UNIT 4 POTENTIAL FOR CONTINUOUS CHARGE DISTRIBUTIONS AND ENERGY

Structure

4.1 INTRODUCTION

In the previous Units, you have calculated the electric field E and potential ϕ due to discrete charge distributions. While calculating ϕ from E, you had to evaluate a line integral. On the other hand, you can also calculate E from ϕ by a simple differentiation. In this Unit, we shall extend these ideas to evaluate ϕ for continuous charge distributions. After computing potential and electric field due to selected continuous charge distributions, you will study equipotential surfaces.

The concept of potential and potential difference have already been introduced in Unit 3. The electrical appliances which we use in our homes work on a potential difference of 220 volts. This concept is also important because physicists do interesting experiments using high voltage sources. If a charged particle is allowed to fall through a potential difference, it accelerates and its kinetic energy increases, Several machines called 'particle accelerators' have been designed to produce high energy charged particles. These high energy charged particles are used in "atom smashing" experiments for studying nuclear structure. You will learn more about this in the nuclear physics course. In this unit, you will also learn the concept of electrostatic energy and the nature of the electrostatic force. These are basic concepts which will help you in understanding not only the nuclear physics course but many other undergraduate level courses.

In the next Block of this course, you will learn about the macroscopic and microscopic properties of the dielectrics in an electric field. There you will come across problems involving potentials and electric fields in dielectrics.

Objectives

After studying this unit you should be able to:

- obtain expressions for potential due to continuous symmetric charge distributions,
- sketch the electric field lines knowing the equipotential surfaces,
- calculate the electrostatic potential energy for a given charge distribution, and
- prove that the electrostatic force is conservative.

4.2 POTENTIAL DUE TO CONTINUOUS CHARGE DISTRIBUTIONS

In the last unit, you have learnt about electrostatic potential and its relation with the electric field E. Both electrostatic potential and electric field are very important quantities. In electrostatics, most of the time, we are interested in calculating either of these two, because knowing any one of them, the other can be determined easily. In this section, we will discuss the evaluation of potential due to infinite line charge and uniformly charged circular disc.

4.2.1 Line Charge

In Unit 2 of this Block, we have calculated the electric field at a point near an infinitely long charged wire (or a line charge). It is given by:

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 r}\,\hat{\mathbf{r}} \qquad \text{...(4.1)}$$

Here, λ is the charge per unit length on the wire; r is the perpendicular distance of the point from the wire, ϵ_0 is the permittivity of free space, and $\hat{\mathbf{r}}$ is a unit vector along the direction of increasing r (Fig. 4.1).

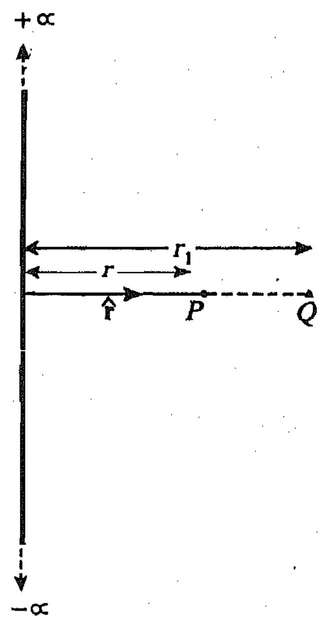


Fig. 4.1: Potential at a point *P* due to an infinitely long charged wire.

The question now is: What is the potential due to this wire at any point P? You have seen in Unit 3 of this block that the negative of the line integral of the electric field between infinity and any point gives the value of the potential at that point, i.e.,

$$\phi_r = -\int_{\infty}^{r} \mathbf{E}\cdot d\mathbf{r} \qquad \text{...(4.2)}$$

We shall evaluate this integral by first taking a finite distance r_1 instead of infinity and then letting r_1 go to infinity. Here r_1 is the distance of the point Q from the wire (see Fig. 4.1). This integral then gives us the difference in potentials between P and Q, i.e.,

$$\phi_r - \phi_{r_1} = -\int_{r_1}^{r} \mathbf{E}\cdot d\mathbf{r}$$

Inserting the expression for E from Eq. (4.1), we get

$$\phi_r - \phi_{r_1} = -\lambda \int_{r_1}^{r} \frac{\hat{\mathbf{r}}\cdot d\mathbf{r}}{r}$$

Since $\hat{\mathbf{r}}$ and $d\mathbf{r}$ are in the same direction, we get

$$\phi_r - \phi_{r_1} = -\frac{\lambda}{2\pi\epsilon_0}\int_{r_1}^{r} \frac{dr}{r}$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r}{r_1}\right) = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_1}{r} \qquad \text{...(4.3)}$$

Let us now try to evaluate the potential with respect to infinity by letting r_1 go to infinity. We notice from Eq. (4.3) that ϕ_r, anywhere in the vicinity of the linear charge distribution (r finite), goes to infinity. This is because the assumption of a uniform and finite charge per unit length over an infinitely long line really amounts to an infinite amount of charge. Therefore, the sum of finite contributions from each part of an infinite amount of charge leads to an infinite potential. However, this does not cause any problem because only the difference in potential enters in practical situations. The choice of infinity for zero potential is only for convenience. It is important to note that only potential differences have any real significance. The absolute value of potential does not have any physical significance. Notice that Eq. (4.3) gives finite values of potential differences for finite distances of r and r_1.

4.2.2 Charged Circular Disc

In the foregoing discussion, we have calculated the potential at a point near an infinitely long charged wire using the value of E and calculating its line integral. The wire is a one-dimensional system. Let us now consider a continuous charge distribution in two-dimensions, namely, the uniformly charged circular disc. What is the potential at any point on the axis of such a disc? We will now calculate the potential at any point on the axis of a uniformly charged circular disc by directly using the general formula for potential due to a point charge derived in the last Unit, i.e.,

$$\phi_r = \frac{q}{4\pi\, \epsilon_0 r} \qquad \text{...(4.4)}$$

For computing potential, we shall divide the disc into a large number of concentric circular strips and then add the potential due to each of them.

Consider a uniformly charged thin circular disc of radius 'a' having a surface charge density σ (charge per unit area) as shown in Fig. 4.2.

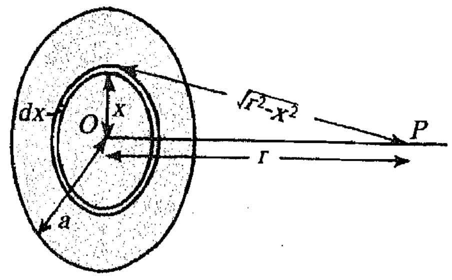


Fig. 4.2: Potential at point P on the axis of a uniformly charged circular disc of radius *a*.

Let us find the potential at some point P lying on its axis. The point P is at a distance r from the centre O of the disc and the line joining P to O is perpendicular to the plane of the disc. For calculating the potential, first consider a narrow circular strip of thickness dx at a distance x from its centre, and write the value of the potential at the point P due to this strip.

