# UNIT 5 MATTER WAVES AND UNCERTAINTY PRINCIPLE

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## 5.1 INTRODUCTION

In Unit 4, you have learnt how quantum physics emerged in order to explain certain experimental results and natural phenomena which could not be accounted for by classical physics. You have also studied the concepts of wave-particle duality and matter waves given by de Broglie. We will explore these concepts further in this unit. You know that waves are spread all over space whereas particles are localised. So a single wave would be inadequate for describing a real particle correctly. The question is: How do we represent matter waves (or wave-particles) in space? For this purpose, we have to introduce the concept of a **wave packet.**

A discussion of wave packets leads us to another fundamental principle of quantum physics, namely, **Heisenberg's uncertainty principle.** You will study some applications of the uncertainty principle, in particular for the microscopic world. The uncertainty principle was met with a great deal of opposition from stalwarts in physics, especially Albert Einstein. The debate amongst physicists, and in particular between Bohr and Einstein about the validity of this principle makes a very interesting reading in the history of quantum mechanics. In this unit we will give you a flavour of some ideal (thought) experiments which provided support for the uncertainty principle and ultimately established it as one of the fundamental principles of quantum mechanics. In the next unit, you will study the Schrödinger equation which is a major pillar of quantum mechanics.

### Objectives

After studying this unit you should be able to

- explain the concept of a wave packet,
- derive the relation between phase velocity and particle velocity,
- apply Heisenberg's uncertainty principle to microscopic systems,
- discuss the y-ray microscope, single slit and double slit interference experiments in support of the uncertainty principle.

## 5.2 MATTER WAVES

From the discussion in Sec. 4.3.2 of Unit 4, you know that classically, a particle can be localised at a single point but a wave cannot. Thus, at least for the microscopic particles for which the wave-particle duality is significant, we are forced to abandon the classical description of a particle. We have to look for a new description which should be consistent with the de Broglie hypothesis. What should this new description of matter waves associated with particles be like? Well, for one, matter waves should always be associated spatially with the particle in such a manner that the resultant amplitude is non-zero only in the neighbourhood of the particle:

Now the de Broglie equation (4.11) yields

$$\frac{\Delta\lambda}{\lambda} = -\frac{\Delta p}{p} \tag{5.1}$$

This equation shows that if $\Delta p = 0$ then $\Delta\lambda = 0$. That is, we are required to represent a *particle* of definite momentum with a single wave *of fixed* wavelength. However, you know that a wave of single wavelength and frequency is spread out in time and space. Thus, it cannot be localised and cannot represent a particle. Can we find an intermediate solution? If we associate some uncertainty with the momentum, **i.e.,** if we take $\Delta p > 0$, then $\Delta\lambda$ is also finite. The following discussion shows that if a finite spread is allowed in the wavelength, we can fruitfully exploit this to represent a microscopic particle.

Consider a simple situation in which two sinusoidal waves with slightly different wavelengths are superimposed. You have studied in the course **PHE-02** (Oscillations and Waves) that the character of the **resultant** wave is quite different from the individual waves. For example, suppose we consider two travelling waves represented by

$$\psi_1 = A \sin (kx - \omega t)$$

and $$\psi_2 = A \sin [(k + dk)x - (\omega + d\omega)t]$$

where A is their amplitude, k $(=2\pi/\lambda)$ is the wave number and $\omega$ $(=2\pi\nu)$ is the angular frequency. The superposition of these waves yields a resultant wave given by

**Using the result**

$\sin\theta + \sin\phi =$

$2\cos\left(\frac{\phi-\theta}{2}\right)\sin\left(\frac{\phi+\theta}{2}\right)$

**we get the value of $\psi$.**

$$\psi = \psi_1 + \psi_2$$

$$= 2A \left[\cos\left(\frac{dk}{2}\right)x - \left(\frac{d\omega}{2}\right)t\right] \sin (kx - \omega t)$$

**where** we have ignored $\frac{dk}{2}$ and $\frac{d\omega}{2}$ in the sine term as they are infinitesimal compared to k and $\omega$. Fig. 5.1 shows a graph of $\psi$. You can see that $\psi$ has an envelope equal to $2A\cos\left(\frac{dk}{2}x - \frac{d\omega}{2}t\right)$ modulating the sine wave given by $\sin (kx - \omega t)$.

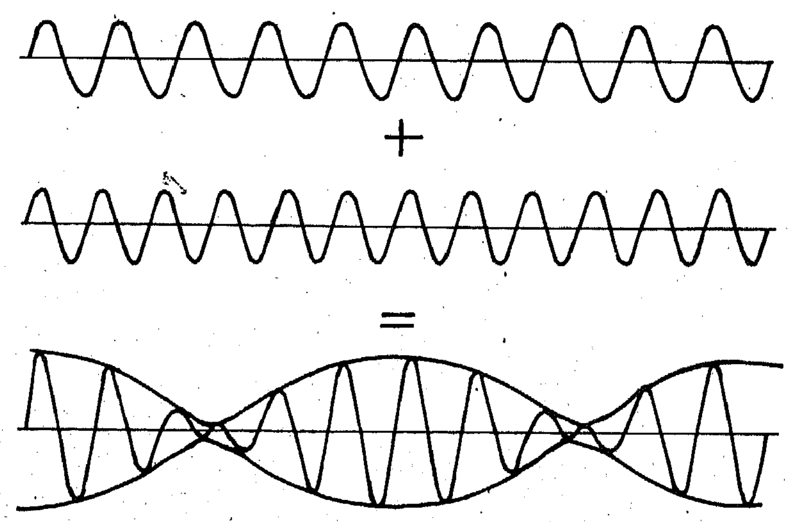

**Fig. 5.1 : A sketch of the resultant of the superposition of two travelling waves.**

Similarly, by superimposing a very large number of waves **having wavelengths** close to a central wavelength $\lambda_0$, a wave packet such as shown in Fig. 5.2 can be constructed. The superposition of these waves results in a variation in amplitude that defines the shape of the wave packet. The wave packet has regular spacing between successive maxima or minima, equal to the central wavelength $\lambda_0$. Thus the wavelength of the wave packet is $\lambda_0$ but at any instant of time it is localized in a finite region of space. Clearly, such a wave packet exhibits both wave and particle aspects,

Thus, in this new representation, a microscopic particle may be represented by a wave packet. To sum up, we may define a wave packet as follows:

> **A wave packet** is a group of'waves with slightly different wavelengths and frequencies interfering with one another in such a manner that the amplitude of the group (i.e., the envelope) is non-zero only in the neighbourhood of the particle.

