UNIT 5 AN INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

Structure

5.1 INTRODUCTION

In Block 1, you have studied ordinary differential equations (ODEs) and learnt various methods of solving them. You have also obtained some understanding of the process of mathematical modelling and used it to solve some simple real-life problems pertaining to physics.

However, the real world around us throws up an astounding variety of probleins which cannot be solved with the knowledge of ODEs alone. For example, suppose a sitar string is plucked at some point. What is the ensuing sound ? Any sitarist will tell you that the sound depends (among other things) upon where the string is plucked. Now if you want to model the motion of the sitar string, you cannot use the techniques you have studied in Block 1. Similarly, if you heat a casting in a furnace and want to know its temperature distribution at a given time, you need to look for new methods;

In order to solve such real-world problems, we need to study **partial** differential equations. This unit being the first in our study of PDEs, we shall discuss some basic concepts related to them.

As you have studied in Unit 1 of Block 1, PDEs arise in connection with various physical problems when the functions involved depend on two or more variables. Recall that, in the study of ODEs, we had asked you to go through calculus. So while modelling physical systems with ODEs, you were in a position to verify that the function of one variable occurring in any ODE was continuous and differentiable in the domain under consideration.

You ought to know similar concepts about functions of more than one variable before you go on to study PDEs. Therefore, we begin this unit by defining a function of more than one variable, and explaining the concepts of limits, continuity and differentiability for such Functions. You will also learn about partial differentiation in this connection. Then, if you wish to go into greater details about these concepts, you may refer to Units 1 to 8 of Blocks 1 and 2 of MTE-07, the mathematics course entitled 'Advanced Calculus'.

Once you have learnt these basic concepts, we will introduce you to PDEs, You will see how PDEs arise in physical problems, and learn to classify them in various ways as you did for ODEs. You will also learn what is meant by the solution of a PDE.

Having become familiar with these concepts, you will be able to solve PDEs arising in problems of physical interest. This forms the subject of Unit 6,

Objectives

After studying this unit, you should be able to:

- verify that a function of more than one variable is continuous and differentiable
- compute the first and higher order partial derivatives of a function of several variables
- classify partial differential equations by way of order and degree, linearity/nonlinearity, homogeneity/nonhomogeneity
- verify that a function is a solution of a given PDE.

5.2 FUNCTIONS OF MORE THAN ONE VARIABLE

So far in this course you have studied differential equations involving functions of one variable. However, many physical quantities depend on several variables. For example, Fig. 5.1 shows the temperature distribution over India on a particular day of summer. The solid lines join up places where the surface temperature was the same at that time. These lines are called isotherms. You must have seen such pictures shown sometimes in the weather report televised every night in the national newscast. Now suppose we set up a coordinate system with New Delhi as the origin and take the x, y-axes in the East and the North directions, respectively. Then each place in India can be represented by its coordinates (x, y). So the variable T representing the temperature at any place at a given instant is a function of two variables x and y, i.e., $T \equiv T(x, y)$.

Remember that the surface of the earth is not planar, and so this example holds only for a small area on the surface of the earth. We cannot represent the temperature distribution over large areas (for instance, aver the area of China or USA) by such a function $T(x,y)$.

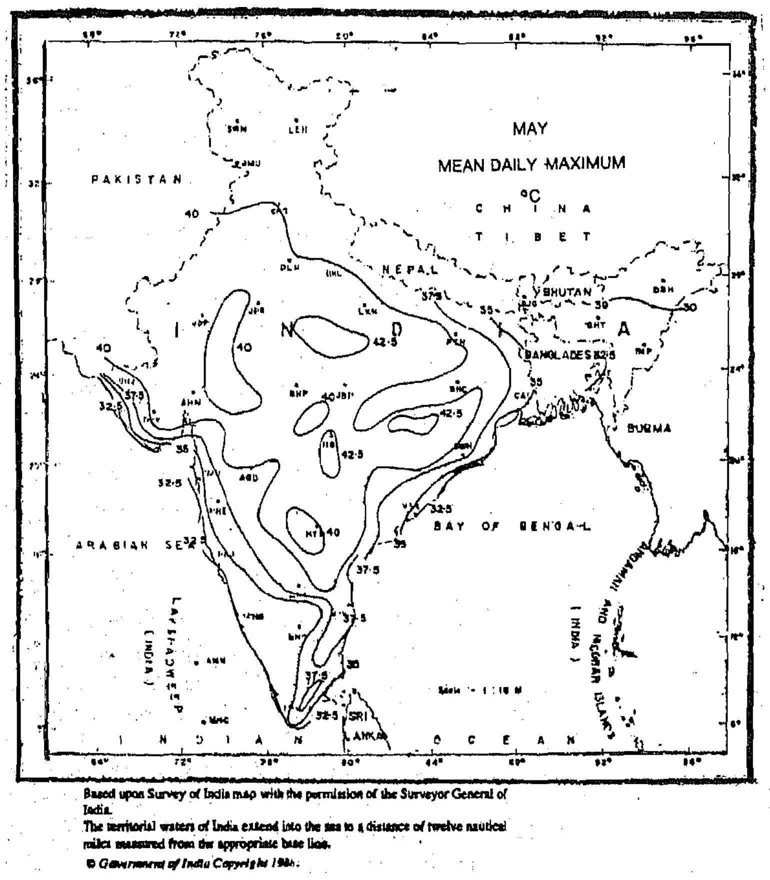


Fig. 5.1 : Temperature distribution at the surface of India on a hot day at a given time

Fig. 5.1 has been reproduced from Agroclimatic Atlas of India (1986), courtesy India Meteorological Department.

Now suppose we have a record of temperature distribution over India at different hours of a day. On how many variables would T depend? In this case T will be a function of x, y and t, i.e., $T \equiv T(x, y, t)$ where t represents the variable time. Once again you can represent $T(x, y, t)$ with the help of isotherms. But now the isotherms will keep changing with time; in this case, the solid lines of Fig. 5.1 would keep wiggling.

Can you now think of some more examples of functions of more than one variable? Remember, you have read about such functions in Unit 2 (Sec. 2.3) of the course Mathematical Methods in Physics-I (PHE-04). You may like to jot down in the margin, some more examples of such functions before you study further.

Our basic purpose in this block is to set up and solve differential equations involving one or more derivatives of functions of several variables, which are continuous and differentiable over a given domain. Therefore, before studying such DEs you should know certain mathematical concepts, such as the limits and continuity, partial derivatives and differentiability of such functions.

5.2.1 Limits and Continuity

In the calculus course you have studied the concepts of limit and continuity of a real-valued function of one variable. Let us now extend these concepts to functions of more than one variable. We will first consider a function of two variables and understand what is meant by its limit.

Recall that a function of one variable is defined as follows:

If A and B are two sets, a function f from A to B is a rule which connects every member of A to a unique member of B. The set A is called the domain and B, the co-domain of f. $f(x)$ denotes that unique element of B which is associated with the element x to A.

If A and B are both subsets of the set of real numbers R, then $f(x)$ is called a real-valued function.