Area of the circular strip is equal to the product of the circumference and thickness.

Let the charge on this strip be dQ, where

$$dQ = (2\pi x\, dx)\, a \qquad \text{...(4.5)}$$

Here, in Eq. (4.5), $2\pi x\, dx$ is the area of the strip. Notice, from Fig. 4.3, that all parts of the strip are equidistant from the point P. The charge dQ on this strip can be written as a sum of a large number of point charges, δq_i such that $dQ = \sum_{i=1}^{n} \delta q_i$, n being very large.

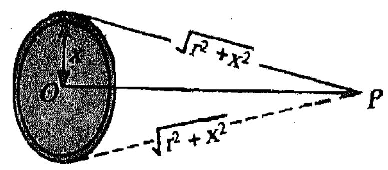


Fig. 4.3: Distances of different parts of strip from point P.

The distance between all the point charges on this strip and the point P is $\sqrt{r^2+x^2}$. The potential $d\phi$ at P due to the charge dQ (i.e., due to the whole strip) using Eq. (4.4), and the principle of superposition is given as:

$$d\phi = \frac{\sum_{i=1}^{n} \delta q_i}{4\pi\, \epsilon_0 \sqrt{r^2 + x^2}} = \frac{dQ}{4\pi\, \epsilon_0 \sqrt{r^2+x^2}}$$

$$= \frac{(2\pi x\, dx)\sigma}{4\pi\, \epsilon_0 \sqrt{r^2 + x^2}} \qquad \text{...(4.6)}$$

As pointed out earlier, the total potential at point P due to the whole disc is obtained by dividing the disc into a very large number of similar but concentric strips and adding their contributions. As this addition contains a large number of terms (infinitesimals), the summation can be replaced by integration. Thus integrating Eq. (4.6) over all the concentric strips, we get the total potential ϕ due to all the charges on the disc as follows:

$$\phi = \int d\phi = \frac{\sigma}{2\epsilon_0} \int_0^a \frac{x}{\sqrt{r^2 + x^2}}\, dx \qquad \text{...(4.7)}$$

Let $x^2 + r^2 = y$; on differentiation, we get $2x\, dx = dy$

$$\therefore \int_b^a \frac{x}{\sqrt{r^2+x^2}}\, dx = \int_{r^2}^{a^2+r^2} \frac{y^{-1/2}}{2}\, dy = \left| y^{1/2} \right|_{r^2}^{a^2+r^2} = \sqrt{a^2 + r^2} - r$$

In Eq. (4.7), a and r are constants for a given charge density (disc) and point P, respectively. The limits of x are from 0 to 'a', as we go from the centre of the disc where $x = 0$, to the edge of the disc where $x =$ a. Carrying out the integration, we obtain

$$\boxed{\phi = \frac{\sigma}{2\epsilon_0}\sqrt{a^2 + r^2} - r} \qquad \text{..(4.8)}$$

Equation (4.8) clearly shows that, for a point at the centre of the disc for which $r = 0$, ϕ reduces to $\phi = \dfrac{a\sigma}{2\epsilon_0}$

For points far off from the centre for which $r >> $ a, the quantity $\sqrt{a^2 + r^2}$ may be approximated using Binomial expansion as follows:

$$\sqrt{a^2 + r^2} = r\left(1 + \frac{a^2}{r^2}\right)^{1/2}$$

$$= r\left(1 + \frac{1}{2}\frac{a^2}{r^2} + \ldots\right)$$

$$\therefore \sqrt{a^2 + r^2} = r + \frac{a^2}{2r} \qquad \text{...(4.9)}$$

Thus, inserting Eq. (4.9) in Eq. (4.8), we get

$$\phi = \frac{\sigma}{2\epsilon_0}\left(r + \frac{a^2}{2r}\right) - r = \frac{\sigma a^2}{4\epsilon_0 r}$$

Multiplying both numerator and denominator by π, we get

$$\phi = \frac{\sigma\, \pi a^2}{4\pi\, \epsilon_0 r} = \frac{q}{4\pi\, \epsilon_0 r} \qquad \text{...(4.10)}$$

Where q $= \sigma\, \pi a^2$ is the total charge on the disc. Compare Eq. (4.10) with Eq. (4.4). You can see that, for points far off on the axis, the disc behaves like a point charge. Further, you may note that the potential ϕ at a point is inversely proportional to r, **i.e.,** as r increases, ϕ decreases.

Do you know why we have considered points only on the axis of symmetry? For points off the axis, the evaluation of definite integral in Eq. (4.7) is complicated and beyond the scope of this course.

The two **SAQs** that follow should convince you that, by using the above ideas, you can solve any related problems.

SAQ 1

From the expression of the potential, **i.e.,** Eq. **(4.8),** obtained above, calculate the value of the electric field near a charged circular disc, Hence, deduce the electric field for an infinite sheet of charge.

SAQ 2

Derive expressions for potentials at points outside and inside of a uniformly charged spherical **shell** (hollow) of negligible thickness. See Fig. 4.4.

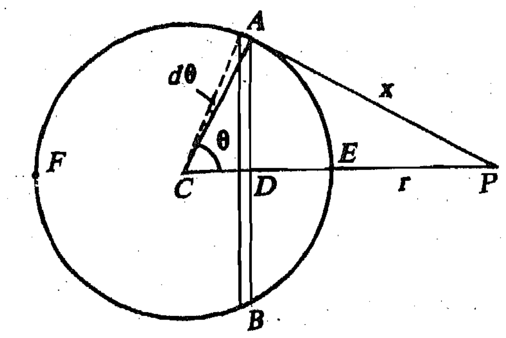


Fig. 4.4: Spherical shell.

[Hint : Divide the shell into a large number of **parallel** rings such as AB shown in Fig. 4.4. The plane of the ring is perpendicular to the line joining the centre C of the shell to the point P where potential is required. The contributions to the potential at P by these **parallel** rings can be summed up by integration. Express the contribution of ring AB in terms of one variable 'x' to do the integration.]

We have so far discussed the electric field and potential which give a **detailed** quantitative description of the electrostatic forces. For a qualitative description, the concepts of lines of force and equipotential surfaces are very useful. These give a geometrical interpretation of the field. In the next **section,** we shall give a description of the equipotential surfaces.

4.3 EQUIPOTENTIAL SURFACES

The locus of all points having the same potential is defined as equipotential surface. For a point charge far away from all other charges, the potential ϕ_r at a distance r is given as

$$\phi_r = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r}\right)$$

From this, you will see that if, on a surface, r is constant, ϕ_r is same everywhere on this surface. Thus the locus of points having the same value of r is a spherical surface (for which r is constant) with the point charge as centre. For a different value of r, we get a different spherical surface. See Fig. **4.5a.** Notice that the electric field lines are everywhere perpendicular to the equipotential surface.