Fig. 5.2 : A wave packet.

The spread of a wavepacket in wavelength (and in frequency) depends on the required degree of localization in space (and time). You should note that the central wavelength $\lambda_0$ is given by the de Broglie equation (4.11).

How do we determine the velocity of a wave packet? Clearly, if the velocities of the individual waves being superimposed are the same, the velocity with which the wave packet travels is the common wave velocity. However, in the case of de Broglie waves, the wave velocity varies with wavelength; the individual waves do not travel at the same velocity. Thus, the wave packet has a different velocity from the waves that compose it. Let us now determine the **phase velocity** $v_p$ and the **group velocity** $v_g$ of the wave packet.

You know that the phase velocity $v_p$ of a wave is given by $v_p = \frac{\omega}{k}$. Hence from Eqs, (4.2 and 4.11) of Unit 4, the phase velocity of the wave packet is given by

$$v_p = \frac{\omega}{k} = \frac{E}{p}$$

Putting $E = mc^2$ and p = mv in this equation, we obtain

$$\boxed{\textbf{phase velocity} \quad v_p = \frac{c^2}{v}} \qquad (5.2)$$

Since v < c, it is clear that the phase velocity of a wave packet associated with a particle is greater than that of light. This should not disturb you because no physical quantity like energy, information or signals etc., associated with the wave, travels with the phase velocity. These entities move with the **group velocity** which is given by

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp} \qquad (5.3)$$

Now from the special theory of relativity

$$E^2 = p^2 c^2 + m_0^2 c^4, \qquad (5.4a)$$

$$E = mc^2, \qquad (5.4b)$$

and

$$p = mv \qquad (5.4c)$$

Hence using these three equations we obtain

$$v_g = \frac{dE}{dp} = \frac{pc^2}{E} = \frac{p}{m} = v$$

or

$$\boxed{\textbf{group velocity} \quad v_g = v} \qquad (5.5)$$

Therefore, the group velocity of the wave packet is nothing but the particle's velocity.

Thus far you have learnt that a single wave is not enough to represent a particle. We need to superimpose a group of waves which yields a wave packet travelling at a group velocity equal to the particle's velocity. You have seen that there is an uncertainty **Ap** in the momentum of the wavepacket and a spread A h in its wavelength. Before we proceed further to analyse the implications of this discussion, we would like you to calculate the phase velocity and group velocity of a wave packet.

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***Spend 5 min***

***SAQ* 1**

The energy of a free electron including its rest mass energy is 1 MeV. Calculate the group velocity and the phase velocity of the wave packet associated with the motion of the electron.

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You have studied so far in this section that a moving particle must be regarded as a wave packet which satisfies the de Broglie relation. We construct a localised wave packet by superimposed waves which leads us to an uncertainty in its momentum and wavelength.

The fact that a moving particle must be represented by a wave packet rather than a localised entity suggests that there is a fundamental limit to the accuracy with which we can measure the particle's position and momentum. For example, the wider the wave packet is, the greater are the number of waves in it, and the better our chances are to determine the particle's wavelength and hence its momentum. But, because the particle can be anywhere in the wave packet, we cannot determine its position with precision. If, however, the wave packet is narrow, the particle's position is better defined, but now its wavelength (or its momentum) is difficult to determine. So the smaller is the uncertainty *Ax* in the particle's position, the larger becomes the uncertainty **Ap** in its momentum, and vice versa,

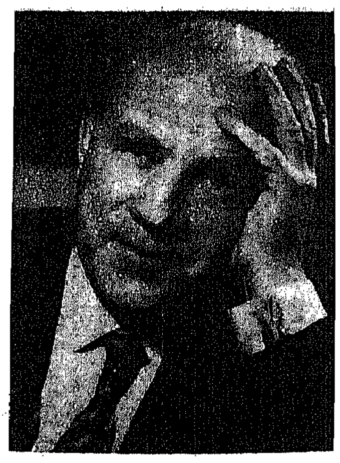

Fig. 5.3 : Werner **Heisenberg, 1901-1976, was** a **German** theoretical physicist. He **was** one of **the** founders of quantum mechanics, and received **the** Nobel **Prize in** 1932.

Thus we can say that ***a direct consequence of the wave-particle duality is the appearance of uncertainties (spreads) in the momentum and the position of a particle.*** If one of them becomes definite, the other becomes completely indefinita. This situation is in sharp contrast to that of classical mechanics according to which it is possible to determine precisely the position and the momentum of a particle at any time ***t***. In 1927, Heisenberg (Fig. 5.3) advanced the above concept in the form of the **uncertainty** principle.

## 5.3 THE UNCERTAINTY PRINCIPLE

Heisenberg discovered that the product of the position and momentum uncertainties of a quantum object such as the wave packet is greater than or equal to Planck's constant *h*. Thus, according to Heisenberg's uncertainty principle

$$\Delta x \, \Delta p_x \geq \hbar \tag{5.6a}$$

where **Ax** and $\Delta p_x$ are the uncertainties in the **x** component of the position and momentum of a microscopic particle, respectively and A = $h/2\pi$.

