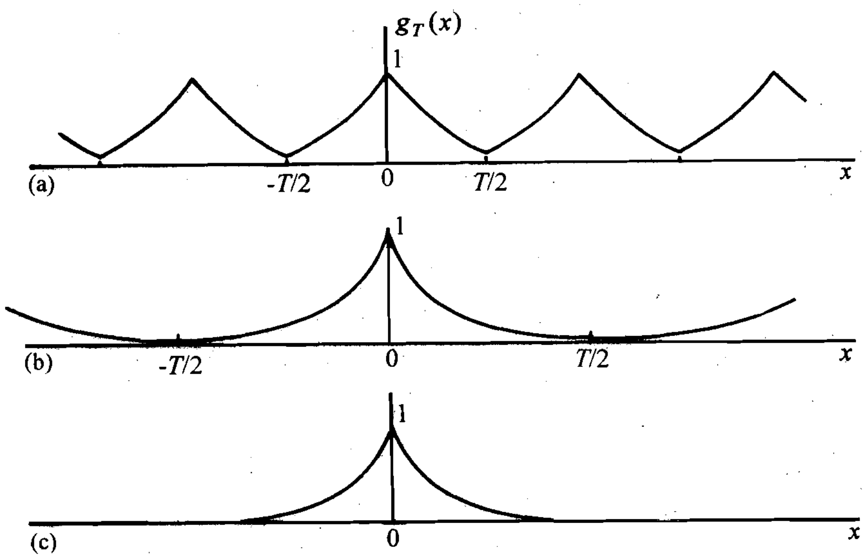

Fig.10.2: (a) The function $g_T(x)$ of Eq. (10.13a); (b) the same function with a larger T; (c) the same function with T going to infinity.

This function is again periodic with period T. We now let T become very large as in (Fig. 10.2b), and finally let it tend to infinity. You can see the resulting aperiodic function in Fig. 10.2c. It is described by the equation

$$g(x) = e^{-|x|}, \text{all } x. \qquad (10.13b)$$

In both the cases, we can express the aperiodic function as the limit of a periodic function as its period tends to infinity. Thus

$$f(x) = \lim_{T\to\infty} f_T(x), \quad g(x) = \lim_{T\to\infty} g_T(x). \qquad (10.14)$$

If the original periodic function contains a finite number of finite discontinuities in one period, the function obtained after taking the limit $T \to \infty$ would possess a finite number of finite discontinuities in the whole interval $-\infty < x < \infty$.
Now what should we do to obtain the expansion of such an aperiodic function? You can see that the answer is contained in the above examples. We can start from the complex Fourier series of a periodic function and take the limit $T \to \infty$. Let us do that and see what we obtain.

Let us consider a general periodic function $f_T(x)$ with period T and write its complex Fourier series in the form of Eq. (10.11a). To simplify further analysis, we define

$$k_n = 2n\pi / T \qquad (10.15)$$

so that the complex Fourier series can be written as

$$f_T(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} \tag{10.16}$$

or substituting for c_n from Eq. (10.10a):

$$f_T(x) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{ik_n x} \int_{-T/2}^{T/2} f(u) e^{-ik_n u} du \tag{10.17a}$$

where we have changed the variable of integration.

In Eq. (10.17a), $\Delta k = k_{n+1} - k_n = 2\pi / T$. Hence we can replace $2/T$ by $\Delta k/\pi$. So we have

$$f_T(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \Delta k\, e^{ik_n x} \int_{-T/2}^{T/2} f(u) e^{-ik_n u} du \tag{10.17b}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \Delta k \left[\int_{-T/2}^{T/2} f(u) e^{-ik_n u} du\right] e^{ik_n x}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \Delta k \int_{-T/2}^{T/2} f(u) e^{ik_n (x-u)} du$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(k_n) \Delta k$$

where

$$F(k_n) = \int_{-T/2}^{T/2} f(u) e^{ik_n (x-u)} du \tag{10.17c}$$

Now as we allow T to tend to infinity, three things happen:

(a) the limits of all integrals become $-\infty$ to $+\infty$,
(b) $1/T$ tends to zero, and
(c) the difference $\Delta k = 2\pi/T$ between successive values of k_n also becomes smaller and smaller.