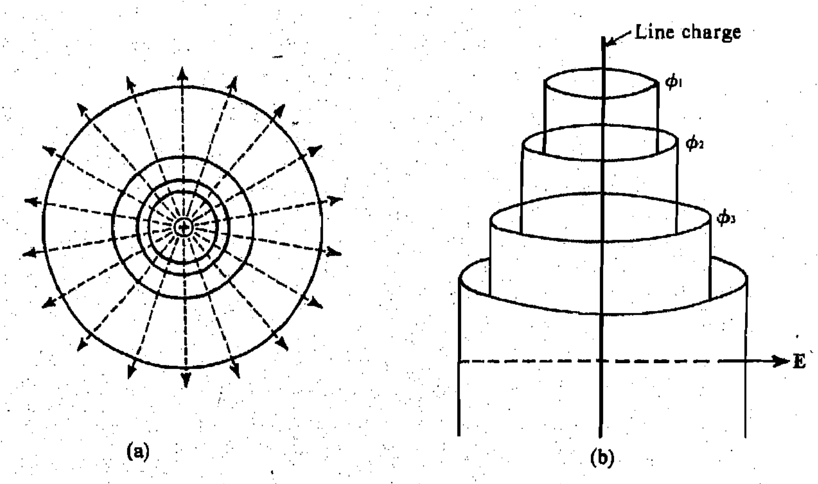


Fig. 4.5 : (a) Equipotential surfaces of a point charge $+q$. The lines of force are radial (dashed). Solid circles are intersections of equipotential surfaces on the plane of paper. (b) The equipotential surfaces (cylindrical surfaces) of a uniform infinite line charge.

For a uniform infinite line charge, the electric potential is the same for points equidistant from the line of charge. Therefore, equipotentials are cylindrical with the line charge as axis of the cylinder. See Fig. 4.5b. Another example of an equipotential surface is a conducting surface. An ideal conducting surface must be an equipotential surface; if any potential difference exists, then charges move from higher to lower potential points until the potential everywhere becomes equal. You will see later in Unit 6 on "capacitors" that this property helps us to compute the field and potential in the space between the plates of a capacitor easily.

Since the equipotential surfaces are "constant potential" surfaces, the potential difference between any two points on them is zero. This implies that the work done in taking a unit charge from any one point to another on' such a surface is also zero.

Using the discussion of Unit 3, you may now say that, since the potential difference between any two points on an equipotential surface is zero, the line integral of the electric field between any two points over such a surface is also equal to zero. See Fig. 4.6. Writing this mathematically, it means that

$$\phi_A - \phi_B = - \int_A^B \mathbf{E} \cdot d\mathbf{r} = 0 \qquad \text{...(4.11)}$$

where ϕ_A and ϕ_B are potentials at A and B respectively.

This is true only when the electric field E and the small displacement vector $d\mathbf{r}$ are perpendicular to each other. Since $d\mathbf{r}$ is an infinitesimal displacement on the equipotential surface, E is at all points perpendicular to such a surface (see 4.6).

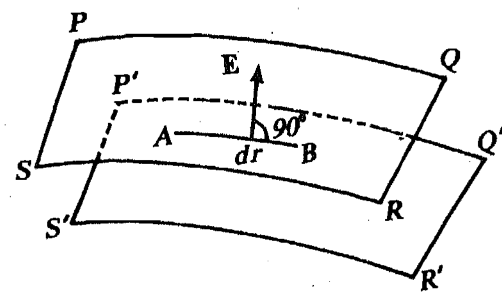


Fig. 4.6: Direction'of electric field vector E relative to equipotential surfaces. *PQRS* and *P'Q'R'S'* are part of equipotential surfaces.

It is for this reason that we have drawn the electric field lines as perpendicular to the equipotential surfaces in Hg. 4.5. For an arbitrary charge distribution, the equipotentials may look like the ones drawn in Fig. 4,7,

Conventionally, the equipotential surfaces are drawn such that there is a constant difference of potential, say A+, between the adjacent surfaces (see Fig, 4.7).

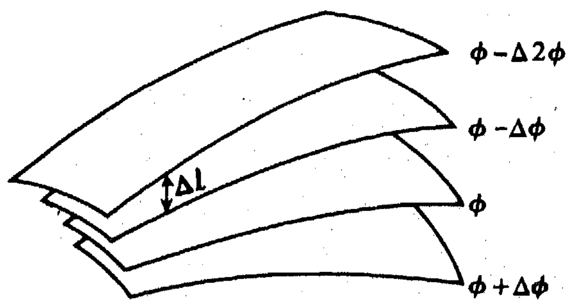


Fig. 4.7: Separation of equipotential surfaces for arbitrary distribution of charges. Portions of four equipotential surfaces arc shorn.

These surfaces may or may not be parallel to each other. They are relatively closer where |E| is large, and are relatively far apart where |E| is small. The reason for this is that the spacing Δl between the equipotential surfaces at any point is given by:

$$\Delta l = \frac{\Delta\phi}{|\mathbf{E}|} \quad ..(4.12)$$

where Δl is along the normal to the equipotentials.

You have seen earlier in Unit 3 that the electric field E and the potential ϕ at a point are related through the relation

$$\mathbf{E} = -\operatorname{grad}\phi \quad ...(4.13)$$

The negative sign in Eq. (4.13), along with the observation that the electric field E is always perpendicular to the equipotential surfaces, indicates that E is always towards the equipotentials of decreasing ϕ. This is elaborated in Fig. 4.8.

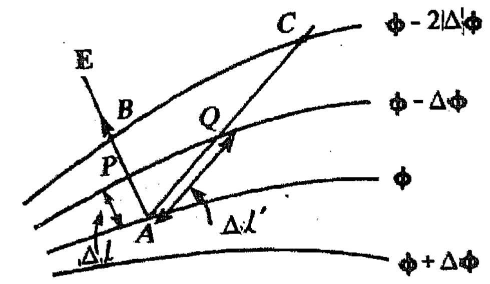


Fig. 4.1: Direction of electric fidd E from equipotential surfaces.

For the equipotentials shown in Fig. 4.8, we find that the resultant electric field vector E is along *AB* because in this direction, the decrease in ϕ is the fastest as determined by the relative ratios

$$\frac{}{\Delta l} \text{ and } \frac{\Delta\phi}{\Delta l'}$$

You may thus remember that the resultant vector E is always along the direction of maximum (or the steepest) decrease of ϕ. Along *AC*, the magnitude of electric field is given by $|\mathbf{E}| \cos\theta$, where θ is the angle between AB and *AC*.

Summing up, you may note from the foregoing discussion that a sketch of the equipotential surfaces gives us a visual picture of both the direction and the magnitude of E in a region of space containing a single charge, a group of charges, or a charge distribution of some particular form (or shape). *So* far, we have described the electrostatic field in terms of electric field vector E, potential and equipotential surfaces. We shall now discuss in the next section the energy d a t e d with assembling of charges, both discrete and continuous. Before moving to the next section, you would like to do an SAQ.