Suppose $f(x, y)$ is a real single valued function of x and y. L is said to be the limit of $f(x, y)$ as the point (x, y) approaches (x_0, y_0), if $f(x, y)$ approaches the value L, as (x, y) approaches (x_0, y_0). It is written as

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L \tag{5.1}$$

Now (x, y) can approach (x_0, y_0) along any one of an infinite number of curves passing through (x_0, y_0). The limit (L) of a function $f(x, y)$ is said to exist, only if the function always approaches the value L, irrespective of the curve along which (x, y) approaches (x_0, y_0). Thus intuitively, we can say that L is the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0), if $f(x, y)$ is as close to L as we wish whenever (x, y) is close enough to (x_0, y_0). You may like to study Fig. 5.2 which shows the geometric interpretation of this limit.

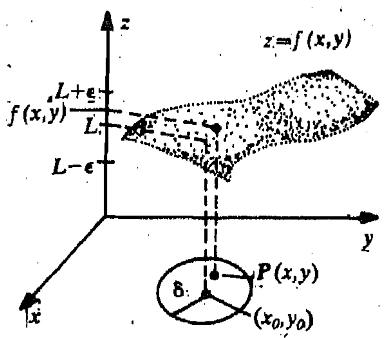


Fig. 5.2

This concept can be extended to functions of three or more variables. For instance, intuitively L is the limit of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0) if $f(x, y, z)$ is as close to L as we wish whenever (x, y, z) is close enough to (x_0, y_0, z_0). It is not possible to represent this limit pictorially because that would require four dimensions.

A natural question follows: When can we say that the limit of a function f does not exist? Let us find the answer. Let f be a function of two variables and let $\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$. Then the concept of the limit as explained above implies that $f(x, y)$ must approach L as (x, y) approaches (x_0, y_0) along each line (or curve) through (x_0, y_0). Thus, to show that $\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$ does not exist, it is enough to show that $f(x, y)$ approaches different numbers as (x, y) approaches (x_0, y_0) along different lines (curves) through (x_0, y_0). This idea can be extended to functions of more than two variables.

Note that the limits

$$\lim_{x \to x_0} \left(\lim_{y \to y_0} f(x, y) \right)$$

and

$$\lim_{y \to y_0} \left(\lim_{x \to x_0} f(x, y) \right)$$

termed the repeated limits are not the same as $\lim_{(x, y) \to (x_0, y_0)} f(x, y)$ which is called the simultaneous limit. For a more detailed study in this regard you may like to go through Block 2 of MTE-07 (Advanced Calculus).

A rigorous mathematical treatment of these concepts is given in an appendix to this unit. You should go through its contents to understand the mathematical basis of these ideas. However, you will not be examined on the material presented in the appendix.

Let us now consider an example to illustrate the concepts you have studied so far.

Example 1

a) Evaluate $\lim_{(x, y) \to (-1, 2)} \dfrac{x^3 + y^3}{x^2 + y^2}$

Solution

Using the results given in Eq. (A.1) of the appendix, we have

$$\lim_{(x, y) \to (-1, 2)} x = -1 \quad \text{and} \quad \lim_{(x, y) \to (-1, 2)} y = 2$$

Using the product formula [(Eq. (A. 3) of the appendix)], we get

$$\lim_{(x, y) \to (-1,2)} x^3 = -1 \quad \text{and} \quad \lim_{(x, y) \to (-1, 2)} y^3 = 8$$

$$\lim_{(x, y) \to (-1, 2)} x^2 = 1 \quad \text{and} \quad \lim_{(x, y) \to (-1, 2)} y^2 = 4$$

Combining the sum and quotient formulas [Eqs.(A. 2 and 4)] we get

$$\lim_{(x,y)\to(-1,2)} \frac{x^3+y^3}{x^2+y^2} = \frac{\lim\limits_{(x,y)\to(-1,2)} x^3 + \lim\limits_{(x,y)\to(-1,2)} y^3}{\lim\limits_{(x,y)\to(-1,2)} x^2 + \lim\limits_{(x,y)\to(-1,2)} y^2} = \frac{-1+8}{1+4} = \frac{7}{5}$$

b) Evaluate $\lim\limits_{(x,y,z)\to(2,1,-1)} \frac{2x^2y - xz^2}{y^2 - xz}$

Solution

You can see that

$$\lim_{(x,y,z)\to(2,1,-1)} x = 2, \quad \lim_{(x,y,z)\to(2,1,-1)} y = 1, \quad \lim_{(x,y,z)\to(2,1,-1)} z = -1$$

Using the sum, product and quotient formulas we get

$$\lim_{(x,y,z)\to(2,1,-1)} \frac{2x^2y - xz^2}{y^2 - xz} = \frac{2.4.1 - 2(-1)^2}{1^2 + 2.1} = \frac{8-2}{3} = 2.$$

You can see that we can obtain the limits of these functions at a point (x_0, y_0) or (x_0, y_0, z_0) effectively by evaluating the value of the function at these points. The only exceptions to this practice will be those functions whose limit does not exist at a point. Let us consider one such function and learn how to find out whether the limit of a function at a point exists or not.

c) Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist for $f(x,y) = \frac{y^2 - x^2}{y^2 + x^2}$

Solution

Let us set $y = mx$. Then

$$\frac{y^2 - x^2}{y^2 + x^2} = \frac{m^2x^2 - x^2}{m^2x^2 + x^2} = \frac{m^2 - 1}{m^2 + 1}$$

The value of $\frac{m^2 - 1}{m^2 + 1}$ will be different for different values of m. This means that $f(x, y)$ approaches different values along the lines corresponding to different values of m as (x, y) approaches (0, 0). Hence the $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

You should now work out an SAQ to concretise these ideas.

Spend 5 minutes

SAQ 1

a) Show that $\lim\limits_{(x,y)\to(-1,1)} \frac{x^2 + 2xy^2 + y^4}{1 + y^2} = 0$

b) Show that $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{(x^2 + y^2)}$ does not exist.

Now that you have understood the concept of limits of functions of several variables, we will define the continuity of such functions.

- A function f of two variables is continuous at (x_0, y_0) if

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$$

- A function f of three variables is continuous at (x_0, y_0, z_0) if

$$\lim_{(x, y, z) \to (x_0, y_0, z_0)} f(x, y, z) = f(x_0, y_0, z_0)$$

A function of several variables is continuous if it is continuous at each point in its domain.

- The sums, products, quotients of continuous functions are continuous.
- The **composite** of continuous functions is continuous.

Consider a pair of functions f and g. Let the co-domain of f be the domain of g, **i.e.,**

$f : X \to Y$ and
$g : Y \to Z$

The function $h : X \to Z$ defined by setting

$h(x) = g(f(x))$

is called the **composite** of f *and* **g.** The function

$g(f(x, y))$

is also a composite of the functions f and g of two variables.

In addition to the definitions and rules you have studied so far, you should **also know** the substitution rule for such functions.

Substitution rule
For the two variable case let us suppose that

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$$

Let g be a function of a single variable t and let g be continuous at $t = L$. Then

$$\lim_{(x, y) \to (x_0, y_0)} g(f(x, y)) = g(L)$$

Let us illustrate these concepts through an example.

Example 2

a) Show that the function **ln** (x/y) is continuous at $(e, 1)$.

Solution
In effect, here we have to show that

$$\lim_{(x, y) \to (e, 1)} \ln(x/y) = \ln(e/1) = 1.$$

Let $f(x, y) = x/y$ and $g(t) = \ln t$.