Because of (a), we can deal with the situation only if the integral

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty, \qquad \text{where } f(x) = \lim_{T\to\infty} f_T(x) \tag{10.18}$$

that is, if the integral is finite. This is expressed by saying that the function $f(x)$ is **absolutely integrable.**

Assuming that this condition is satisfied, we are now justified in taking the limit $T \to \infty$ for $f_T(x)$. Its left hand side can be replaced by $f(x)$ after taking the limit. In the integral on the right hand side, we can replace k_n by k and the summation over n from $-\infty$ to ∞ by integration over k from $-\infty$ to ∞. You should note that these limits are consistent with the definition $k_n = 2n\pi / T$ through which we have made the transformation from n to k. Thus we have

$$F(k) = \int_{-\infty}^{\infty} f(u) e^{ik(x-u)} du$$

and

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} g(k)\,dk=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{ik(x-u)}du\,dk$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk\int_{-\infty}^{\infty} f(u)e^{-iku}du \qquad (10.19)$$

If we define $g(k)$ by

$$g(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

then

$$f(x)=\int_{-\infty}^{\infty} g(k)e^{ikx}dk$$

For reasons of symmetry, we multiply both $f(x)$ and $g(k)$ by $\sqrt{\frac{1}{2\pi}}$ instead of having the factor $\frac{1}{2\pi}$ in only one function. Thus, we obtain the definition of Fourier transforms as follows:

Fourier Transforms

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(k)\,e^{ikx}dk \qquad (10.20a)$$

$$g(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx \qquad (10.20b)$$

The function $g(k)$ is called the **Fourier transform** of $f(x)$ and $f(x)$ is called the **inverse Fourier transform** of $g(k)$. Notice that the two integrals differ in form only in the sign of the exponent. Therefore it is also quite common to call either of them a **Fourier transform** of the other.

Compare Eqs. (10.20a,b) and (10.11a,b). The function $g(k)$ corresponds to c_n, k corresponds to n and $\int_{-\infty}^{\infty}$ corresponds to $\sum_{-\infty}^{\infty}$. The quantity k is a continuous analog of the integer n, and so the set of discrete coefficients has become a continuous function $g(k)$: the sum over n has become an integral over k.

We can state this result in the form of the **Fourier integral theorem** as follows:

Fourier Integral Theorem

If a function $f(x)$ possesses no more than a finite number of finite discontinuities and if the integral $\int_{-\infty}^{\infty}|f(x)|\,dx$ is finite, then it can be expressed in the form of an integral as in Eq. (10.20a) with the coefficients given by Eq. (10.20b).

Now you may want to know: What is the physical meaning of Eqs. (10.20a,b)? Let us interpret these results from the point of view of physics.

Physical Interpretation

The Fourier series of a periodic function tells us which frequencies (or wave vectors) of sinusoidal waves we should superimpose, and what should be the amplitude of each wave, so as to get the original function. Similarly, the Fourier transform of a function tells us which waves (what frequencies and what amplitudes) should be combined to get the original aperiodic function. The only difference is that *whereas it requires discrete frequencies* (multiples or harmonics of the frequency of the periodic function) *to generate a periodic function,* **we require a continuous range of frequencies (all frequencies) to generate an aperiodic function.**

We would also like to put in a word about the dimensions of parameters in Fourier transforms.

As far as abstract mathematics is concerned, the algebraic parameters such as x, k, f, g need not have any physical meaning and hence need not be assigned any physical dimensions. But as you know, in physics we use mathematical models to describe physical situations. When we do that, each parameter must denote some physical quantity. Since the product kx appears as the argument of the exponential or the trigonometric function, it is clear that k must have the dimensions of the reciprocal of x. Thus, if x stands for length, k stands for reciprocal of length (wavenumber), if x denotes energy, k must denote some quantity which is the reciprocal of energy, if x denotes time, k denotes angular frequency and so on.

Further, Eqs. (10.20a and b) show that the dimensions of g should be those of the product of dimensions of f and x, and the dimensions of f should be the same as the product of the dimensions of g and k. For example, you may be dealing with the problem of the vibrations of a string. In that case, $f(x)$ will denote the instantaneous displacement of the string at the point x. Thus x and f are both lengths (dimension L). Then k would have the dimensions of L^{-1} and g those of L^2. It is important to have these considerations in the back of your mind when you physically interpret the final results of a mathematical calculation. You will appreciate these points better when you study the next unit.

Let us now determine Fourier transforms of some simple functions.

10.2.3 Fourier Transforms of Simple Functions

As a first example, let us obtain the Fourier transform of the single square pulse of Eq. (10.12b).