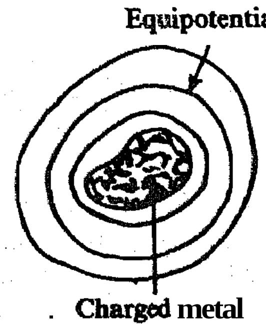


Fig. 4.9

SAQ 3

a) Suppose you are given a sketch of electric field lines due to a group of charges and are asked to draw the equipotential surfaces. List the various points which you will keep in mind while attempting to draw equipotential surfaces.

b) The equipotentials for a charged solid metal object are shown in Fig. 4.9. Draw the electric field lines.

4.4 ELECTROSTATIC POTENTIAL ENERGY

Work done in assembling charges is stored as potential energy of the charges. Suppose, there are two charges q_1 and q_2 which are initially very far apart.

Suppose q_1 is fixed at r_1 and q_2 is brought from infinity to a position r_2 (see Fig. 4.10).

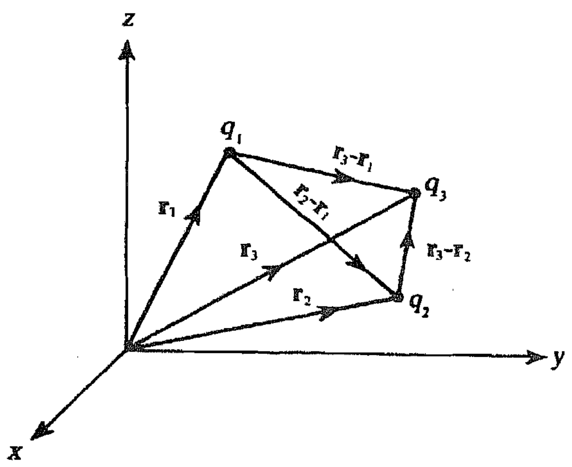


Fig. 4.10: Assemblage of three charges q_1, q_2 and q_3 .

You may ask how much work has been done in bringing q_2 from infinity to a position r_2. From Unit 3, you know that it is equal to the charge q_2 multiplied by the potential at r_2 due to q_1. that is

$$dW = q_2 \frac{1}{4\pi \epsilon_0} \frac{q_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \qquad ..(4.14)$$

This, in effect, is equal to the work done in assembling the two point charges q_1 and q_2 at r_1 and r_2 by bringing them close together. The work done is stored in the system and is usually interpreted as the electrostatic potential energy (or simply as potential energy) of the system of two charges. If, however, these charges are pulled apart such that q_2 is taken back to infinity, an equal amount of energy is supplied back by the system.

You may be wondering as to where this potential energy is stored. Is it at the location r_1 of q_1. or at r_2 of q_2? This energy is neither at r_1 nor at r_2, but in the system as a whole. It is not located at any particular point.

When both the charges are either positive or negative, they repel each other. In that case, work has to be done in assembling the system together. This work m y be recovered by allowing the charges to move apart on account of repulsion. The stored potential energy in that case gets converted into kinetic energy of the charges. When one charge is positive and the other is negative, work has to be done in separating them from each other. In that case, work done can be recovered by allowing the charges to come close together due to attraction.

If the product of the two charges q_1 and q_2 is positive, i.e., their polarities are same, the potential energy is positive. A positive potential energy means that work has to be done to assemble the Like charges' together. If the product $q_1 \times q_2$ is negative, i.e., their polarities are opposite, the potential energy is negative. A negative potential energy means that work has to be done to pull the charges away from each other. It is then easy to say that a positive potential energy corresponds to repulsive electric forces while a negative potential energy corresponds to attractive electric forces.

Imagine what happens when one has to assemble a system of many charges instead of just two. To begin with, start with just three charges q_1, q_2 and q_3 which have to be assembled at positions r_1, r_2 and r_3 as shown in Fig. 4.10. The assembling may be done step by step. First bring q_1 to r_1 and q_2 to r_2. For this, work will have to be done as given in Eq. (4.14). Now bring q_3 to r_3 against the force that q_1 and q_2 exert on it. The work done for this stage is:

$$dW' = q_3 \frac{q_1}{4\pi \epsilon_0 |\mathbf{r}_3 - \mathbf{r}_1|} + q_3 \frac{q_2}{4\pi \epsilon_0 |\mathbf{r}_3 - \mathbf{r}_2|} \qquad ...(4.15)$$

This is because the total force on q_3 is equal to the sum of two individual forces. The total work done including the first stage is then

$$W = dW + dW'$$

$$= q_2 \frac{q_1}{4\pi \epsilon_0 |\mathbf{r}_2 - \mathbf{r}_1|} + q_3 \frac{q_1}{4\pi \epsilon_0 |\mathbf{r}_3 - \mathbf{r}_1|} + q_3 \frac{q_2}{4\pi \epsilon_0 |\mathbf{r}_3 - \mathbf{r}_2|}$$

$$= \sum_{\text{all pairs}} \frac{q_j q_k}{4\pi \epsilon_0 |\mathbf{r}_j - \mathbf{r}_k|} \quad \text{for } j = 1 \text{ to } 3 \text{ and } k = 1 \text{ to } 3 \text{ but } j \neq k$$

$$= \frac{1}{2} \sum_{j=1}^{3} \sum_{\substack{k=1 \\ j \neq k}}^{3} \frac{q_j q_k}{4\pi \epsilon_0 |\mathbf{r}_j - \mathbf{r}_k|} \qquad \text{...(4.16)}$$

The last step in Eq. (4.16) has been written with a factor 1/2 before the summation sign to make sure that the contribution from each pair of charges is **included** only once. For example, for pair q_1 and q_2, we get contribution when $j = 1$ and $k = 2$; and similarly when $k = 1$ and $j = 2$. The factor of 1/2 thus reduces this double contribution to a single contribution. Further, you may note that we have written $j \neq k$ below the second summation sign. This is to avoid the force between a charge with its **ownself.**

Generalising the above discussion for the assemblage of N point charges $q_1, q_2, \ldots, q_N$ at $r_1, r_2, \ldots, r_N$, the expression for electrostatic potential energy (P.E.) may be written as:

$$\text{P.E.} = \frac{1}{2} \sum_{j=1}^{N} \sum_{\substack{k=1 \\ j \neq k}}^{N} \frac{q_j q_k}{4\pi \epsilon_0 |\mathbf{r}_j - \mathbf{r}_k|} \qquad \text{...(4.17)}$$

You may note in Eq. (4.17) that for each value of j (as fixed by the first' summation), the summation on k avoids that value of k which is equal to j. This amounts to considering the potential at charge q_j by **all** the other charges except its own. In terms of potentials ϕ_j at the position of the charge q_j, Eq. (4.17) may be written as

$$\boxed{\text{P.E.} = \frac{1}{2} \sum_{j=1}^{N} q_j \phi_j} \qquad \text{...(4.18)}$$

This **equation** implies that, for calculating the electrostatic potential energy for a group of point charges, one may consider each charge turn by turn, and the corresponding potential at its position due to all other charges except the one in question.