Similar relationships hold for the y and z components of the positions with their respective momenta of the object. However, you should note that the Heisenberg uncertainty principle does not impose restriction on the simultaneous and precise measurements of **x,** y and $p_z$ or y, $p_x$ and $p_z$ etc. The restrictions are only on what are called as conjugate variables, **i.e., x** along with $p_x$, *y* along with $p_y$ and *z* along **with** $p_z$. Thus, we have

$$\Delta y \, \Delta p_y \geq \hbar, \tag{5.6b}$$

$\Delta z A p, \geq A,$ (5.6c)

and $\Delta r \Delta p_r \geq \hbar$ (5.6d)

A general statement of Heisenberg's uncertainty principle can be given as follows:

> **The Uncertainty Principle**
>
> The values of two (canonically conjugate) variables cannot be simultaneously measured with infinite accuracy (zero error) for a microscopic particle. The product of uncertainties in the simultaneous measurement of conjugate variables always has a value above a certain minimum (which is approximately equal to Planck's constant).

You should note that the uncertainty relation $\Delta x \Delta p_x$ 2 $A$ has been obtained purely as a mathematical property of a wave packet. Hence this relation is as fundamental as wave-particle duality. Like wave-particle duality, the uncertainty principle, though universally applicable is of significance only for microscopic systems.

According to Eq. (5.6a), for a microscopic system there cannot be a state in which $p_x$ as well as $x$ have definite values. We can never simultaneously ascertain values of both position and momentum with arbitrary accuracy. If the position of the microscopic particle is defined (measured) precisely then the uncertainty in its momentum will be infinite, i.e., we will not have any idea of what its momentum is. Similarly, if we are able to precisely measure the particle's momentum, we will have no knowledge of its position. Thus, for instance, in quantum mechanics, a microscopic particle's motion cannot be described by equations like $x = a \sin \omega t$ because it implies definite velocity at a definite position. In other words, the uncertainty principle does not allow the concept of a trajectory. *Thus, unlike classical physics, a definite path of a microscopic particle with definite velocity at every point on the path is not possible in quantum meclzanics.* We will take up this point once again.

**In several text books you will come across the following fornn of the uncertainty principle**

$$\Delta x \Delta p_x \geq \frac{h}{2}$$

**You should keep in mlnd that the right hand side expresses the order of $h$. The lower limit of $h/2$ for A$x$ A$p_x$ is rnrely nttnined; Eq. (5.6a) holds more usually or even $\Delta x \Delta p_x \geq h$ holds.**

Another form of the uncertainty principle finds use in atomic processes. Sometimes we might wish to measure the energy emitted in an atomic process in a time interval *At*. Then we require an uncertainty relation between energy and time. For this we write Eq. (5.6a) as

$$\frac{m\,\Delta x}{p}\,\frac{p\,\Delta p}{m} \geq \hbar$$

The first factor is $\frac{\Delta x}{v}$ or $A\,t$, and since $E = \frac{p^2}{2m}$, $\Delta E = \frac{p\,\Delta p}{m}$.

Thus, we get

$$\boxed{\Delta E \Delta t \geq \hbar} \qquad (5.7a)$$

where *At* is the uncertainty in the time localizability of the wave packet and *AE* is the uncertainty in its energy. A more precise calculation based on the nature of wave packets changes this result to

$$\Delta E \Delta t \geq \frac{\hbar}{2} \qquad (5.7b)$$

Eqs. (5.7a and b) tell us that in *order for a microscopic particle to have a well defined energy state, the state must last for a very long time — it must be stationary,* If the energy state is shortlived, e.g., the excited state of an atom, its energy is uncertain. This is revealed in the width of spectral lines. Suppose the excited atom having life time *At* makes a transition to a lower state. Then according to Eq. (5.7a) the energy (or the 'frequency) of the radiation emitted by the atom is uncertain by an amount $\hbar/\Delta t$. Thus the radiation will not be monochromatic as $\Delta \nu = A\,Elh$, It will contain frequencies between $\nu + \Delta\nu$ and $\nu - \Delta\nu$. And the **line width $\Delta\nu$** of the spectral line, also known as natural width, will be

$$\boxed{\Delta\nu = \frac{1}{2\pi\,\Delta t}} \tag{5.7c}$$

Let us now consider an application of the uncertainty principle.

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Example **1**

Calculate the minimum uncertainty in the momentum of a 4He atom confined to a 0.40 nm region.

**Solution**

We know only that the 4He atom is somewhere in the 0.40 nm region; therefore $\Delta x = 0.40$ nm. Equation (5.7a) gives us $\Delta p_x \geq \hbar/\Delta x$. Using the equal sign to obtain the minimum, we have

$$(\Delta p_x)_{min} = \frac{\hbar}{\Delta x} = \frac{6.626 \times 10^{-34}\ \text{J s}}{2\pi \times 0.40 \times 10^{-9}\ \text{m}} = 2.64 \times 10^{-25}\ \text{kg m s}^{-1}.$$

This example gives us a reasonable picture of what happens to $^4$He atoms at low temperatures if we try to make them stay in one region by solidifying helium, Even at temperatures approaching absolute zero, the $^4$He atoms have considerable momentum. Since $^4$He has a mass of $6.7 \times 10^{-27}$ kg, a momentum spread of $2.64 \times 10^{-25}$ kg m s$^{-1}$ means that at some time the 4He atom probably has a momentum of at least that much, or a speed of at least

$$v = \frac{\Delta p}{m} = \frac{2.64 \times 10^{-25}\ \text{kg m s}^{-1}}{6.7 \times 10^{-27}\ \text{kg}} = 394\ \text{m s}^{-1}$$

which is over 1400 km h$^{-1}$! So even as $T \to 0$ K, this large zero-point motion persists because of the Heisenberg uncertainty principle. The associated kinetic energy is so large that $^4$He will not solidify even as $T \to 0$ K, unless more than 20 atm of external pressure are applied, This pressure pushes the atoms close enough together so that their attractive binding forces will be large enough to hold the solid crystal together.