Example 1: Fourier transform of a single square pulse

The single square pulse (Fig. 10.1c) may represent an impulse in mechanics (i.e., a force applied only over a short time, for example, a bat hitting a ball), or a sudden short surge of electric current or a short pulse of sound and light which is not repeated. This function has two discontinuities on the entire real line: a discontinuity of +1 at $x = -1$ and another one of -1 at $x = 1$. Moreover, the integral $\int_{-\infty}^{\infty} |f(x)|\, dx$ comes out to be 2 (this is just the total area enclosed between the function and the x-axis since the function is non-negative everywhere). Thus you see that this function satisfies both the requirements for a function to be expressed as a Fourier integral.

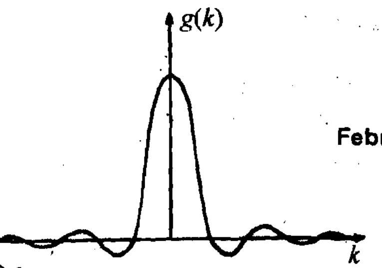

Fig.10.3: Fourier transform of a single square pulse.

Using Eq. (10.20b) we have

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$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ikx}\, dx,$$

since the integrand is finite over the interval $[-1, 1]$. Thus

$$g(k) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ikx}}{-ik}\Bigg|_{-1}^{+1} = \frac{\sqrt{2}}{\sqrt{\pi}k} \frac{e^{-ik} - e^{ik}}{-2i} = \frac{\sin k}{\sqrt{\pi}k} . \sqrt{2}$$

The function $g(k)$ is shown in Fig. 10.3.

Now

$$f(x) = \frac{\sqrt{2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin k}{\sqrt{\pi}k} e^{ikx} dk$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin k(\cos kx + i \sin kx)}{k} dk$$

Since $\frac{\sin k}{k}$ is an even function, $\frac{\sin k \sin kx}{k}$ will be odd. Recall from Unit 7 of PHE-05 that the integral of an odd function over a symmetric interval about the origin (here, $-\infty$ to $+\infty$) is zero. Therefore the integration of sin k sin kx from $-\infty$ to $+\infty$ will be zero. Further, recall that the integral of an even function over a symmetric interval is twice the integral over positive values of the variable of integration. Thus

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin k \cos kx}{k} dk$$

But since the function $f(x)$ is given by Eq. (10.12b), we now see that

$$I(x) = \int_{0}^{\infty} \frac{\sin k \cos kx \, dk}{k} = \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \qquad (10.21)$$

where we have denoted the integral by the symbol $I(x)$. In particular, at $x = 0$, cos $kx = 1$, and therefore Eq. (10.21) reduces to

$$\int_{0}^{\infty} \frac{\sin k \, dk}{k} = \frac{\pi}{2} \qquad (10.22)$$

which is the mid point of the jump in $f(x)$ at $|x| = 1$.

Did you notice in Example 1 that the Fourier integral has provided us with a method to obtain certain definite integrals? To evaluate the above integral, we would have otherwise been compelled to use methods of complex integration, which are somewhat more tedious.

You should also note that the integral $I(x)$ on the left hand side of Eq. (10.21), regarded as a function of x, is discontinuous at $x = 1$ and $x = -1$. But it **does not mean** that the integral does not exist or does not possess a unique value at these two points. It is important to realize that $I(x)$ exists and is unique for every (finite) value of x.

The value of the integral in Eq. (10.21) for $x = 1$ or $x = -1$ is $\pi/4$. This is half the sum of the left hand limit and the right hand limit of the function $f(x)$ at the point of discontinuity. From this experience, we arrive at an important result:

Whenever a function has a discontinuity, its Fourier integral at that point equals the average of the left hand limit and the right hand limit of the function at that point.

Recall that in Unit 7 of PHE-05, we have obtained a similar result for Fourier series.

In Eq. (10.21), you will notice that we have not mentioned the value of the integral at $|x| = 1$. With this result, we can write the value of $I(x)$, which is known as **Dirichlet's discontinuous factor**, for all values of x:

$$\int_{0}^{\infty} \frac{\sin k \cos kx \, dk}{k} = \begin{cases} \pi/2, & |x| < 1 \\ \pi/4, & |x| = 1 \\ 0 & |x| > 1 \end{cases} \qquad (10.23)$$

In order to further clarify the evaluation of Fourier transforms, we consider an example from physics.

Example 2: Fourier transform of the electric field in a radiated wave

The electric field in a radiated wave may be represented by a function like the following:

$$f(t) = \begin{cases} 0 & (t < 0) \\ e^{-t/T} \sin \omega_0 t & (t > 0) \end{cases}$$

This function (Fig. 10.4) might also represent the displacement of a damped harmonic oscillator or the current in an antenna.