Now suppose, we take a simple example of adding point charges on an isolated conductor gradually in steps, then the work done can be evaluated as follows. Let the charge on the conductor at a given time be q. Then the potential 9 of this charge is proportional to q. The work done δW in adding an additional charge δq on q is then

$$SW = \phi \delta q.$$

We can write ϕ as

$$\phi = kq.$$

where k is the constant of proportionality. Hence

$$\delta W = kq\, \delta q.$$

As we go on adding more and more charges to this conductor, the total work done is stored as potential energy in the charged body. This total work can be found by integration (equivalent to summation). Thus, the potential energy is given as follows (if Q is the final charge on the body):

$$\text{P.E.} = \int \delta W = \int k\, q\, \delta q = k \left[\frac{q^2}{2}\right]_0^Q$$

$$= k \frac{Q^2}{2} = \frac{Q}{2} \phi_f$$

where $\phi_f = k$ Q **is** the final potential of the charged body.

For continuous charge distributions, summation has to be replaced by integration. If in an infinitesimal volume dV, we assemble point charge such that the volume charge density is ρ and the potential is ϕ, then the potential energy may be written as

$$\text{P.E.} = \frac{1}{2} \int_{\text{volume}} \rho\phi \, dV$$

For a charge **distribution on** a surface if σ is the charge per unit area on the element of surface area dS, then

$$\text{P.E.} = \frac{1}{2} \int_{\text{surface}} \sigma\phi \, dS$$

For a line charge distribution, if λ is the charge per unit length, then the potential energy P.E. is

$$\text{P.E.} = \frac{1}{2} \int_{\text{line}} \lambda\phi \, dl$$

Example 1

Three charges are arranged as shown in Fig. 4.11. What is their electrostatic potential energy? Assume $q = 1.0 \times 10^{-5}$ C, and $d = 0.1$m.

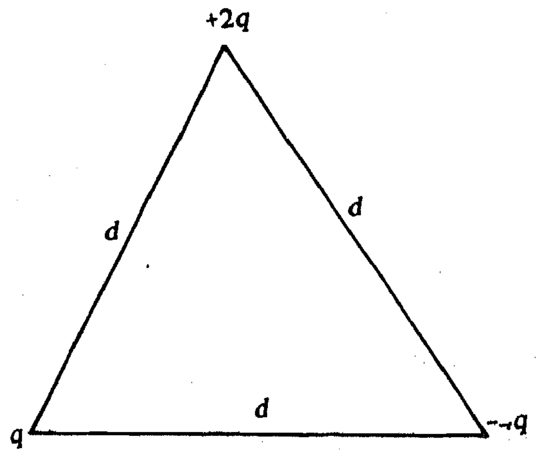


Fig. 4.11: Example 1.

Solution: The total potential energy (P.E.) of the system is the algebraic sum of the potential energies of all pair of **charges, viz.,**

$$\text{P.E.} = \frac{1}{4\pi\epsilon_0}\left[\frac{(+q)(-4q)}{d} + \frac{(+q)(+2q)}{d} + \frac{(-4q)(+2q)}{d}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{-10q^2}{d}\right]$$

$$= \frac{-(9.0 \times 10^9 \text{ Nm}^2\text{C}^{-2})(10) \times (1.0 \times 10^{-5} \text{ C})^2}{0.1\text{m}}$$

$$= -90\text{J}$$

SAQ 4

After drawing a diagram, estimate the number of terms that will contribute to the electrostatic potential energy for a system of five point charges.

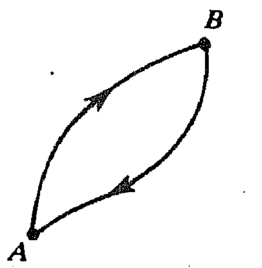
So far, we have discussed the electrostatic potential, equipotential surfaces and electrostatic potential energy. We now discuss the nature of the **electrostatic force.**

4.5 NATURE OF ELECTROSTATIC FORCE

You have seen in Unit 3 that the work W done in moving a charge q from A to B in the region of the electric field E is written as

$$W = -\int_A^B \mathbf{F} \cdot d\mathbf{r} = -q\int_A^B \mathbf{E} \cdot d\mathbf{r} \qquad \ldots(4.20)$$

where $\mathbf{F}$ is the electrostatic force on q. You have also seen in the same unit that the line integral of the electric field, i.e., $\int_A^B \mathbf{E} \cdot d\mathbf{r}$, is independent of the path between A and B. This implies that the line integral of the electrostatic force, viz., $\int_A^B \mathbf{F} \cdot d\mathbf{r}$, is also independent of the path between A and B. That is the work done on a charged particle in moving it against the electrostatic force $\mathbf{F}$ is independent of the path between A and B, and depends only on the end points A and B. This also implies that work done in taking a path around a closed loop (Fig. 4.12) is zero. If this was not so, then one can find a loop, traversing which yields negative work, i.e., energy to us. Thus, one could recover any amount of energy by going around this. That this does not happen is related to conservation of energy. Thus, path independence of work done in an electrostatic field and concept of potential are essentially related to the fact of energy conservation. Hence, the electrostatic force is conservative just as gravitational force. The conservation of energy holds good for a conservative force.



Ng. 4.12: Work done around this loop is zero.

4.6 METHOD OF IMAGES

The concept of equipotential surface is useful in solving a few problems involving charges and conducting surfaces. This is best illustrated in the method of images. We shall discuss these while evaluating the force on a charge Q placed in front of an infinitely large grounded conducting plate. By grounding, the conducting plate is kept at zero potential.

A charge Q placed at a distance r from an infinitely large grounded conducting plate experiences a force because of the induced charges on the conducting plate. See Fig. 4.13. You may ask a question that why does induced charges appear on the conducting surface facing charge Q? These induced charges on conducting plate are a must for ensuring the absence of electric field inside the conductor. These induced charges are negative if the charge Q is positive.

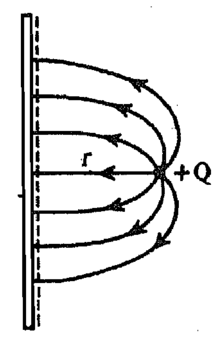


Fig. 4.13: Induced charges on a grounded conducting plate due to a point charge Q.

To evaluate the force on Q, we must know the distribution of induced charges on the conducting plate. Once we know the charge density of the induced charges on the conducting plate, we can use Coulomb's law for finding the force on Q. However, we can arrive at the solution in a very easy manner. Suppose, we place an equal and opposite charge $-Q$ on the other side of the conducting plate as shown in Fig. 4.14 at an equal distance r from the place. This charge $-Q$ is like the mirror image of Q produced by regarding the conducting plate as a mirror. Since every point on the conducting plate is equidistant from the two charges, it is an equipotential plane. As far as the field produced on the right side of the conducting plate is concerned, the field produced by this dipole and the earlier field produced by the point charge Q and the induced charges on the plate is identical.