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You may now like to work out an SAQ.

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*Spend 10 min*

**SAQ 2**

(a) The average life time of an excited atom is about $10^{-8}$s. What is the order of the natural width ($\Delta\nu$) of the line emitted by the atoms?

(b) The radius of an atomic nucleus is typically $5 \times 10^{-15}$m. What is the lower limit of the energy that an electron must have to be in the atomic nucleus?

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You should understand that the (theoretical) limits set by the uncertainty principle have nothing to do with the accuracy of our measuring instruments. Even the most sophisticated instruments shall be limited by the uncertainty principle. This concept was found difficult to accept by many a leading scientist, including Albert Einstein. Hence, a number of thought (ideal) experiments were proposed and debated in the Solvay Congress held at Brussels in 1930 to disprove the above principle but without any success. The analysis of some of the thought experiments illustrates very well the physical implications of the principle. Therefore, we discuss them briefly in the next section.

### 5.3.1 Some Thought Experiments

We will describe here some thought experiments that help us understand the uncertainty principle better. All these attempts reflect a search for ways of violating this principle.

In this direction, they seek to determine the position and momentum of a microscopic particle to an arbitrary accuracy.

Measurement of the **position** of an **electron: The gamma ray microscope**

Let us consider a conceptual experiment first discussed by Heisenberg which attempts to measure the position of electron as accurately as possible. This experiment consists of an arrangement in which an electron is illuminated and its image is observed through a microscope (Fig. 5.4). Electrons travel in a given direction (the positive x-direction) in the form of a well defined monoenergetic beam, i.e., the velocity of the electrons is known exactly. The position of an electron can be located by observing the light (photons) scattered by the electron into the microscope. Clearly, the precision with which the position of an electron can be determined is equal to the resolving power of the microscope. Thus it is equal to the minimum distance by which the microscope can resolve two objects, i.e.,

$$\Delta x = \lambda/\sin\phi$$

where $\lambda$ is the wavelength of the photon used to observe the electron and $\phi$ is the half angle subtentled by the aperture of the microscope at the position of the electron. This result is a standard result from optics. Thus, to obtain as accurate a position measurement as possible, light of short wavelengths must he chosen, such as $\gamma$-rays.

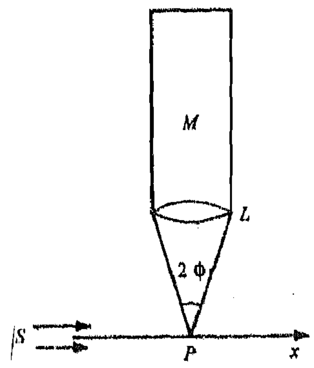


**Fig. 5.4 : Position measurement of electron by Heisenberg's $\gamma$-ray microscope. Photons from a source $S$ are scattered into a microscope $M$ from an electron located at $P$.**

Now in order that an electron be observed, it should scatter at least one photon into the microscope. In the process of scattering the photon would transfer momentum to the electron, causing it to recoil. For instance, if the photon is scattered by 90°, the momentum imparted to the recoiled electron along x-direction would be equal to that of the incident photon which is given by $h/\lambda$. But the photon can be scattered at any angle between 0° and $\phi$. Hence, the x-component of its momentum after scattering can lie anywhere between 0 and p sin $\phi$, where $p$ is its total momentum. Since momentum is conserved, the magnitude of the electron's recoil momentum along x-direction is uncertain by the same or a greater amount, i.e.,

$$\Delta p_x \geq p \sin\phi = \frac{h}{\lambda}\sin\phi$$

Thus the product of the two uncertainties is

$$\Delta x\,\Delta p_x \geq h$$

which is consistent With Heisenberg's uncertainty relation. It is evident that by taking $\lambda$ small enough (i.e., by using y-rays) $\Delta x$ may be made quite small, This, howevar, would increase $\Delta p_x$ such that the product of $\Delta x$ and $\Delta p_x$ is always finite and given by the uncertainty relation.

To gain further insight into the uncertainty relation let us look at one of the most famous of these thought experiments : the *single slit diffraction experiment.*

Single slit diffraction experiment

Consider a highly collimated beam of photons moving along the $x$-direction, such that $p_x = p_0 = h/\lambda$ and $p_y = 0$. Let the beam be incident upon a single slit of width $d$ (Fig, 5.5). The photons are diffracted by the slit and the diffraction pattern is shown in Fig. 5.5.

Since the slit is of finite width $d$, the position of the photons along the $y$-direction is uncertain by an amount $d$, i.e., $\Delta y = d$. What can we say about the component of their momentum in the y-direction?

All we know is that the photon will arrive at the screen somewhere within the diffraction pattern but we don't know where. Thus the uncertainty in momentum is given by the angular spread of the pattern. Since most of the photons hit the screen within the central maximum, we can obtain a rough estimate of the spread in $p_y$ (i.e., $\Delta p_y$) by confining ourselves to the analysis of the central maximum. From Fig. 5.5, you can see that for the central maximum, $p_y$ can take values ranging from $-p_0 \sin\theta$ to $p_0 \sin\theta$. Therefore,