Using the quotient formula for limits

$$\lim_{(x, y) \to (e, 1)} (x/y) = e.$$

Since $\lim_{t \to e} g(t) = \ln(e) = 1 \equiv g(e)$, therefore $g(t)$ is continuous at $t = e$. Thus it follows from the substitution rule that

$$\lim_{(x, y) \to (e, 1)} \ln(x/y) = g(e) = 1$$

Hence the function **ln** (x/y) is continuous at (e, 1).

b) Let us consider an **example from** physics. The electric field at a point P due to a point charge q is given by

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q\hat{\mathbf{r}}}{r^2}$$

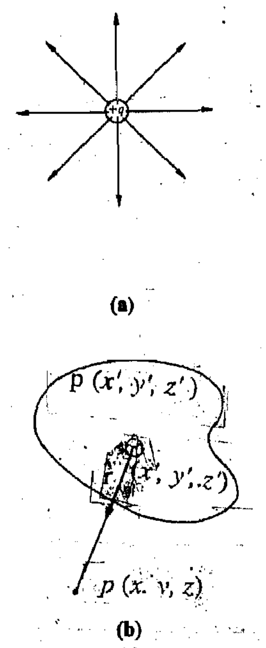


Fig. 5.3 : (a) E for a point charge is not continuous at the point where the charge is located; (b) Each element of the charge distribution **ρ(x', y', z') makes a contribution to the field E at the point (x, y, z). The total field at this point is the sum of all such contributions.**

E for a point charge is shown in Fig. **5.3a.** You can see that as $r \to 0$, the magnitude of the electric field tends to infinity. Thus E for a point charge is not continuous at the point r = 0, **i.e.,** at the location of the charge. This is the reason why we do not talk about the electric field at the point at which the charge is located.

However, the electric field due to a finite continuous charge distribution is continuous. The field at a point $P(x, y, z)$ due to a continuous charge distribution $\rho(x', y', z')$ (Fig. 5.3b) is given as

$$\mathbf{E}(x, y, z) = \int_V \frac{\rho(x', y', z')\hat{\mathbf{r}}dV}{r^2}$$

Here $\hat{\mathbf{r}}$ points from (x', y', z') to (x, y, z). Now you know from Unit 3 of PHE-04 that, in spherical polar coordinates, dV is given as

$$dV = r^2 dr \sin\theta \, d\theta \, d\phi$$

$$\therefore \ \mathbf{E}(r, \theta, \phi) = \int_V \rho(r', \theta', \phi') \, \hat{\mathbf{r}} dr \sin\theta \, d\theta \, d\phi$$

This integral is finite at every point in space and in the limit as r tends to any point in space, the integral tends to the value of E at that point. Therefore, so long as ρ remains finite, E is continuous everywhere, even in the interior or on the boundary of a charge distribution.

You may now by the following SAQ.

Spend 5 minutes

SAQ 2

Show that $f(x, y) = \sin \dfrac{xy}{1 + x^2 + y^2}$ is a continuous function.

So far you have studied the basic concepts of limits and continuity of functions of several variables. We are now ready to consider the following questions: How do we differentiate such functions? As an example, consider a vibrating guitar string of length L at time t (Fig. 5.4). The string is fixed at points A and B, and vibrates in the xy plane in such a way that each point on it moves in a direction perpendicular to the x-axis (transverse vibrations). Suppose $u(x, t)$ is the vertical displacement of a point P on the string, measured from the x-axis at time $t > 0$. We may like to determine how fast the point P is moving, i.e., the velocity of the string along the vertical line with abscissa x. We may also want to find the slope of the curve in Fig. 5.4 at P. In the former case, we keep x fixed and differentiate u with respect to t. In the latter case, we differentiate u with respect to x, keeping t fixed. Thus, a function of several variables can be differentiated with respect to one variable at a time keeping other variables fixed.

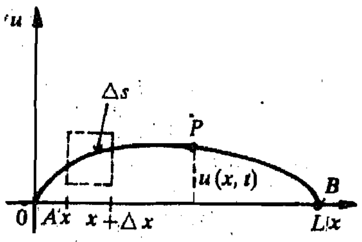


Fig. 5.4 : The displacement $u(x, t)$ of a vibrating guitar string at an instant $t > 0$

This example gives us an intuitive idea that the rate of change of a function of several variables is not just a single function. This is because the independent variables may vary in different ways. All the rates of change for a function of n variables are described by n functions, called its **partial derivatives.** Let us learn about partial derivatives in some detail.

5.2.2 Partial Differentiation

In this section we will define partial derivatives and practice computing them. Consider a function of two variables $f(x, y)$ and let (x_0, y_0) be in the domain of f; The first order partial derivatives of f with respect to x at (x_0, y_0) is defined by

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y) - f(x_0, y_0)}{\Delta x} \quad (5.2a)$$

provided that this limit exists. Similarly, the first order partial derivative of f with respect to y at (x_0, y_0) is defined by

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \quad (5.2b)$$

provided that this limit exists.

So if these limits do not exist at any given point for a function, its partial derivatives also do not exist. The functions f_x and f_y that arise through partial differentiation and are defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \qquad (5.3a)$$

and

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \qquad (5.3b)$$

are called partial derivatives of f with respect to x and y, respectively. These are also denoted by $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, respectively. First order partial derivatives of functions of three or more variables are defined in the same way.

We can think of $f_x(x_0, y_0)$ as the rate of change of $f(x, y)$ at (x_0, y_0) with respect to x, when y is held constant. For example, let $T(x, y)$ be the temperature at any point (x, y) on a flat metal plate lying in the xy plane. Then $T_x(x_0, y_0)$ is the rate at which temperature changes at (x_0, y_0) along the line $y = y_0$ (Fig. 5.5). Similarly, the partial derivative $T_y(x_0, y_0)$ is the rate at which the temperature changes at (x_0, y_0) along the line $x = x_0$.

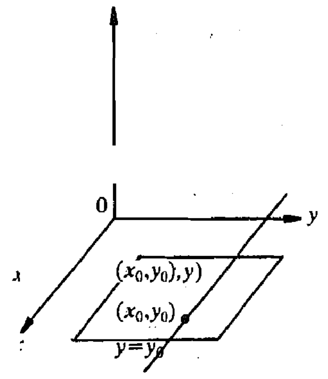


Fig. 5.5

A notation for partial derivatives which is frequently used in applications (particularly in thermodynamics) is $(\partial z/\partial x)_y$. It represents the partial derivative of $z(x, y)$ w.r.t. x when y is held constant. For example, in thermodynamics we use the notation

$$\left(\frac{\partial T}{\partial p}\right)_V, \left(\frac{\partial T}{\partial p}\right)_U$$

etc., where T, p, V and U are the thermodynamical variables, temperature, pressure, volume and internal energy, respectively. You can see that these two partial derivatives are different.

Thus, computing partial derivatives is no more difficult than finding derivatives of functions of a single variable. The following rule will help you compute partial derivatives:

> To calculate the partial derivative of a function f of several variables with respect to a certain variable
>
> - treat the remaining variables as constants
> - differentiate f as usual by using the rules of one variable calculus.