As seen by the point charge Q the induced charges of the metal plate produce

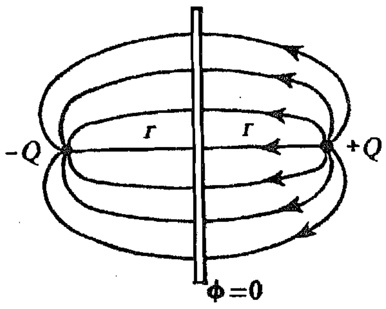


Fig. 4.14: The field of a dipole resulting in an identical field to the right of the conducting plate.

exactly the same field as would a point charge $-Q$ placed at a distance 2r away. from Q. Hence, the force between conducting plate and charge Q is obtained by applying Coulomb's law between the charges Q and –Q. This is given by :

$$F = \frac{Q^2}{4\pi\epsilon_0 (2r)^2}.$$

This method of images can be used in a number of similar situation. Instead of using the method of images, if we had computed the charge density on the. conducting plate at all points and then used Coulomb's law for evaluating the force, the problem would have been very tedious. You may wonder why we have grounded the conducting plate in the above problem. By grounding, we ensure that the potential ϕ of the conducting plate is kept constant and only the induced charges contribute to the force between the plate and the point charge Q.

Let us now sum up what we have learnt in this unit.

4.7 SUMMARY

- The potential difference between two points at distances r and r_1 from an infinitely long charge wire is given by

  $$\phi_r - \phi_{r_1} = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_1}{r}\right)$$

  where λ is the charge per unit length of the wire.
- The potential ϕ at a point at a distance r on the axis of a charged circular disc of radius 'a' it given by:

  $$\phi = \frac{\sigma}{2\pi\epsilon_0}\left(\sqrt{a^2 + r^2} - r\right)$$

  where σ is the charge per unit area on the disc.
- Equipotential surfaces are surfaces on which the potential at each point is constant.
- The electric field E is always directed perpendicular to an **equipotential** surface. It is always along the direction of the fastest decrease of the electric potential ϕ.
- Equipotential surfaces are close **together** 'in regions of strong electric field and are relatively far apart in **regions** of weak electric field.
- 'The electrostatic potential energy is the energy stored in a system of charges. It

is equal to the amount of work done in assembling the system together by bringing the charges from infinity. This energy is recoverable in the form of kinetic energy of the charges, if the charges move away from each other on account of repulsion.

- The electrostatic potential energy for a group of charges is written as:

$$\text{P.E.} = \frac{1}{2}\sum_{j=1}^{N} q_j\,\phi_j$$

where ϕ_j is the potential at the position of charge q_j due to all the charges except the q_j.

- The electrostatic force is conservative, which is a consequence of the fact that the work done in taking a charge around a closed path is zero.

4.8 TERMINAL QUESTIONS

1) If electric field E equals zero at a given point, must ϕ (potential) equal zero for that point? Give one example to prove your answer.

2) An infinite charged sheet has a surface charge density σ of 1.0 x 10^{-7} Cm^{-2}. How far apart are the equipotential surfaces whose potentials differ by 5.0 volts?

3) Show that the electric potential at a point distant *x* on the axis of a ring of charge of radius a is

$$\phi = \frac{1}{4\pi\epsilon_0}\,\frac{q}{\quad}$$

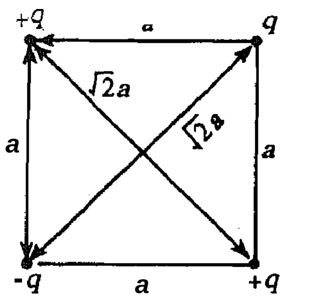


Fig. 4.15

4) Derive an expression for the work required to put the four charges together as indicated in Fig. 4.15.

5) Calculate the gain or loss of electrostatic energy when a droplet of radius R carrying a charge Q splits into two equal sized droplets of charge $Q/2$ and radius R'. Assume droplets are repelled to a large distance compared to R' because of electrostatic repulsion.

6) There are two charged conducting spheres of radii a and b. Suppose, they are connected by a conducting wire? Using the result from this arrangement, explain why charge density on sharp and pointed ends of a conductor is higher than on its flatter portions.

   (Hint: Charges redistribute till potentials are same. Sharper ends have smaller radii, while flatter ends have larger radii.)

7) Devise an arrangement of three point charges, separated by finite distances, that has zero potential energy.

8) Derive expressions for potentials at points outside and inside a uniformly charged non-conducting sphere.

   (Hint: Divide the sphere into concentric shells and use the results of a spherical shell.)

4.9 SOLUTIONS AND ANSWERS

SAQ 1 $\phi = \dfrac{\sigma}{2\epsilon_0}\left[\sqrt{a^2 + r^2} - r\right]$

$$\mathbf{E} = -\,\text{grad}\,\phi = -\hat{r}\,\frac{\partial\phi}{\partial r}$$

$$= -\hat{r}\frac{\partial}{\partial r}\left\{\frac{\sigma}{2\epsilon_0}\sqrt{a^2+r^2}-r\right\}$$

$$= -\hat{r}\frac{\sigma}{2\epsilon_0}\left\{\frac{r}{\sqrt{a^2+r^2}}-1\right\}$$

$$\therefore \mathbf{E} = \hat{r}\frac{\sigma}{2\epsilon_0}\left[1-\frac{r}{\sqrt{a^2+r^2}}\right]$$

By symmetry one can see in the case of a infinitely long charged wire that the electric field is along r. Hence $E = -\hat{r}\frac{\partial\phi}{\partial r}$ equation has to be used. See Appendix given at the end of Unit 3 of this block.

For an infinite sheet of charge the radius 'a' of the disc goes to infinity. Then the electric field is given as

$$E = \hat{r}\frac{\sigma}{2\epsilon_0} \quad \text{as } \frac{r}{\sqrt{a^2+r^2}} \to 0 \text{ when } a \to \infty$$

(This is the same expression as obtained earlier in Unit 2 for E at a point near an infinitely charged plane sheet.)

SAQ 2 For point P outside the shell

The spherical shell of radius 'a' has a uniform charge density σ (charge per unit area). To find the potential at P distant r from the centre C of the shell, we divide the shell into a large number of thin rings such as AB. The axis of this ring is along CP. The contribution to potential at P from this ring of charge is first evaluated. Then the contributions from all other parallel rings are added to evaluate the potential due to the spherical shell. This addition can be done by integration.