$$\Delta p_y = 2p_0 \sin\theta$$

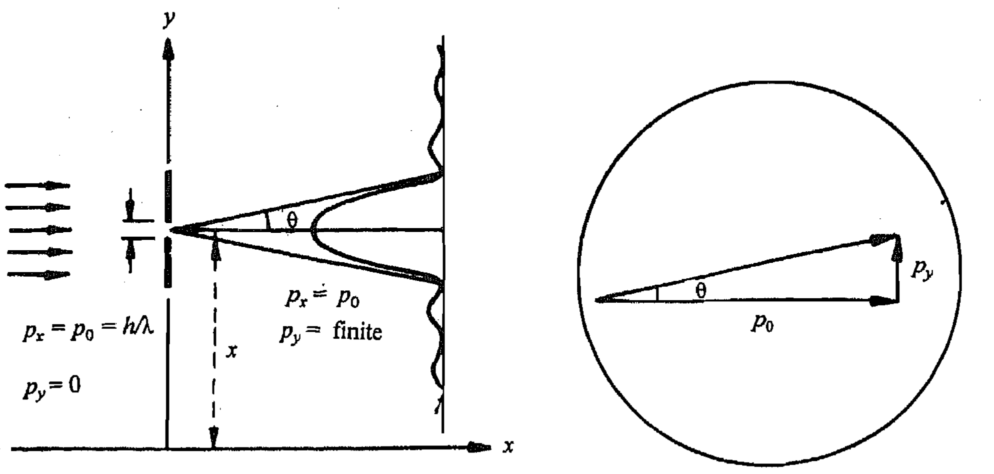


**Fig. 5.5 : Single slit diffraction experiment.**

Now we know from the diffraction theory that the **angular** spread of the pattern is inversely proportional to the width of the slit.

$$\sin\theta = \frac{\lambda}{d} = \frac{\lambda}{\Delta y}$$

where $\lambda$ is the wavelength of incident light. Hence, we obtain

$$\text{Ay}\ \Delta p_y = d\,(2p_0 \sin\theta) = 2p_0\,\lambda$$

or $$\Delta y\ \Delta p_y = 2h \qquad \left(\because \lambda = \frac{h}{p_0}\right)$$

This is consistent with the uncertainty relation **Ay** $\Delta p_y \geq$ A. Trying to reduce the width of the slit (to reduce Ay) leads to a greater spread of diffraction pattern increasing the momentum uncertainty. Thus it is impossible to measure the position and momentum of a microscopic particle precisely at the same time.

Finally, we describe the double slit experiment which is another milestone in establishing the uncertainty principle.

The double slit experiment

In the double-slit experiment, a beam of monoenergetic microscopic particles (such as photons, electrons, protons etc.) are allowed to pass **through** two slits before falling on a fluorosceet screen placed nearby (see Fig. 5.6).

If after some time we plot the total number of particles arriving at the screen as a function of position, we observe an interference pattern. This is a characteristic of waves and can be explained as follows: The matter **waves** corresponding to the particle are split at the two slits and then interfere with one another, But beware of thinking of these matter waves as classical waves, because the particles do arrive at the **fluoroscent** screen in a particle like way: We get one localised flash everytime a particle strikes the screen. However, the totality of spots made by a large number of particles looks like the wave interference pattern. But then, is the wave-like behaviour seen only when we observe a group of particles? What happens when only one particle arrives at the slit?

To answer this question, suppose **we make** the particle beam **very** weak so that at any one instant only one particle arrives at the slit. Do we still get an interference pattern? Quantum mechanics says yes to this and experimental data seems to agree with this viewpoint. It is not easy to accept this picture, **You** may **ask:** Can a single **particle** split, pass through **both** slits and the two halves interfere with one another? Quantum mechanics says yes to all these questions. As Paul **Dirac,** one of the pioneers of quantum mechanics, put it, "Each photon [or a microscopic particle] interferes only with itself'. Why don't we find out whether it is correct by making a measurement?

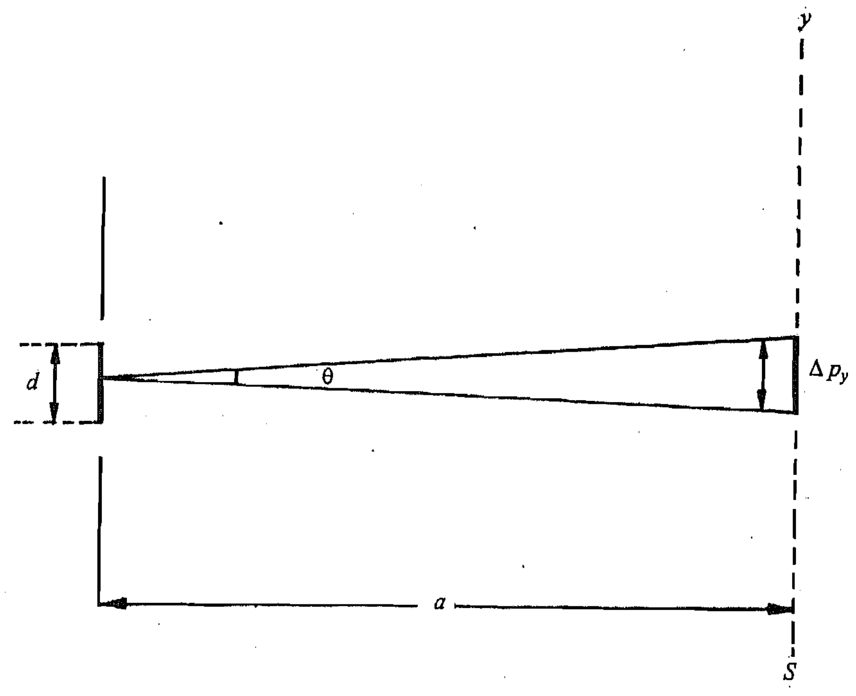


**Fig. 5.6 : The double slit experiment.**

The simplest way of doing this is by means of a thought experiment: Look with a flashlight! We can focus a flashlight on the slits to see which slit the particle is really passing through. What do we obtain? We find that the interference pattern is destroyed. How do we explain this effect? This effect can be explained by the uncertainty principle. As soon as we try to locate the particle and determine the slit (A or B) through which it passed, we lose information about its momentum. As we have seen in the γ-ray microscope experiment, the collision of the particle with the photon we are using to observe it, affects its momentum and introduces an uncertainty in it. Mathematically, to observe whether the particle goes through one of the slits, the photon wavelength must be smaller than at least half the distance d between the slits: ($\Delta y < d/2$) Therefore, its momentum ($=h/\lambda$) must be larger than $2h/d$ (as per the de Broglie relation). The interaction of this photon with the particle will make its momentum uncertain by an amount $\Delta p_y$ given by the uncertainty relation.