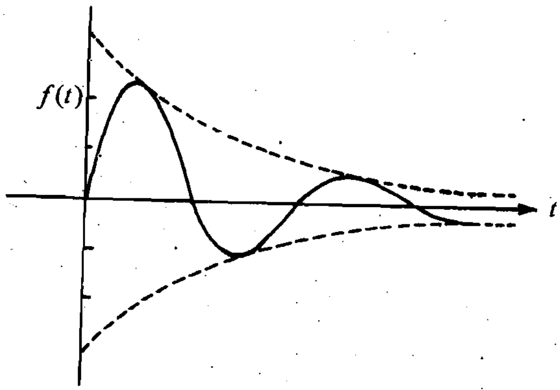

Fig.10.4: Electric field in a radiated wave

The Fourier transform of $f(t)$ is

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

Substituting for $f(t)$ we get

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-t/T} e^{-i\omega t} \sin \omega_0 t \, dt = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-t/T} e^{-i\omega t} \left(\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} \right) dt$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \left(\frac{1}{\omega + \omega_0 - \frac{i}{T}} - \frac{1}{\omega - \omega_0 - \frac{i}{T}} \right)$$

This was easy to obtain, was it not? Would you like to understand the physical meaning of $g(\omega)$? If $f(t)$ is a radiated electric field, the radiated power is proportional to $|f(t)|^2$ and the total energy radiated is proportional to $\int_0^{\infty} |f(t)|^2 dt$. This is equal to

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |g(\omega)|^2 d\omega,$$

a result you can take as given. Then $|g(\omega)|^2$ may be interpreted as the energy radiated per unit frequency interval (times some constant).

Now suppose T is large ($\omega_0 T >> 1$). Then $g(\omega)$ is peaked sharply near $\omega = \pm\omega_0$. For example, near $\omega = \omega_0$, the second term will dominate the first term and we have

$$g(\omega) \approx -\frac{1}{2}\frac{1}{\omega - \omega_0 - \frac{i}{T}}$$

and

$$|g(\omega)| \approx \frac{1}{2}\frac{1}{\left[(\omega-\omega_0)^2 + \frac{1}{T^2}\right]^{1/2}}$$

When $\omega = \omega_0 \pm \frac{1}{T}$, $g(\omega)$ is scaled down by a factor $\sqrt{1/2}$ and the radiated energy $|g(\omega)|^2$ is reduced by a factor $\frac{1}{2}$ (see Fig. 10.5). In other words, the width Γ at half (power) maximum is given by $\Gamma \times 2/T$.

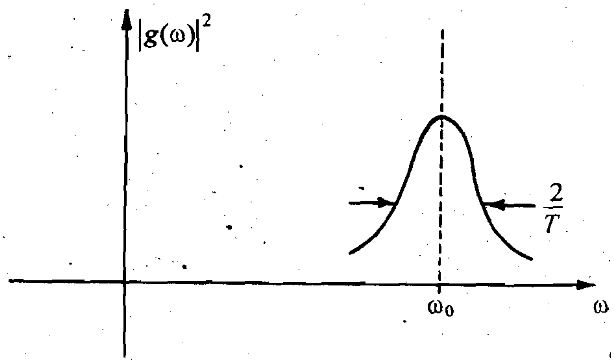

Fig.10.5: Spectrum of radiated energy

This result is a typical uncertainty principle. Recall from Unit 5 of PHE-11, Heisenberg's uncertainty principle relating the energy of a state with its life time. The time period (T) is inversely proportional to the width Γ which is a measure of "uncertainty" in the energy or the frequency.

You should now calculate the Fourier transform of some functions yourself to get some more practice.

SAQ 1

Spend 15 min

Obtain the Fourier transforms of the following functions:

(a) $$f(t) = \begin{cases} 0 & t < 0 \\ e^{-\lambda t} & t \geq 0, \quad \lambda > 0 \end{cases}$$

(b) $$f(x) = \begin{cases} x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

In Unit 7 of the course PHE-05, you have dealt with Fourier cosine and sine series expansions of even or odd functions. You have seen that if the periodic function is an even function of x, the coefficients b_n in the Fourier series Eq. (10.1d) vanish. On the other hand, if the function is an odd function of x, the coefficients a_0 and a_n vanish. In fact, this is clear from the form of Eqs. (10.1b to d). In a similar manner, Eqs. (10.20a and b) suggest that the Fourier integral would simplify in case the function $f(x)$ is even or odd. We shall mention the two cases more explicitly for convenience and ready reference. In the next section, we shall assume that the function can be expressed in the form of a Fourier integral and obtain **Fourier sine and cosine transforms**.