The ring AB has a thickness, a $d\theta$ and radius a $\sin\theta$ (which is AD in Fig. 4.4). Thus, the surface area of ring is $2\pi(a\sin\theta)\,a\,d\theta$ and the total charge on this ring is

$(2\pi a^2 \sin\theta\, d\theta)\,\sigma$

All the points of this ring are at the same distance x from P. Therefore, the potential $d\phi$ at P due to this ring of charge is

$$d\phi = \frac{(2\pi a^2 \sin\theta\, d\theta)\sigma}{4\pi\epsilon_0 x}$$

To get the potential ϕ, we integrate this for the whole of the shell. In order to integrate easily, we rewrite it in terms of one variable. Using the geometry, one can see that in triangle ACP

$$x^2 = a^2 + r^2 - 2ar\cos\theta$$

$$2x\,dx = 2ar\sin\theta\, d\theta$$

Thus, substituting for $a\sin\theta\, d\theta$, we get

$$d\phi = \frac{\sigma 2\pi\, a\, x\, dx}{4\pi\epsilon_0 xr} = \frac{\sigma a}{2\epsilon_0}\frac{dx}{r}$$

This expression can be easily integrated over the whole of the spherical shell. The lower limit of x is $(r-a)$ for the ring at E and the upper limit of x is $r+a$ for the ring at F. Thus, for including the contributions from all the rings of the shell, we integrate $d\phi$ and obtain

$$\phi = \frac{\sigma a}{2\epsilon_0 r}\int_{r-a}^{r+a} dx = \frac{\sigma a}{2\epsilon_0 r}\Big[x\Big]_{r-a}^{r+a}$$

$$= \frac{\sigma}{2\epsilon_0}\frac{2a^2}{r} = \sigma\frac{a^2}{\epsilon_0 r}$$

Multiplying the numerator and denominator by 4π, we get

$$\phi = \frac{\sigma 4\pi a^2}{4\pi\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0 r}$$

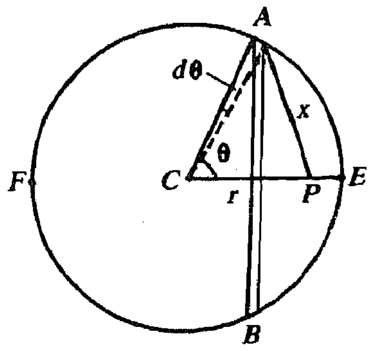


Fig. 4.16: Potential due to a spherical shell at a point P inside the shell.

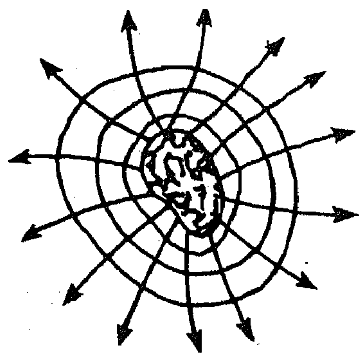


Fig. 4.17

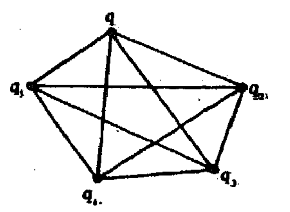


Fig. 4.18

where Q is the total charge on the shell. The expression is the same as that for a point charge Q at C.

For point P inside the shell

When P is inside the spherical shell, the limits of x vary from $(a - r)$, **i.e., *PE*** to $(a+r)$, **i.e.,** *PF*. See Fig. 4.16. Then the potential at P is:

$$\phi = \frac{\sigma a}{2\epsilon_0 r}\int_{a-r}^{a+r} dx = \frac{\sigma a}{2\epsilon_0 r}\Big[x\Big]_{a-r}^{a+r} = \frac{\sigma a}{\epsilon_0}$$

$$= \frac{4\pi a^2}{4\pi \epsilon_0 a} = \frac{Q}{4\pi \epsilon_0 a}$$

Thus, potential is constant inside the shell and is equal to its value at the surface.

SAQ3 a) **i)** Equipotentials are always perpendicular in the electric field lines.

ii) Equipotentials never cross **each** other.

iii) Separation between the **equipotentials** depends on the strength of the electric field.

b) Electric field lines for the charged metal object are shown in Fig. 4.17.

SAQ 4 The diagram is shown in Fig. **4.18.** Since each pair of charge has a potential energy and there are 10 pairs between 5 point charges, 10 terms would be contributing to the potential energy of 5 charges.

Rule : If there are n charges, the number of terms (pairs) contributing to the potential energy is $\frac{n(n-1)}{2}$

Terminal Questions

1) $|\mathbf{E}| = -\frac{d\phi}{dx}$ shows that for $|\mathbf{E}| = 0$, **ϕ has** to **be** a **constant. It** is not necessary that 4 **be equal** to zero when $|\mathbf{E}| = 0$. Consider, for example, two equal charges separated by a distance $2a$. At the mid-point between the charges.

$|\mathbf{E}| = 0$, **but** $\phi = \frac{1}{2\pi \epsilon_0}\frac{q}{a}$.

2) The **electric** field intensity $|\mathbf{E}|$ near **an** infinite charged sheet is **given** by (see Unit 2).

$$|\mathbf{E}| = \frac{q}{2\epsilon_0}.$$

where σ is the surface charge density.
Therefore, in the problem under consideration,

$$|\mathbf{E}| = \frac{1.0 \times 10^{-7}\ \mathrm{Cm^{-2}}}{2 \times 8.9 \times 10^{-12}\ \mathrm{C^2 N^{-1} m^{-2}}}$$

$$= 5.6 \times 10^3\ \mathrm{NC^{-1}}$$

The spacing Δl between the **equipotential** surface is given by

$$\Delta l = \frac{\Delta\phi}{|\mathbf{E}|}$$

where A4 is the potential difference between the adjacent surfaces, With $\Delta\phi = 5.0\mathrm{V}$

$$\Delta l = \frac{5.0\mathrm{V}}{5.6 \times 10^3\ \mathrm{NC^{-1}}} = 0.89 \times 10^{-3}\mathrm{m} = 0.89\mathrm{mm}$$

3) A ring is just like a strip which we have cut-out from a charged circular disc in Sub-section **4.4.2.** Follow similar steps assuming a small portion of the ring as an element of charge.

4) The work required to assemble four charges together as shown in Fig. 4.15 is equal to the potential energy of the system. It may be obtained by considering the charges in pairs. It is

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{(q)(-q)}{a} + \frac{(-q)(q)}{a} + \frac{(-q)(q)}{a} + \frac{(-q)(q)}{a} + \frac{(-q)(-q)}{\sqrt{2}a} + \frac{(q)(q)}{\sqrt{2}a}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{-4q^2}{a} + \frac{2q^2}{\sqrt{2}a}\right]$$

$$= \frac{1}{4\pi\epsilon_0}(-4 + \sqrt{2})\frac{q^2}{a}$$

5) Total volume of 2 droplets after splitting $= 2 \times \frac{4}{3}\pi R'^3$

Volume of original droplet $= \frac{4}{3}\pi R^3$

Since volumes have to be equal

$$2 \times \frac{4}{3}\pi R'^3 = \frac{4}{3}\pi R^3$$

$$\therefore R' = \left(\frac{1}{2}\right)^{1/3} R \qquad \text{...(1)}$$

Electrostatic energy (E.E.) of original droplet with charge Q

$$= \frac{1}{2}Q\phi = \frac{1}{2}Q\frac{Q}{4\pi\epsilon_0 R} = \frac{Q^2}{8\pi\epsilon_0 R} \qquad \text{...(2)}$$