This uncertainty in the particle momentum introduces an uncertainty in its position on the screen. As shown in Fig. 5.6 it is given by

$$\frac{\Delta y}{a} = \frac{\Delta p_y}{p_0} \gtrsim \frac{2\hbar}{dp_0} = \frac{\lambda_p}{d\pi} \qquad \left(\because \Delta y\,\Delta p_y \gtrsim \hbar,\ \Delta y < d/2\right)$$

or

$$\Delta y \gtrsim \frac{a\lambda_p}{d\pi}$$

where $\lambda_p$ ($=h/p_0$) is the de Broglie wavelength of the particle. Now the condition for constructive interference is

$$d \sin\theta_n = n\,\lambda_p$$

so that the distance between two adjacent maxima is

$$y_m = a \sin\theta_{n+1} - a \sin\theta_n = \frac{a\lambda_p}{d}$$

Thus

$$\Delta y \gtrsim \frac{y_m}{\pi}$$

In other words, the uncertainty in the position of the electron (produced as a result of attempting to detect it near the slit) is of the order of the distance between the two adjacent maxima. This uncertainty is enough to shift the interference pattern observed at the screen up and down in the y-direction, by a distance roughly equal to the distance between the two maxima. Such a random shift is just enough to smear out the interference pattern so that no interference is observed. So if we attempt to determine the slit through which the particle passes, the interference pattern is destroyed. The fact of the matter is that as soon as we lose information about the particle's momentum, we must also lose information about its wavelength (as per de Broglie's relation). But if there were interference fringes, from their spacing we would be able to measure the wavelength. Thus the fringe pattern cannot exist any more — the interference pattern is destroyed.

**Complementarity Principle**

The point is that the position and momentum measurements are really ***complementary,*** as Bohr first pointed out; they are mutually exclusive processes. This means that we can concentrate on the momentum and measure the wavelength of the particle from the interference pattern and hence its momentum. But then we cannot tell which slit the particle went through. Or we can concentrate on the position and lose information about the wavelength and momentum. You have seen that when we try to find out which slit the particle passes through, we lose the interference pattern and hence, the information about its wavelength and momentum. In his **complementarity principle,** introduced in **1928,** Bohr described this situation by stating that the wave and particle aspects of a physical system are complementary — when we localize (find out which slit the particle goes through), we reveal the particle aspect; and when we don't localize (don't worry aboul which slit the particle goes through), we reveal the wave aspect. However, we cannot reveal both the aspects at the same time — they are complementary.

You may well ask: Is it that the microscopic particles are both wave and particle and we can see only one attribute with a particular experimental arrangement? That is, they possess well defined position and Iinear momentum at each instant but we are unable to measure them simultaneously? Or, is it that the particles just do not possess well defined position and momentum simultaneously'? While Einstein was of the former view which he never gave up, Bohr and Heisenberg took the latter viewpoint. Their interpretation of quantum mechanics is also referred to as the Copenhagen interpretation. Thus, the uncertainty relation propagates the view that these uncertainties arise as a result of an inherent limitation of nature; these are intrinsic to the nature of the quantum world. However precise may be the measuring devices or the method of measurement, there is no escape from these uncertainties. The Heisenberg uncertainty principle is clearly a consequence of wave-particle duality. It reflects the fact that quantum mechanics, although a complete theory, provides a less detailed description of a physical system than does classical physics. This description is governed by the complementarity principle,

The uncertainty principle is a fundamental principle of quantum mechanics. You have noticed the role of Planck's constant — it is so small that the limitations imposed by the uncertainty principle are significant only in the domain of microscopic particles, namely atoms, molecules, subatomic, nuclear and subnuclear particles. On this scale, however, this principle is of great help in understanding many phenomena. Let us now **study** some interesting applications of this principle.

## 5.3.2 Some Applications of the Uncertainty Principle

(a) **The path of an object**

To define the path of a particle in an exact manner we must know its exact position and velocity simultaneously. Such a knowledge is not permitted by the uncertainty principle, Hence the path (or the orbit) of an object in quantum mechanics is not defined. This invalidates Bohr's theory of the hydrogen atom which assigns position and velocity simultaneously to the orbiting electron.

(b) **Tho angular momentum of an object**

The angular momentum L of an object is defined as a cross product of its position

**r** and momentum **p**. Since **r** and **p** are not known simultaneously, **L** is also uncertain. However, as you will learn later, $L^2$ (=**L.L**) can have well defined values.

(c) **The size of an atom**

You can use uncertainty principle even to determine the approximate size of an atom. Let us take the example of the hydrogen atom. A hydrogen atom has a proton and an electron. If we assume the size of the atom to be a then the uncertainty in the position of the electron is about $a$ (the electron is inside the atom). Hence according to Heisenberg uncertainty principle, the uncertainty in the electron's momentum is given by $\Delta p = \hbar/a$. The total (non-relativistic) energy of the electron is equal to

$$E = \frac{p^2}{2m_0} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{a}$$

For a stable atom, E will be minimum. Hence we replace p by Ala and equate $dE/da$ to zero. This yields

$$a = 4\pi\varepsilon_0 \frac{\hbar^2}{m_0 e^2} \simeq 0.5 \text{ Å}$$

and the corresponding value of the energy E is

$$E = -\frac{1}{2}\frac{m_0 e^4}{(4\pi\varepsilon_0)^2 \hbar^2} = -13.6 \text{ eV}$$

The negative sign of the energy shows that the electron is bound to the proton. You will note that these values are in good agreement with the experimental data.