10.3 FOURIER SINE AND COSINE TRANSFORMS

If $f(x)$ is an odd function, then $g(k)$ is odd too, and Eqs. (10.20a and b) reduce to a pair of **Fourier sine transforms**. Let us see how. We substitute

$$e^{-ikx} = \cos kx - i\sin kx$$

in Eq. (10.20b) and get

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\left(\cos kx - i\sin kx\right)dx \qquad (10.24)$$

Since cos kx is even and $f(x)$ is odd, the product $f(x)$ cos kx is odd. Therefore, the term $\int_{-\infty}^{\infty} f(x)\cos kx\, dx$ in Eq. (10.24) is zero. The product $f(x)\sin kx$ is even (product of two odd functions). Substituting these results in Eq. (10.24) we have

$$g(k) = -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\sin kx\, dx = -i\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x)\sin kx\, dx \qquad (10.25)$$

If we now replace k by $-k$, the sign of sin kx and hence of $g(k)$ changes. Thus

$$g(-k) = -g(k)$$

that is, $g(k)$ is an odd function. Then using the same procedure we can obtain the inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)e^{ikx}\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)\left(\cos kx + i\sin kx\right) dk$$

or

$$f(x) = \sqrt{\frac{2}{\pi}}\, i \int_{0}^{\infty} g(k)\sin kx\, dk \qquad (10.26)$$

If we substitute $g(k)$ from Eq. (10.24) into Eq. (10.26) we get an equation like Eq. (10.19),

$$f(x) = \sqrt{2}\, i\left(-\frac{i}{\sqrt{\pi}}\right) \int_{0}^{\infty} \sin kx\, dk \int_{0}^{\infty} f(u)\sin ku\, du$$

Notice that the numerical multiple outside the integral becomes $\left(-i^2\right)(2/\pi)^{1/2} = \sqrt{2/\pi}$. Thus the imaginary factors are not needed. Thus we obtain

Fourier Sine Transforms

The pair of **Fourier sine transforms** representing **odd functions** is defined by the equations

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} g_s(k)\sin kx\, dk \qquad (10.27a)$$

$$g_s(k) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f_s(x)\sin kx\, dx \qquad (10.27b)$$

We can discuss the even functions in a similar way and obtain Fourier cosine transforms.

Fourier Cosine Transforms

The pair of **Fourier cosine transforms** representing **even functions** is defined by the equations

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_c(k) \cos kx \, dk \qquad (10.28a)$$

$$g_c(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(x) \cos kx \, dx \qquad (10.28b)$$

You should verify Eqs. (10.28a and b) before studying further.

SAQ 2

Spend 5 min

Prove Eqs. (10.28a and b).

So far we have seen that the evaluation of Fourier transforms requires the full knowledge of the given function in the entire interval from $-\infty$ to ∞. We may occasionally come across functions which are defined only on the positive real axis. Can we express such functions in terms of Fourier integrals?

A partial knowledge of the function as in the above case is **not a disadvantage** or a stumbling block. It is, on the contrary, an advantage. We can extend the function on the negative real axis in any manner we choose (subject to the conditions that the Fourier integral theorem is satisfied) and obtain the Fourier integral of the extended function. The procedure is similar to the even and odd periodic Fourier extensions you learnt in Unit 7 of PHE-05. Let us study the same procedure for Fourier transforms.

Fourier transforms for functions defined only over $x > 0$

Let $f(x)$ be the given function, defined only for $x > 0$. We define an auxiliary function $g(x)$ such that

i) $g(x) = f(x)$ for $x > 0$, and
ii) $g(x)$ is chosen arbitrarily for $x < 0$,

subject to the two conditions that the complete function $g(x)$ has no more than a finite number of finite discontinuities and the integral of the modulus of the function over the entire real line is finite. Then we determine the Fourier integral of $g(x)$ and call it $I(x)$, say. It is clear that for values of x on the positive real line, the integral $I(x)$ will also represent the original function $f(x)$. Thus we would have

$$f(x) = I(x), \qquad x > 0 \qquad (10.29)$$

The two most common methods of choosing the extended function $g(x)$ are to take either an **even continuation** or an **odd continuation** of $f(x)$ itself. Mathematically, we obtain it as follows:

$F(x) = f(-x)$ for $x < 0$: **even continuation;** (10.30a)

$F(x) = -f(-x)$ for $x < 0$: **odd continuation.** (10.30b)

Due to this arbitrariness in the choice of $F(x)$, you will notice that a function $f(x)$ which is defined only in the half-interval $x > 0$ can be expressed in terms of a Fourier integral **in two ways.** Let us take an example to illustrate this concept.