The **sum, product,** and **quotient rules** for ordinary derivatives have counterparts for partial derivatives. Thus, if $f(x, y)$ and $g(x, y)$ have partial derivatives then

$$\frac{\partial}{\partial x}(f \pm g) = \frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial y}(f \pm g) = \frac{\partial f}{\partial y} \pm \frac{\partial g}{\partial y} \qquad (5.4a)$$

$$\frac{\partial}{\partial x}(fg) = \frac{\partial f}{\partial x} g + f\frac{\partial g}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial y}(fg) = \frac{\partial f}{\partial y} g + f\frac{\partial g}{\partial y} \qquad (5.4b)$$

$$\frac{\partial}{\partial x}\frac{f}{g} = \frac{\frac{\partial f}{\partial x} g - f\frac{\partial g}{\partial x}}{g^2} \quad \text{and} \quad \frac{\partial}{\partial y}\frac{f}{g} = \frac{\frac{\partial f}{\partial y} g - f\frac{\partial g}{\partial y}}{g^2} \qquad (5.4c)$$

As an example, consider $f(x, y) = x^2y^3 - x^3y^2$. We hold y constant and differentiate f with respect to x to get

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} x^2y^3 - \frac{\partial}{\partial x} x^3y^2 = 2xy^3 - 3x^2y^2$$

Similarly, we hold x constant and differentiate with respect to y and get

$$\frac{\partial f}{\partial y} = x^2 \frac{\partial}{\partial y} y^3 - x^3 \frac{\partial}{\partial y} y^2 = 3x^2y^2 - 2x^3y$$

Let us consider an example from physics to illustrate these concepts.

Example 3

Consider the variation of current i in a circuit as we change the resistance r for different values of the applied voltage v (Fig. 5.6). The relation between these quantities is given by the familiar Ohm's law $i = \frac{v}{r}$.

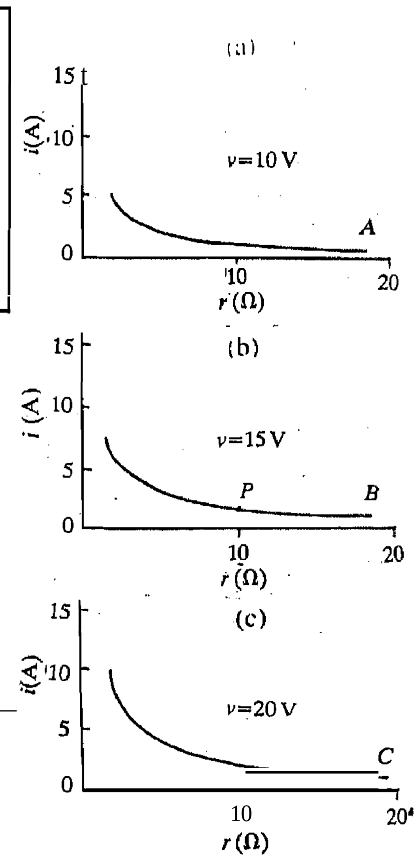


Fig. 5.6 : The variation of current in a resistive circuit element with the resistance for different values of applied voltage

Now suppose we are asked to find the slope at point P on the curve B (Fig. 5.6 b). Treating v as constant we get

$$\frac{\partial i}{\partial r} = -\frac{v}{r^2}$$

For the curve B, v = 15V and at point P, r = 10Ω.

Thus $\left.\frac{\partial i}{\partial r}\right|_P = -\frac{15}{100} \text{A } \Omega^{-1} = -0.15 \text{ A } \Omega^{-1}$

We say that the current varies with resistance at a negative rate of 0.15 ampere per ohm, other things being equal.

You will find several other applications of partial derivatives in physics. For example, in the physics course PHE-06 entitled 'Thermodynamics and Statistical Mechanics' you will study about thermodynamic potentials (which involve an extensive use of partial derivatives).

However, you **must** keep in mind that it is not always possible to compute partial derivatives of functions' in this manner. In some exceptional cases we have to use the limiting process. We shall deal with such cases as and when we come across them. In the study of **PDEs**, you will also come across higher order partial derivatives and you should know about them too.

Higher order partial derivatives

Since the **partial** derivatives are themselves functions, we can take their partial derivatives to obtain higher order partial derivatives. There are four ways to take a second derivative of $f(x, y)$. We may compute

$$f_{xx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \quad f_{yy}(x, y) = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}$$

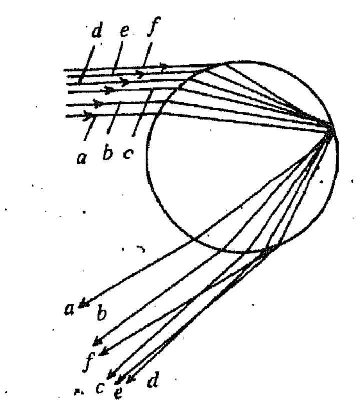
$$f_{xy}(x, y) = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}, \quad f_{yx}(x, y) = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} \qquad (5.5)$$

f_{xy} and f_{yx} are called mixed partial derivatives or mixed partials.

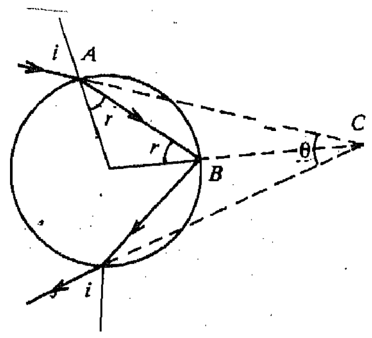
If $f(x, y)$ has continuous second partial derivatives then the mixed partial derivatives are equal, i.e.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad f_{yx} = f_{xy} \qquad (5.6)$$

You should now compute some first and second partials.



(a)



(b)

Fig. 5.7 : (a) The main rainbow you see in the sky is created by various monochromatic rays that have been refracted, reflected and finally refracted again by water droplets (b) the angle θ formed by the path of a monochromatic ray before it enters the droplet and the path after it leaves the droplet depends on both μ and i

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Spend 15 minutes

SAQ 3

a) Find all the first-order partial derivatives of $f(x, y, z) = x^4 - 2x^2y^2z^2 + 3yz^4$ and $h(x, y, t) = xe^t - y^2e^{2t}$. What are the values of $\frac{\partial f}{\partial y}$ (1, 1, 1) and $\frac{\partial h}{\partial t}$ (4, 1, 0) ?

b) Show that the function $z = \ln(x^2 + y^2)$ satisfies the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

c) On a rainy day, you may have observed a rainbow in the sky. A rainbow is formed due to the refraction, reflection and another refraction of various monochromatic rays in the sunlight by water droplets suspended in air (see Fig. 5.7a). The angle θ shown in Fig. 5.7b for one monochromatic ray is given as

$$\theta(\mu, i) = 4 \sin^{-1}\left(\frac{\sin i}{\mu}\right) - 2i$$

where μ is the index of refraction of water for the ray and i, its angle of incidence. For any given μ, find the angle i_μ for which

$$\frac{\partial \theta}{\partial i}(\mu, i_\mu) = 0$$

d) The entropy S of a gas is given by

$$S = C_v \ln P + C_p \ln V + A \qquad \text{(i)}$$

where C_p, C_v and A are constants. We can substitute for V from the ideal gas law PV = RT to obtain

$$S = (C_v - C_p) \ln P + C_p \ln T + B \qquad \text{(ii)}$$

where B is a constant. Compute $\partial S / \partial P$ from (i) and (ii). Why do the **two** expressions differ?