(d) **The** existence **of** electrons **inside the nucleus**

In SAQ 2(b) you have used the uncertainty principle to show that the electrons do not exist inside the nucleus. The size of a nucleus is of the order of one Fermi ($10^{-15}$m). Therefore, if electrons are present inside nucleus, then the maximum uncertainty in their position is $Ax = 10^{-15}$m. Hence Ap will be $A/\Delta x = 10^{-19}$ Js $m^{-1}$. The total energy may be obtained from the relation $E^2 = p^2c^2 + m_0^2c^4$, or $E = pc$ as $m_0c^2$ is much smaller than pc. Thus we obtain

$E = 3 \times 10^{-11}$ J = (311.6) $x$ $10^8$ eV = 200 MeV.

However, experimentally we find that during β-decay of a nucleus, electrons of energies between 2–3 MeV are ejected. Hence we conclude that electrons were not present in the nucleus before the decay.

(e) Zero point **energy**

According to kinetic theory, the kinetic energies of atoms oscillating about their positions in crystals are proportional to the absolute temperature. Hence, at absolute zero, the atoms, according to this theory, would stop oscillating and would remain fixed in their lattice position. But, according to uncertainty relation both position and the momentum cannot be specified at the same time with complete accuracy. This means that the atomic oscillators even at absolute zero would retain a certain amount of oscillatory motion enough to obey the uncertainty relation. The energy possessed by the atomic oscillator at absolute zero is termed as zero **point** energy. Experimental studies of the motion of the atom at a tempernturc (0.001 K), quite close to absolute zero, have shown the reality of the zero-point energy.

You may like to end this section with an SAQ.

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**SAQ 3**

Spend *5 min*

A linear harmonic oscillator of mass m oscillates with a frequency $\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, where k is its force constant. Use the uncertainty principle to show that the minimum energy of the oscillator is $h\nu/2$.

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Let us now summarise what you have studied in this unit.

## 5.4 SUMMARY

- Wave-particle duality and the localization of the particles leads to the representation of **a** particle by a group of waves called a wave packet. The group velocity $v_g$ of the wave packet is equal to the particle velocity v and the phase velocity $v_p$ is given by $c^2/v$.
- The concept of **a** wave packet leads to Heisenberg's uncertainty principle according to which two canonically conjugate variables like $x$ and $p_x$ or E and $t$ cannot be simultaneously determined with perfect accuracy. The product of the uncertainties associated with these variables, **i.e.,** $\Delta x\, \Delta p_x$ and $\Delta E\, \Delta t$ is of the order of the Planck constant ***h:***

$$\Delta x\, \Delta p_x \geq \hbar,$$
$$\Delta E\, \Delta t \geq \hbar$$

- Some of the notable **consequences** of the uncertainty principle are as follows:
  - The path of a particle is not defined in quantum physics.
  - Electrons do not exist inside the nucleus.
  - Atomic oscillators possess a certain amount of energy, known as the zero-point energy, even at absolute zero temperature.
- Several thought experiments, such as **the γ-ray** microscope experiment, the single slit diffraction experiment and the double slit experiment have helped in firmly establishing the validity of the uncertainty principle.

## 5.5 TERMINAL QUESTIONS

**Spend 30 *min***

1. Show that the uncertainty principle can be expressed in the form $\Delta L\, \Delta\theta \geq \hbar$, where $\Delta L$ is the uncertainty in the angular momentum of the particle and $\Delta\theta$ is the uncertainty in its angular position.
2. The radius of a hydrogen atom is $5.3 \times 10^{-11}$ m. Estimate the minimum kinetic energy of the electron in this atom using the uncertainty principle.
3. **An** atom remains in an excited state for $10^{-8}$ s. Calculate the uncertainty in its energy.
4. Consider that a microscopic object is moving along the x-axis and the **uncertainities** in its position are $\Delta x_0$ and $\Delta x$, respectively, at $t = 0$ and $t = t$. Show that $\Delta x$ is directly proportional to $t$ and inversely proportional to $\Delta x_0$. From this problem what do you learn about the spreading of the waves associated with the motion of an object?

## 5.6 SOLUTIONS AND ANSWERS

Self-Assessment Questions

1. Phase velocity $v_p = \dfrac{c^2}{v_g}$

and

$$v_g = \frac{p}{m}$$

where $p = (m^2c^2 - m_0^2c^2)^{1/2} = (m^2 - m_0^2)^{1/2}\, c$

**and** $m = \dfrac{E}{c^2} = \dfrac{10^6 \times 1.6 \times 10^{-19}\ \text{J}}{9 \times 10^{16}\ \text{m}^2\text{s}^{-2}} = 1.778 \times 10^{-30}$ **kg.**

Therefore, $p = [(17.8)^2 - (9.11)^2]^{1/2} \times 10^{-31} \times 3 \times 10^8$ kg m s$^{-1}$

$= 4.58 \times 10^{-22}$ kg m s$^{-1}$

$$\therefore\ v_g = \frac{4.58 \times 10^{-22}\ \text{kg m s}^{-1}}{1.778 \times 10^{-30}\ \text{kg}}$$

$= 2.576 \times 10^8$ m s$^{-1}$

and $$v_p = \frac{9}{2.576} \times 10^8\ \text{m s}^{-1} = 3.5 \times 10^8\ \text{m s}^{-1}$$

2. (a) The order of the natural line width is

$$\Delta \nu = \frac{1}{2\pi\,\Delta t} = \frac{10^8}{2\pi}\ \text{Hz} = 1.6 \times 10^7\ \text{Hz}$$