Example 3: Obtain the Fourier cosine transform of the function

$$f(x) = e^{-px}, \quad 0 < x < \infty, \quad p > 0. \tag{10.31}$$

Solution

To obtain the Fourier cosine transform of the given function $f(x)$, we define an auxiliary function $F(x)$ such that it is the same as $f(x)$ for $x > 0$, and it is an even continuation of $f(x)$ for $x < 0$ (as in Eq. (10.30a)). In fact, the complete function $F(x)$ will look like that shown in Fig. 10.6.

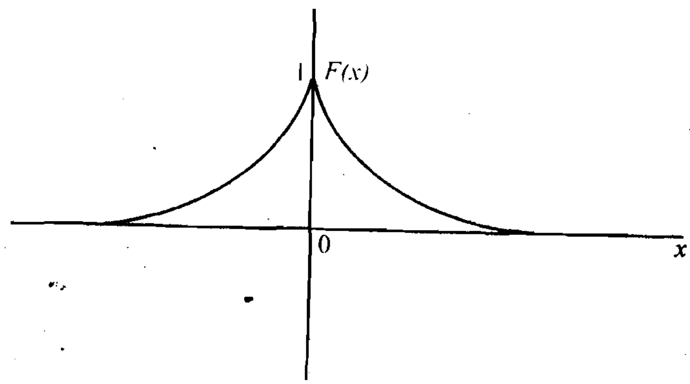

Fig.10.6: Even periodic extension of $f(x)$

This is an even function, so that it will have the pair of Fourier transforms

$$F_c(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_c(k) \cos kx \, dk, \tag{10.32a}$$

and

$$g_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(y) \cos ky \, dy. \tag{10.32b}$$

(Note that these are the same as Eqs. (10.28a and b) with the function f replaced by F). For $y > 0$, $F_c(y) = e^{-py}$ from Eq. (10.31). Substituting this in Eq. (10.32b), we get

$$g_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-py} \cos ky \, dy = \sqrt{\frac{2}{\pi}} \frac{p}{p^2 + k^2}. \tag{10.33}$$

Having thus found $g_c(k)$, we shall substitute it back in Eq. (10.32a) to obtain $F_c(x)$. But since $F_c(x)$ is equal to e^{-px} for $x > 0$, we shall finally have the Fourier cosine integral for this function. You should note that $F(x)$ is continuous at $x = 0$ (there is no discontinuity in $g(x)$). Hence the Fourier integral would be equal to the function e^{-px} not only for $x > 0$ but for $x \geq 0$. Thus

$$e^{-px} = \frac{2p}{\pi} \int_0^\infty \frac{\cos kx \, dk}{p^2 + k^2}, \quad x \geq 0, \quad p > 0. \tag{10.34}$$

You may now like to work out an SAQ on these concepts.

SAQ 3

Obtain the Fourier sine transform of the function $f(x)$ defined in Eq. (10.31) and show that

Spend 10 min

$$e^{-px} = \frac{2}{\pi} \int_0^\infty \frac{k \sin kx \, dk}{p^2 + x^2}, \quad x > 0, \quad p > 0. \tag{10.35}$$

Hint: Choose an auxiliary function $h(x)$ which is the same as $f(x)$ for $x > 0$ and which is given by Eq. (10.30b) for $x < 0$ (odd continuation of $f(x)$). Obtain the Fourier sine transforms by using Eqs. (10.27a and b).

In SAQ 3, what happens at $x = 0$? Why have we excluded $x = 0$ from the range of validity of Eq. (10.35)? To answer this question, you should plot a graph (rough, qualitative) of $h(x)$ defined in this problem. You will find that it looks somewhat like that shown in Fig. 10.7. By definition it is an odd function and has a discontinuity at $x = 0$. The left hand limit of $h(x)$ at $x = 0$ is -1 and the right hand limit is 1, the average of these two being 0. This is the value of the Fourier integral on the right hand side of Eq. (10.35) for $x = 0$.