Certain results follow logically from the discussion so far. We state them without proof. ***The mere existence of partial derivatives does not imply the continuity of a function of several variables. Also a function of several variables which is continuous at a point need not have any of the partial derivatives at the point.*** You can read about these ideas in detail in Block 2 of the mathematics course MTE-07. With this background we would now like to consider these questions : When can we say that a function of several variables is 'differentiable'? Is a continuous function $f(x, y)$ differentiable at a point ? Or is $f(x, y)$ differentiable at a point provided its **partial derivatives** exist? Let us answer these questions very briefly.

5.2.3 Differentiability

Recall that a real-valued **continuous** function of one variable need not be differentiable. The same applies to functions of several variables. Similarly, since the existence of partial derivatives **does** not even guarantee continuity, it cannot guarantee differentiability, So we need additional **conditions. We** will not go into a formal mathematical definition of a **differentiable function.** Instead we state here the sufficient conditions which, if satisfied, ensure that a function of several **variables** is differentiable.

> If $f(x, y)$ has partial derivatives on a disc centred at (x_0, y_0), and if f_x and f_y are **continuous** at (x_0, y_0), then f is differentiable at (x_0, y_0).

However, these conditions are not necessary. Thus, a function can be differentiable at a point even when none of its partial derivatives is continuous at that point.

Having studied these concepts, you should know what the term 'a real-valued, continuous differentiable function of several variables' means, whenever you come across it. With this mathematical background, we are ready to discuss partial differential equations, **i.e.,** equations involving the partial derivatives of functions of several variables.

5.3 PARTIAL DIFFERENTIAL EQUATIONS

Let us begin by considering examples of how some special partial differential equations **(PDEs)** arise in physical situations. We will then classify PDEs and understand what is meant by their solutions. Consider a steadily flowing **stream** of water with velocity field $\mathbf{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$. Let the velocity field be incompressible and irrotational. Then you know from Unit 2 (**Sec.** 2.4) of the **course Mathematical** Methods in Physics-I (PHE-04) that for this velocity field $\nabla \cdot \mathbf{v} = 0$ and $\nabla \times \mathbf{v} = 0$, **i.e.** $\mathbf{v} = \nabla\phi$ **where** ϕ is a scalar field. Now you can use the definitions of divergence and **gradient** to express **these** relations in a Cartesian coordinate system in their **differential** form

$$\nabla \cdot \mathbf{v} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0 \qquad (5.7a)$$

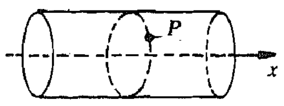
$$\mathbf{v} = \nabla\phi, \text{ i.e., } \quad v_1 = \ldots., \quad v_2 = \ldots.., \quad v_3 = \ldots. \qquad (5.7b)$$

Then, substitute Eq. (5.7b) in Eq. (5.7a) and you **will get**

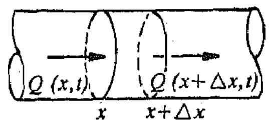
$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad (5.8)$$

Thus, you have set up a well known **PDE** in physics termed the **Laplace** equation. This equation is satisfied by the velocity potential function of any incompressible and irrotational flow. It also finds applications in such diverse fields as gravitation, electrostatics, elasticity and steady state heat conduction. Let us take up another example.

Consider the flow of heat along a cylindrical metal rod (Fig. 5.8a). We choose the x-axis to be along the axis of the cylinder. Let us assume that heat can flow only in a direction parallel to the x-axis. This means that at any instant t, the temperature T is the same at all points of a cross-section x = constant (see Fig. 5.8a). Then $T(x, t)$ describes the temperature of any point $P(x)$ in the rod at time t. We also assume that no heat is generated within the rod. We can model heat flow in the rod according to two experimentally verified laws: Fourier's law and the principle of conservation of heat. Fourier's law states that the rate of heat flow per unit area, $Q(x, t)$ perpendicular to the flow is proportional to the temperature gradient. Would you like to express this law mathematically? Give it a try.



(a)



(b)

Fig. 5.8 : Flow of heat in a cylindrical metal rod

Since the flow is one-dimensional, you should get a relation of the following kind.

$$Q(x, t) = -AK\frac{\partial T}{\partial x}(x, t), \qquad (5.9)$$

where $Q(x$, t) is the rate of heat flow in the positive xdirection across the section x = constant at time t, Here K is the thermal conductivity of the metal and **A** is the rod's cross-sectional area. The **minus** sign appears because heat flows from hotter to colder regions, so that Q is positive where the temperature gradient is negative, and vice versa.

The principle of conservation of heat states that the rate at which heat accumulates in a region containing no heat sources is equal to the net rate at which heat enters that region through its boundaries. Let us apply this principle to a small portion of the rod between x and x + Δx (See Fig. 5.8b). The rate at which heat accumulates in this portion at a time t is

$$Q(x, t) - Q(x + \Delta x, t)$$

Now you can write the expression for the heat accumulated in this portion in the short time interval between t and t + Δt in the space below :

.... (5.10a)

You know that the **heat energy** required to **raise** the average temperature of a body of mass m and **specific heat** s, from T_1 to T_2 is ms $(T_2 - T_1)$. For the small portion of the rod, we have $m = \rho A\Delta x$, where p is the density of the metal. Let $T(x$, t) and $T(x$, t + $\Delta t)$ be the average **temperatures** of the portion at time t and (t + Δt), respectively and let s be the metal's'specific heat. Then you can write down the heat energy added to the portion between these times:

.... (5.10b)

The principle of conservation of heat demands that the expressions contained in Eqs. (5.10a) and **Eq.** (5.10b) shouid be equal. Thus, we have

$$[Q(x, t) - Q(x + \Delta x, t)]\,\text{At} = (\rho A\Delta x)s\,[T(x, t + \text{At}) - T(x, t)] \qquad (5.10c)$$

Now divide both sides by (Δx **At**) and take the simultaneous limit as Δx and At tend to zero. What do you get ? Write down the result :

.... (5.10d)

Substituting Eq. (5.9) in Eq. (5.10d) you should get

$$\frac{\partial}{\partial x}\left[AK\frac{\partial T}{\partial x}(x, t)\right] = \rho As\frac{\partial T}{\partial t}(x, t) \qquad (5.10e)$$

On simplifying the equation further we get

$$\frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2} = 0, \qquad (5.11)$$

where $k = \frac{K}{\rho s}$ is called the **thermal diffusivity** of the material of which the rod is made. Remember that in writing Eq. (5.11) we have assumed K to be a constant, which need not be true always.

Eq. (5.11) is termed the one-dimensional **diffusion** equation. It is so called because it models the 'diffusion' or gradual change of various physical quantities that are continuous functions of time and space coordinates in one dimension. This equation is also used to describe the diffusion of liquid or gas concentrations. A slight variation of Eq. (5.11) describes the diffusion of neutrons in a nuclear reactor. In the situations when Eq. (5.11) models heat flow in a one-dimensional object as in the example considered above it is also termed the one- dimensional heat flow equation.