(b) The uncertainty in the electron's position is

Ax = $5 \times 10^{-15}$ m. Therefore,

$$\Delta p \geq \frac{\hbar}{\Delta x} \geq \frac{6.626 \times 10^{-34}\ \text{Js}}{2\pi \times 5 \times 10^{-15}\ \text{m}} \geq 2.11 \times 10^{-20}\ \text{kg m s}^{-1}$$

The momentum would also be of the same order if this is the uncertainty in it. This suggests that the **K.E.** of the electron is far greater than its rest energy and we can write

K.E. = pc so that

K.E. = pc $\geq (2.11 \times 10^{-20}$ kg ms$^{-1}) \times (3 \times 10^8$ ms$^{-1})$

$\geq 6.33 \times 10^{-12}$ J

$\geq 39$ MeV

Thus the K.E. of an electron must exceed 39 MeV for it to be a nuclear constituent. Experiments indicate that electrons in an atom have only a fraction of this energy. Thus we can conclude that electrons are not present in atomic nuclei.

3. The energy of the linear harmonic oscillator is

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

This is a constant of motion. We can represent the constant value of E by means of averages of the kinetic and potential energies over a cycle of motion by writing

$$E = \frac{<p^2>}{2m} + \frac{1}{2}k<x^2>$$

The average values of x and p should vanish for an oscillating particle. So we can identify $<p^2>$ and $<x^2>$ with the squares of the corresponding uncertainties:

$$<x^2> = <x>^2 + (\Delta x)^2 \equiv (\Delta x)^2$$

and $$<p^2> = <p>^2 + (\Delta p)^2 = (\Delta p)^2 = \left(\frac{\hbar}{2\,\Delta x}\right)^2$$

Thus

$$E = \frac{(\Delta p)^2}{2m} + \frac{1}{2}k\,(\Delta x)^2 = \frac{\hbar^2}{8\,m\,(\Delta x)^2} + \frac{k}{2}(\Delta x)^2$$

since from the uncertainty principle $\Delta x \Delta p \geq \hbar/2$. To determine the minimum energy of the oscillator we put

$$\frac{dE}{d(\Delta x)} = 0$$

or $$-\frac{\hbar^2}{4m\,(\Delta x)^3} + k(\Delta x) = 0$$

or $$(\Delta x)^2 = \left(\frac{\hbar^2}{4mk}\right)^{1/2}.$$

The minimum energy is

$$E_{min} = \frac{\hbar^2}{8m}\left(\frac{4mk}{\hbar^2}\right)^{1/2} + \frac{1}{2}k\left(\frac{\hbar^2}{4mk}\right)^{1/2}$$

$$= \frac{\hbar}{4}\left(\frac{k}{m}\right)^{1/2} + \frac{\hbar}{4}\left(\frac{k}{m}\right)^{1/2} = \frac{\hbar}{2}\left(\frac{k}{m}\right)^{1/2} = \frac{h}{2\,(2\pi)}\left(\frac{k}{m}\right)^{1/2}$$

or $$E_{min} = \frac{h\nu}{2}, \text{ since } \nu = \frac{1}{2\pi}\left(\frac{k}{m}\right)^{1/2}.$$

**Terminal Questions**

1. Consider a particle moving in a circle of radius $r$. If $\Delta x$ is the arc length corresponding to angular position $\Delta\theta$, then we can rewrite Eq. (5.6) as

$$r\Delta\theta\; m\Delta v \geq \hbar$$

or $$\Delta\theta\; mr\Delta v \geq \hbar$$

But $L = mvr$ for the particle and $\Delta L = m\Delta vr$, since $m$ and $r$ are constant. Hence we obtain

$$\Delta L\; \Delta\theta \geq \hbar$$

2. The uncertainty in the electron's position is

$$\Delta x = 5.3 \times 10^{-11}\ \text{m}$$

and

$$\Delta p \geq \frac{\hbar}{\Delta x} = \frac{1.054 \times 10^{-34}\ \text{Js}}{5.3 \times 10^{-11}\ \text{m}} = 1.99 \times 10^{-24}\ \text{kg m s}^{-1}$$

An electron with such a low magnitude of momentum behaves almost like a classical particle [since $\lambda = \frac{h}{p} \simeq 10^{-10}$ m] and its kinetic energy is

$$\text{K.E.} = \frac{p^2}{2m} = \frac{(1.99 \times 10^{-24})^2\ \text{kg}^2\text{m}^2\ \text{s}^{-2}}{2 \times (9.1 \times 10^{-31}\ \text{kg})} = 2.2 \times 10^{-18}\ \text{J} = 13.7\text{eV}$$

3. The energy of the atom is uncertain by an amount

$$\Delta E \geq \frac{\hbar}{\Delta t} = \frac{1.054 \times 10^{-34}\ \text{Js}}{10^{-8}\text{s}} = 1.054 \times 10^{-26}\ \text{J}$$

4. If $v_g$ is the group velocity of the wave packet associated with the microscopic particle then at time t

$$\Delta x = v_g t = \frac{p_0}{m}\, t = \frac{h}{\lambda_0 m}\, t$$

where $\lambda_0$ is the initial wavelength of the wave packet at time $t = 0$. This is equal to $\Delta x_0$, the uncertainty in the particle's position at time $t = 0$. Thus, we have

$$\Delta x = \frac{h}{m} \frac{t}{\Delta x_0}$$

This result tells us that Ax, i.e., the spread of the wave-packet increases with time. The narrower the packet is initially, the quicker it spreads. This is the hidden influence of the uncertainty principle. If the confinement length $\Delta x_0$ is small, the uncertainty in its momentum and hence, its velocity is large $\left( Av \simeq \frac{\hbar}{m \Delta x_0} \right)$. This means that the wave-packet will contain many waves of high velocity much greater than the average group velocity $p_0/m$. Due to the fluctuation in velocity, the distance covered by the particle will also be uncertain by an amount $\Delta x(t)$, i.e., its spread will be large.