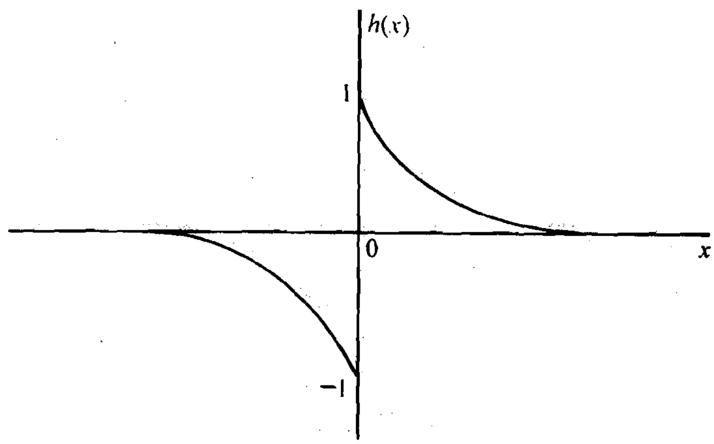

Fig.10.7: The function e^{-px} for $x > 0$ and its odd continuation for $x < 0$, for some (positive) value of p.

The two integrals in Eqs. (10.34) and (10.35) are known as **Laplace integrals**.
We have seen that Eqs. (10.34) and (10.35) are valid for $x > 0$. (Of course, the first one is also valid for $x = 0$). Can we obtain corresponding equations for $x < 0$? Suppose we replace x by $-x$ everywhere in these two equations. The left hand side will then become e^{px} (with $x < 0$, remember). The right hand side will remain unchanged in Eq. (10.34) but will be multiplied by a negative sign in Eq. (10.35) (because $\sin(-kx) = -\sin kx$). You may like to obtain these results yourself.

SAQ 4

Spend 5 min

Obtain the two equations in the manner described above, which will correspond to Eqs. (10.34) and (10.35) but will be valid for $x < 0$.

Let us now summarize what you have studied in this unit.

10.4 SUMMARY

- A **periodic function** $f(x)$ having period T and having a finite number of finite discontinuities in one period can be expanded in a complex Fourier series with complex Fourier coefficients given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi n i x/T}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\, e^{-2\pi n i x/T} dx, \quad n = \text{any integer.}$$

- The **Fourier transform** of an **aperiodic function** $f(x)$ having a finite number of finite discontinuities on the entire real line and which is absolutely integrable between $-\infty$ and ∞ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(k)\, e^{ikx} dk$$

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

- An **even aperiodic function** can be represented by a **Fourier cosine transform**:

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_c(k) \cos kx \, dk$$

$$g_c(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(x) \cos kx \, dx$$

- An **odd aperiodic function** can be represented by a **Fourier sine transform**:

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_s(k) \sin kx \, dk$$

$$g_s(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(x) \sin kx \, dx$$

- At a point of discontinuity in the function, its Fourier integral has a value which is the average of the left hand limit and the right hand limit of the function at that point.

10.5 TERMINAL QUESTIONS *Spend 30 min*

1. Show that $f(x) = 1, x > 0$ cannot be represented by a Fourier integral.

2. Determine the Fourier cosine integral representation of the function $f(x)$ which equals p (a positive constant) for $0 < x < a$ and 0 for $x > a$.

3. Obtain the Fourier sine and cosine transforms of the following functions as indicated:

(a) $f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ 0 & x > \pi/2 \end{cases}$, sine and cosine transform

(b) $f(x) = \begin{cases} \sin x & |x| < \pi/2 \\ 0 & |x| > \pi/2 \end{cases}$, sine transform

(c) $f(x) = \begin{cases} \cos x & -\pi/2 < x < \pi/2 \\ 0 & |x| > \pi/2 \end{cases}$, cosine transform

10.6 SOLUTIONS AND ANSWERS

Self-assessment Questions

1. (a) $$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$= \frac{1}{2\pi} \int_0^{\infty} \exp(-\lambda t) e^{-i\omega t} dt$$

$$= \frac{1}{2\pi} \int_0^{\infty} \exp[-(\lambda + i\omega) t] \, dt$$

$$= \frac{1}{2\pi} \frac{1}{\lambda + i\omega} = \frac{1}{2\pi} \frac{\lambda - i\omega}{\lambda^2 + \omega^2}$$

(b) $$g(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$= \frac{1}{2\pi} \int_0^1 x\, e^{-ikx} dx$$

Integrating by parts, we obtain

$$g(k) = \frac{1}{2\pi k^2} \left[e^{-ik} (ik + 1) - 1 \right]$$

2. We know that for an even function

$$f(-x) = f(x)$$

If $f(x)$ is even then $g(k)$ is even too. Substituting $e^{-ikx} = \cos kx - i \sin kx$ in Eq. (10.20b) we get

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) (\cos kx - i \sin kx)\, dx$$

Since $f(x)$ is even, the product $f(x) \sin kx$ is odd and the integral over the second term will be zero:

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos kx\, dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(x) \cos kx\, dx$$

which is Eq. (10.28b).