Eqs. (5.8) and (5.11) are two PDEs which occur quite often in physics. You will come across other PDEs in Unit 6 of this course and other physics courses. Our main aim, of course, is to learn the methods of solving the PDEs which occur in physics. However, before you learn these methods you should know how to classify PDEs. You must also understand what constitutes the solution of a PDE. These will be our concerns in the next two sub-sections.

5.3.1 Classification of PDEs

We classify PDEs in much the same way as ODEs, i.e., in terms of their order, degree and linearity/nonlinearity. Linear PDEs are further classified as homogeneous/nonhomogeneous, and as elliptic, parabolic and hyperbolic PDEs. Let us see what these terms mean.

Order and Degree

Just as in the case of ODEs, the order of a PDE is the order of the highest derivative occurring in the equation.

For example, the equation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=0 \tag{5.12}$$

is a first order PDE. A first order PDE for a function $f(x, y)$ contains at least one of the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ but no partial derivative of order higher than one. A second order PDE for $f(x, y)$ contains at least one of the partial derivatives f_{xx}, f_{yy}, f_{xy} or f_{yx} but no partial derivatives of order higher than two. Eqs. (5.8) and (5.11) are second order PDEs. Second order PDEs may also contain first order terms like $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, as in Eq. (5.11).

The degree of a PDE is the degree of the highest derivative in the equation.

For example, Eqs. (5.8), (5.11) and (5.12) are PDEs of degree one. The PDE

$$\left(\frac{\partial f}{\partial x}\right)^3+\frac{\partial f}{\partial t}=0 \tag{5.13}$$

is a first order PDE of degree 3. You may turn to SAQ 4 and write down the order of the PDEs listed there right away if you so wish.

Linear and nonlinear **PDEs**

Just as in the case of ODEs, we say that a PDE is linear if (i) it is of the first degree in the unknown function (the dependent variable) and its partial derivatives, (ii) it does not contain the products of the unknown functions and either of its partial derivatives and (iii) it does not contain any transcendental functions. Otherwise it is nonlinear. For example, the PDEs given by Eqs. (5.8), (5.11), (5.12) and the following PDEs are all linear:

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial y^2} \tag{5.14}$$

and

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x, y) \tag{5.15}$$

The PDE given by Eq. (5.13) is nonlinear as it is of degree 3.

A linear PDE can further be classified as homogeneous, or nonhomogeneous.

Homogeneous and nonhomogeneous linear PDEs
If each term of a PDE contains either the unknown function or one of its partial derivatives, it is said to be homogeneous: otherwise it is nonhomogeneous. Which of the Eqs. (5.8), (5.11), (5.12), (5.14) and (5.15) are homogeneous and which ones nonhomogeneous? You are right. All the equations except Eq. (5.15) are homogeneous.

You can practise classifying PDEs further by working out SAQ 4.

SAQ 4

Spend 5 minutes

Write down the order and degree of each of the PDEs listed below. Determine which of the PDEs are linear, nonlinear. Classify the linear PDEs as homogeneous, nonhdmogeneous.

i) $x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2} = 0$

ii) $xy \frac{\partial^2 f}{\partial x^2} + x \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + y \frac{\partial^2 f}{\partial y^2} = x^2 + y^2$

iii) $\left(\frac{\partial y}{\partial x}\right)^3 + \frac{\partial y}{\partial t} = 0$

iv) $x^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = e^{xy}$

v) $\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial^3 u}{\partial x \partial y^2} - 6\left(\frac{\partial u}{\partial y}\right)^4 = 0$

In this course we shall restrict ourselves to linear second order partial differential equations because these occur most frequently in physics. The most general form of such an equation, for a function $u(x, y)$ is

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g(x, y) \tag{5.16}$$

where a, b, c, d, e and f are functions of (x, y). If the coefficients a, b, c, d, e, f are constants, Eq. (5.16) is termed a linear, second order, constant coefficient **PDE**. Equations of the form (5.16) with constant coefficients, are further classified as elliptic, hyperbolic and **parabolic,** depending on the relationship between the second-order coefficients a, b, c:

if $ac - b^2 > 0$, the equation is elliptic,

if $ac - b^2 < 0$, the equation is hyperbolic,

if $ac - b^2 = 0$, the equation is parabolic.

You can verify that the **Laplace equation** Eq. (5.8) and Eq. (5.15) known as Poisson's equation are elliptic. The diffusion equation Eq. (5.11) is parabolic and Eq. (5.14), known as the wave equation is hyperbolic. Check these results before studying further. We have introduced you to the PDEs (Eqs. 5.8, 5.11, 5.14, 5.15) that occur most frequently in physics. You will learn the methods of solving these PDEs under specified boundary and initial conditions in Unit 6. But before that you must know what is meant by the solution of a PDE and some properties of the solutions. This is the subject of Sec. 5.3.2.

5.3.2 What is a Solution of a PDE ?

In part (b) of SAQ 3 you have verified that the function $u = \ln(x^2 + y^2)$ satisfies the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{5.17}$$

The function $\ln (x^2 + y^2)$ is a **solution** of this PDE. The process of solving a PDE involves finding all the functions which satisfy it. A more formal definition of a solution is as follows.

A solution of a PDE in some region R of the space of independent variables is a function, all of whose partial derivatives appearing in the equation exist in some domain containing R and which satisfies the equation everywhere in R.

In general, there can be a large number of solutions of a PDE. For example, the functions $u = x^2 - y^2$ and $u = e^x \cos y$ are also solutions of Eq. (5.17). A unique solution of a PDE corresponding to a given physical problem is obtained by applying appropriate initial conditions and boundary conditions.

Let us now ask: What is involved in solving a PDE? Consider the following rather simple PDE

$$\frac{\partial u(x, t)}{\partial x} = 1 \tag{5.18a}$$

To find $u(x, t)$ we may keep t constant and integrate with respect to x. We then obtain

$$u(x, t) = x + C \tag{5.18b}$$

However, C is a constant only if t is kept fixed. For different values of t, C will be different, i.e., C is a function of t. Thus, the most general solution of Eq. (5.18a) is

$$u(x, t) = x + f(t) \tag{5.18c}$$

where f is an arbitrary function of t. You can verify that $u(x, t)$ of Eq. (5.18c) does satisfy Eq. (5.18a).

So you see that while the solution of an ODE involves arbitrary constants, **the** solution of **a** PDE involves arbitrary functions. As we increase the order of the partial derivatives in a PDE, we introduce more arbitrary functions.

Recall that a linear combination of the linearly independent solutions of an ODE is also its solution. The same principle applies to the solutions of a linear homogeneous partial differential equation. Thus, if u_1 and u_2 are any Linearly independent solutions of a linear homogeneous PDE in some region then

$$u = C_1 u_1 + C_2 u_2$$

where C_1 and C_2 are arbitrary constants, is also a solution of that equation in that region. This principle is called the principle of superposition and it can be extended to the case where n solutions of a PDE exist. To sum up, in this section we have studied partial differential equations, their classification and the meaning of their solutions. We would like to end this section with an exercise for you.