Total electrostatic energy of 2 droplets after splitting

$$= 2 \times \frac{1}{2}\frac{Q}{2}\frac{Q/2}{4\pi\epsilon_0 R'} = \frac{Q^2/2}{8\pi\epsilon_0 R'}$$

Using Eq. **(1)**, E.E. $= \frac{Q^2}{8\pi\epsilon_0 R}\left(\frac{1}{2}\right)^{2/3}$ ...(3)

$\therefore$ Loss in electrostatic energy after splitting $= \frac{Q^2}{8\pi\epsilon_0 R}\left[1 - \frac{1}{(2)^{2/3}}\right]$

6) When two charged conducting spheres are connected by a wire as shown in Fig. 4.19, the charges redistribute themselves till both spheres are at the same potential, **i.e.,**

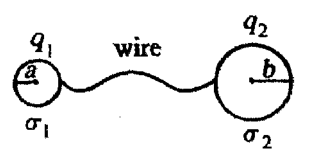


Fig. 4.19

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{q_1}{a} = \frac{1}{4\pi\epsilon_0}\frac{q_2}{b}$$

where q_1 and q_2 are charges on spheres of radii a and b respectively. This gives

$$\frac{q_1}{q_2} = \frac{a}{b} \qquad \text{...(1)}$$

The surface charge densities σ_1 and σ_2 on these spheres are:

$$\sigma_1 = \frac{q_1}{4\pi a^2} \text{ and } \sigma_2 = \frac{q_2}{4\pi b^2}$$

Dividing one by the other we get

$$\frac{\sigma_1}{\sigma_2} = \frac{q_1}{q_2}\cdot\frac{b^2}{a^2} \qquad \text{...(2)}$$

Combining **Eqs.** (1) and (2), we get

$$\frac{\sigma_1}{\sigma_2} = \frac{a}{b} \times \frac{b^2}{a^2} = \frac{b}{a}$$

That is, the surface charge densities are inversely proportional to their radii.

For sharp and pointed ends, the radii are small, resulting in high surface charge densities. For flatter ends, the radii are larger. These result in low surface charge densities. See Fig. 4.20.

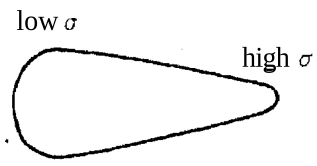


Fig. 4.20

7) If we devise an arrangement as shown in Fig. 4.21, the potential energy (P.E.) turns out to be zero

$$\text{P.E.} = \frac{1}{4\pi\epsilon_0}\left[\frac{(-q)\,q}{2a} + \frac{(-q)\,(q)}{2a} + \frac{(q)\,(q)}{a}\right] = 0$$

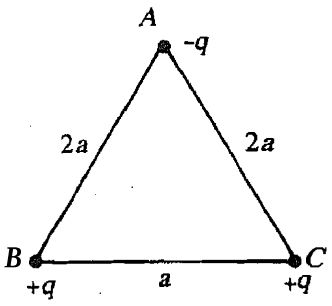


Fig. 4.21

8) **Potential due to a non-conducting sphere (uniformly charged)**

Let ρ be the charge density (charge per unit volume) of the uniformly charged non-conducting **sphere. The** radius of the sphere is equal to **'a'.** See Fig. 4.22.

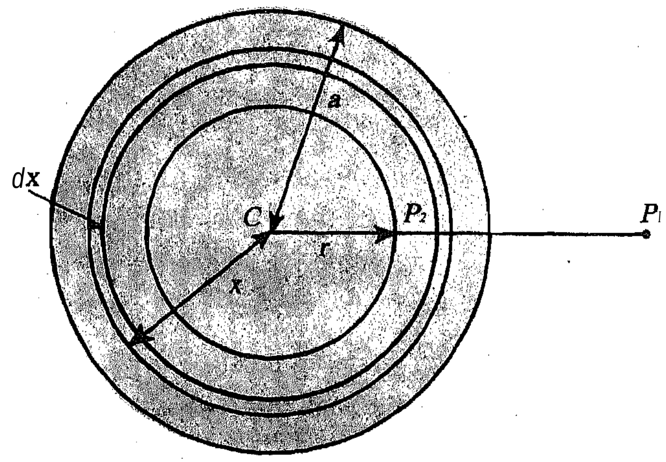


Fig. 4.22: Potential due to a non-conducting sphere.

For points outside the sphere

For points outside the sphere such as P_1 (distance r from the centre C), the whole charge behaves like a point charge at the centre. This can easily be deduced from the derivation of the potential due to a spherical shell. We can divide the non-conducting sphere into a large number of thin concentric shells. For each of these shells, the charge can be regarded as concentrated at the centre C for points outside the shells. Thus, the whole charge can be regarded as a point charge at the centre C. Hence, for points outside the sphere, the potential ϕ is given as

$$\phi = \frac{Q}{4\pi\epsilon_0 r}$$

where Q is the total charge and r is the distance from the centre.

For points inside the sphere

Let the point be at a distance r from the centre C (P_2 in the Fig. 4.22). If we divide the **sphere** into a large number of concentric shells with centre C as before, then for shells with radii $\leq$ r, the point P_2 **is** outside and for shells which have radii between r and a, the point P_2 **is** inside. For shells with radii less than or equal to r potential ϕ_1 at P_2 is given as:

$$\phi_1 = \frac{4\pi}{3}\frac{r^3\rho}{4\pi\epsilon_0 r} = \frac{\rho r^2}{3\epsilon_0}$$

To evaluate the contribution to potential by shells for which P_2 is inside, consider a shell of radius x and thickness dx. See Fig. 4.22. For this shell, the total charge is equal to volume times charge density, **i.e.,**

$$4\pi x^2\, dx\rho$$

This charge contributes a (constant) potential $d\phi$ at any inside point given by

$$d\phi = \frac{4\pi\, x^2\, dx\, \rho}{4\pi\, \epsilon_0\, x} = \frac{\rho x\, dx}{\epsilon_0}$$

For adding the contributions from all such shells, we integrate this expression for x varying from r to a. This gives the potential ϕ_2 at P_2 due to shells for which P_2 is inside as

$$\phi_2 = \frac{\rho}{\epsilon_0} \int_r^a x\, dx = \frac{\rho}{\epsilon_0} \left(\frac{a^2 - r^2}{2} \right)$$

Thus, the potential at P_2 due to the whole non-conducting sphere is:

$$\phi_1 + \phi_2 = \frac{\rho}{3\epsilon_0} r^2 + \frac{\rho}{\epsilon_0} \left(\frac{a^2 - r^2}{2} \right)$$

$$= \frac{\rho}{3\epsilon_0} \left(\frac{3a^2 - r^2}{2} \right)$$

$$= \frac{4\pi\, a^3}{3\; 4\pi\, \epsilon_0} \left(\frac{3a^2 - r^2}{2a^3} \right)$$

$$= \frac{Q\; (3a^2 - r^2)}{4\pi\, \epsilon_0\; 2a^3}$$

where Q is the total charge on the sphere.