If we now replace k by $-k$, we get

$$g(-k) = g(k)$$

i.e., $g(k)$ is even. Then in the same way we obtain

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) (\cos kx + i \sin kx)\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) \cos kx\, dk$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_c(k) \cos kx\, dk$$

which is Eq. (10.28a). Hence proved.

3. We define the auxiliary function $h(x)$ such that $h(x)$ is an odd extension of $f(x)$:

$$h(x) = f(x) \qquad \text{for} \qquad x > 0$$

and

$$h(x) = -f(-x) \qquad \text{for} \qquad x < 0$$

The function $h(x)$ is an odd function so it will have the Fourier transform:

$$g(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty h(x) \sin kx \, dx$$

Then

$$h(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(k) \sin kx \, dk$$

Now

$$g(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-px} \sin kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{k}{p^2 + k^2}, \quad x > 0, \quad p > 0$$

Therefore

$$h(x) = \frac{2}{\pi} \int_0^\infty \frac{k}{p^2 + k^2} \sin kx \, dk$$

Substituting for $h(x)$, we get

$$e^{-px} = \frac{2}{\pi} \int_0^\infty \frac{k \sin kx \, dk}{p^2 + k^2}, \qquad x > 0, \quad p > 0$$

4. Replacing x by $-x$ in Eqs. (10.34) and (10.35), we get

$$e^{px} = \frac{2p}{\pi} \int_0^\infty \frac{\cos(-kx) \, dk}{p^2 + k^2}, \qquad x < 0, \quad p > 0$$

$$= \frac{2p}{\pi} \int_0^\infty \frac{\cos kx \, dk}{p^2 + k^2}, \qquad x < 0, \quad p > 0$$

$$e^{px} = \frac{2}{\pi} \int_0^\infty \frac{k \sin(-kx) \, dk}{p^2 + k^2} = -\frac{2}{\pi} \int_0^\infty \frac{k \sin kx}{p^2 + k^2} dk$$

Terminal Questions

1. The Fourier integral representation of $f(x) = 1, x > 0$ is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx} dk$$

where

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ikx} dx$$

Thus

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_0^\infty dx \; e^{-ikx} e^{ikx}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx$$

which will diverge for the given limits. Hence $f(x) = 1, x > 0$ cannot be represented by a Fourier integral.

2. $$f(x) = p \quad , \quad 0 < x < a \quad , \quad p > 0$$

$$= 0 \qquad x > a.$$

The Fourier cosine integral representation of $f(x)$ is obtained by substituting Eq. (10.28b) in Eq. (10.28a):

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_c(k) \cos kx\, dk$$

where

$$g_c(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(k) \cos kx\, dk$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{a} p \cos kx\, dk \quad , \quad p > 0$$

$$g_c(k) = -\sqrt{\frac{2}{\pi}} \frac{p \sin ka}{k}$$

Thus

$$f_c(x) = -\frac{2p}{\pi} \int_0^{\infty} \frac{\sin ka}{k} \cos kx\, dk$$

which is the Fourier cosine integral representation of the given $f(x)$.

3. (a) The Fourier sine transform of $f(x)$ is

$$g_s(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(x) \sin kx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\pi/2} \sin kx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{\cos kx}{k}\bigg|_0^{\pi/2}$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\cos k\pi/2}{k} - \frac{1}{k} \right)$$

$$= \frac{1}{k} \sqrt{\frac{2}{\pi}} \left(\cos \frac{k\pi}{2} - 1 \right)$$

The Fourier cosine transform of $f(x)$ is

$$g_c(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(x) \cos kx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\pi/2} \cos kx \, dx = -\sqrt{\frac{2}{\pi}} \frac{\sin kx}{k} \bigg|_0^{\pi/2}$$

$$= -\sqrt{\frac{2}{\pi}} \left(\frac{\sin k\pi/2}{k} \right)$$

(b) The Fourier sine transform of $f(x)$ is

$$g_s(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(x) \sin kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\pi/2} \sin x \sin kx \, dx = \sqrt{\frac{2}{\pi}} \left[\frac{\sin(k-1)\pi/2}{2(k-1)} - \frac{\sin(k+1)\pi/2}{2(k+1)} \right]$$

(c) The Fourier cosine transform of $f(x)$ is

$$g_c(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(x) \cos kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\pi/2} \cos x \cos kx \, dx = \sqrt{\frac{2}{\pi}} \left[\frac{\sin(k+1)\pi/2}{2(k+1)} + \frac{\sin(k-1)\pi/2}{2(k-1)} \right]$$