SAQ 5

Spend 10 minutes

a) Verify that $u_1 = \cos x \cos cy$ and $u_2 = \sin x \sin cy$ are both solutions of the PDE

$$\frac{\partial^2 u}{\partial y^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Show that $\cos(x + cy)$ and $\cos(x - cy)$ are also solutions of this PDE.

b) Show that for each integer n, the function

$$u = e^{-k n^2 y} \sin nx$$

is a solution of the PDE

$$\frac{\partial u}{\partial y} - k \frac{\partial^2 u}{\partial x^2} = 0$$

Deduce that for any positive integer N and real numbers, $a_1, a_2, \ldots, a_N$, the function

$$\sum_{n=1} a_n e^{-k n^2 y} \sin nx$$

is also a solution of'the PDE.

Let us now sum up what you have studied in this unit.

5.4 SUMMARY

We summarise below the concepts for a function of two variables. These can be extended to functions of more than two variables.

- A function f (x, y) of two variables x and y is one whose value is determined by the values of x and y. We call x and y the ***independenr variables:*** a variable equal to f (x, y) is called the ***dependent variable.***
- The limit $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

  if $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$ then $|f(x, y) - L| < \varepsilon$
- A function f of two variables is continuous at (x_0, y_0) if

  $$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$
- The limits

  $$\frac{\partial f}{\partial x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \lim_{\Delta y\to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

  if they exist, are called the first order partial derivatives off with respect to x and y, respectively.,
- The partial derivative of a function of several variables w.r.t. a variable is calculated by differentiating the function w.r.t. that variable alone, treating other variables as constant.
- A function f (x, y) is said to be differentiable at (x_0, y_0) if it has partial derivatives on a disc centred at (x_0, y_0) and if the partial derivatives are continuous at (x_0, y_0).
- Differential equations involving functions of more than one variable are termed partial differential equations (PDEs).
- PDEs occur quite often in physics. In this unit, we have discussed the setting up of **Laplace's** equation and one-dimensional diffusion equation.
- Like **ODEs**, PDEs are also classified by way of order and degree, linearity and nonlinearity. Linear PDEs are further classified as homogeneous and nonhomogeneous.
- A linear, second order, constant coefficient PDE can also be classified as elliptic, hyperbolic and parabolic depending on the relationship between the coefficients of second order partial derivatives occurring in the PDE.
- A solution of a PDE in some region R of the space of independent variables is a function, which satisfies the PDE everywhere in R.
- A linear combination of the linearly independent solutions of a PDE is also a solution of the PDE.

5.5 TERMINAL QUESTIONS

Spend 15 minutes

1) According to Newton's law of gravitation the magnitude of the force of attraction between two particles of mass m is

$$F = -\frac{Gm^2}{r^2}$$

where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$ is the distance between the two particles. Determine whether F is continuous and differentiable at all points in space. The potential of this gravitational force field is given as

$$f(x, y, z) = -\frac{Gm^2}{r} \quad (r > 0)$$

Show that f satisfies the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

2) Given below are some PDEs that appear in physics. Classify them by way of order and degree, linearity (L)/nonlinearity(NL), homogeneity(H)/nonhomogeneity(NH).

i) $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + 2\beta \frac{\partial u}{\partial t} + \alpha u = 0$ (the telegraph equation)

ii) $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$ (the wave equation)

iii) $-\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) + V(r)\psi = \frac{\partial \psi}{\partial t}$ (Schrödinger's time-dependent equation)

iv) $\frac{\partial \rho}{\partial t} + \rho\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\right) = 0$ (the continuity equation)

v) $\frac{\partial u}{\partial t} - k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$ (the two-dimensional diffusion equation)

vi) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\varepsilon_0}\rho(x, y, z)$ (Poisson's equation)

5.6 SOLUTIONS AND ANSWERS

1) a) $\lim\limits_{(x,y)\to(-1,1)} \frac{x^2 + 2xy^2 + y^4}{1 + y^2}$

Using Eq. (A.1) of the Appendix and the sum, product, quotient rules we get

$$\lim_{(x,y)\to(-1,1)} \frac{x^2 + 2xy^2 + y^4}{1 + y^2}$$

$$= \frac{\lim\limits_{(x,y)\to(-1,1)} x^2 + 2 \lim\limits_{(x,y)\to(-1,1)} x \lim\limits_{(x,y)\to(-1,1)} y^2 + \lim\limits_{(x,y)\to(-1,1)} y^4}{1 + \lim\limits_{(x,y)\to(-1,1)} y^2}$$

$$= \frac{(-1)^2 + 2[-1]\,[1]^2 + 1^4}{1 + 1^2} = \frac{1 - 2 + 1}{2} = 0$$

b) Let $y = mx$, then $f(x, y) = \frac{xy}{x^2 + y^2} = \frac{mx^2}{x^2 + m^2x^2} = \frac{m}{1 + m^2}$

This will have different values for different values of m. This means that $f(x, y)$ approaches different values along the lines corresponding to **different** values of m as (x, y) approaches (0, 0). Hence $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

2) Let $g(x, y) = \left(\frac{xy}{1 + x^2 + y^2}\right)$ and $u(t) = \sin t$

Then $f(x, y) = u(g)$, i.e., $f(x, y)$ is a composite of $g(x, y)$ and u. We hav'e shown in Eq. (A.1) of the Appendix that $\lim\limits_{(x,y)\to(x_0,y_0)} x = x_0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} y = y_0$. Since x_0 and y_0 can be any points in the domain of x and y, $p(x, y) = x$ and $q(x, y) = y$, are continuous. The sums, products and quotients of continuous functions are continuous. Hence $g(x, y)$ is continuous. Similarly, you can verify that $u(t)$ is continuous. Therefore, their composite $f(x, y)$ is also continuous.

3) a) $f(x, y, z) = x^4 - 2x^2y^2z^2 + 3yz^4$

$$\frac{\partial f}{\partial x} = 4x^3 - 4xy^2z^2$$

$$\frac{\partial f}{\partial y} = -4x^2yz^2 + 3z^4 ; \frac{\partial f}{\partial y}(1, 1, 1) = -4 + 3 = -1$$

$$\frac{\partial f}{\partial z} = -4x^2y^2z + 12yz^3$$

$h(x, y, t) = xe^t - y^2e^{2t}$

$$\frac{\partial h}{\partial x} = e^t$$

$$\frac{\partial h}{\partial y} = -2ye^{2t}$$

$$\frac{\partial h}{\partial t} = xe^t - 2y^2e^{2t}$$

$$\frac{\partial h}{\partial t}(4,1,0) = 4.1 - 2.1^2.1 = 2$$

b) $z = \ln(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}, \frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2}$$

$$= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2}{x^2 + y^2} - \frac{4y^2}{(x^2 + y^2)^2}$$

$$= \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2y^2 - 2x^2 + 2x^2 - 2y^2}{(x^2 + y^2)^2} = 0$$

Thus, $z = \ln(x^2 + y^2)$ satisfies the equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

c) $\theta(\mu, i) = 4 \sin^{-1}\left(\frac{\sin i}{\mu}\right) - 2i$

$$\frac{\partial \theta}{\partial i} = \frac{4}{\sqrt{1 - \frac{\sin^2 i}{\mu^2}}} \frac{\cos i}{\mu} - 2$$

$$= \frac{4\mu}{\sqrt{\mu^2 - \sin^2 i}} \cdot \frac{\cos i}{\mu} - 2$$

$$= \frac{4\cos i}{\sqrt{\mu^2 - \sin^2 i}} - 2$$

$$\frac{\partial \theta}{\partial i} = 0 \text{ for } i = i_\mu$$

$$\text{or } \frac{4\cos i_\mu}{\sqrt{\mu^2 - \sin^2 i_\mu}} = 2$$

$$4\cos^2 i_\mu = \mu^2 - \sin^2 i_\mu$$

$$\text{or } 4\cos^2 i_\mu + 1 - \cos^2 i_\mu = \mu^2$$

$$\text{or } 3\cos^2 i_\mu = \mu^2 - 1$$

$$\text{or } \cos i_\mu = \sqrt{\frac{\mu^2-1}{3}}$$

$$\therefore\ i_\mu = \cos^{-1}\left(\sqrt{\frac{\mu^2-1}{3}}\right)$$

d) From (i)

$$\frac{\partial S}{\partial P} = \frac{C_v}{P}$$

and from (ii)

$$\frac{\partial S}{\partial P} = \frac{C_v - C_p}{P}$$

In finding $\frac{\partial S}{\partial P}$ from (i), we keep Veonstant, whereas in computing $\frac{\partial S}{\partial P}$ from (ii), we keep T constant. Therefore, these two partial derivatives are different.

4) We have used the notations L for linear, NL for nonlinear, H for homogeneous and NH for nonhomogeneous PDEs in the answer. The first term gives the order and the second, the degree of each PDE.

i) 2, 1, L, H

ii) 2, 1, L, NH

iii) 1, 3, NL

iv) 2, 1, L, NH

v) 3, 1, NL

5) a) The partial derivatives of $u_1 = \cos x \cos cy$ and $u_2 = \sin x \sin cy$ are

$$\frac{\partial u_1}{\partial x} = -\sin x \cos cy, \quad \frac{\partial^2 u_1}{\partial x^2} = -\cos x \cos cy$$

$$\frac{\partial u_1}{\partial y} = -c\cos x \sin cy, \quad \frac{\partial^2 u_1}{\partial y^2} = -c^2 \cos x \cos cy$$

$$\text{and } \frac{\partial u_2}{\partial x} = \cos x \sin cy, \quad \frac{\partial^2 u_2}{\partial x^2} = -\sin x \sin cy$$

$$\frac{\partial u_2}{\partial y} = c\sin x \cos cy, \quad \frac{\partial^2 u_2}{\partial y^2} = -c^2 \sin x \sin cy$$

Substituting the relevant partial derivatives of u_1 and u_2 in the given PDE we get two identities implying that both u_1 and u_2 are its solutions. Now from the principle of superposition a linear combination of linearly independent solutions of a PDE is also its solution. Since u_1 and u_2 are linearly independent we get that

$$\cos(x + cy) = \cos x \cos cy - \sin x \sin cy = u_1 - u_2$$

$$\text{and } \cos(x - cy) = \cos x \cos cy + \sin x \sin cy = u_1 + u_2$$

are also solutions of the PDE.

b) Let us first compute the partial derivatives of u_n :

$$\frac{\partial u_n}{\partial y} = -kn^2 e^{-kn^2 y} \sin nx$$

$$\frac{\partial u_n}{\partial x} = n\, e^{-kn^2 y} \cos nx$$

$$\frac{\partial^2 u_n}{\partial x^2} = -n^2 e^{-kn^2 y} \sin nx$$

Substituting $\frac{\partial u_n}{\partial y}$ and $\frac{\partial^2 u_n}{\partial x^2}$ in the PDE gives us an identity. Therefore, u_n is a solution of the PDE. Again, since $u_1, u_2, u_3, \ldots, u_n$ are linearly independent functions, we get from the principle of superposition that their linear combination, i.e. $u = a_1u_1 + a_2u_2 + \ldots + a_nu_n$ is also a solution of the PDE.

In concise form we may write.

$$u = \sum_{n=1}^{N} a_n u_n = \sum_{n=1}^{N} a_n e^{-kn^2 y} \sin nx$$

Terminal Questions

1) F is not continuous and therefore not differentiable at the point $r = 0$ for reasons explained in part (b) of Example 2.

$$\frac{\partial f}{\partial x} = -\frac{\partial}{\partial x}\left[\frac{Gm^2}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\right]$$

$$= \frac{1}{2}\frac{2(x-x_0)\,Gm^2}{\left\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right\}^{3/2}} = \frac{(x-x_0)Gm^2}{r^3}$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{3}{4}\frac{4(x-x_0)^2 Gm^2}{\left\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right\}^{5/2}} + \frac{Gm^2}{r^3} = -\frac{3(x-x_0)^2 Gm^2}{r^5} + \frac{Gm^2}{r^3}$$

Similarly $\frac{\partial^2 f}{\partial y^2} = -\frac{3(y-y_0)^2 Gm^2}{r^5} + \frac{Gm^2}{r^3}$ and $\frac{\partial^2 f}{\partial z^2} = -\frac{3(z-z_0)^2 Gm^2}{r^5} + \frac{Gm^2}{r^3}$

Thus $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -\frac{3Gm^2}{r^5}[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2] + \frac{3Gm^2}{r^3}$

$$= -\frac{3Gm^2}{r^5}r^2 + \frac{3Gm^2}{r^3} \equiv 0$$

Therefore, $f = -\frac{Gm^2}{r}$ $(r > 0)$ satisfies the given PDE.

2)
- i) 2, 1, L, H
- ii) 2, 1, L, H
- iii) 2, 1, L, H
- iv) 1, 1, L, H
- v) 2, 1, L, H
- vi) 2, 1, L, NH

APPENDIX A LIMITS OF A FUNCTION OF MORE THAN ONE VARIABLE

We will give here the formal mathematical definitions of the limits of a function of two or more variables and state the rules for evaluating limits of such functions.

Recall that the distance between two points (x, y) and (x_0, y_0) in the plane containing these points is less than 6 if

$$\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$$

Let f (x,y) be a real-valued function defined throughout a set containing a disc centred at (x_0, y_0) exceptpossibly at (x_0, y_0) itself (see Fig. 5.2). Let L be a real number: Then L is the limit off at (x_0, y_0) if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\text{if } 0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta, \text{ then } |f(x,y)-L|<\varepsilon$$

Then we write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L$$

and say that $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists.

Fig. 5.2 shows the geometric interpretation of the limit.

We can extend the definition to a function of three variables.

Once again, recall that the distance between the points (x, y, z) and (x_0, y_0, z_0) is less than δ if

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}<\delta$$

The formal definition of the limit of $f(x, y, z)$ is then given as follows:

Let f be defined throughout a set containing a ball centred at (x_0, y_0, z_0) except possibly at (x_0, y_0, z_0) itself. Then L is the limit off at (x_0, y_0, z_0) if for every $\varepsilon > 0$, there is a 6 > 0 such that

$$\text{if } 0<\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}<\delta, \text{ then } |f(x,y,z)-L|<\varepsilon$$

In this case we write

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)=L$$

and say that $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z)$ exists.

Let us consider a simple example to evaluate the limits using these basic definitions. Let us show that

$$\lim_{(x,y)\to(x_0,y_0)} x=x_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} y=y_0 \qquad (A.1)$$

Let $\varepsilon > 0$. Now $(x-x_0)^2 \le (x-x_0)^2+(y-y_0)^2$

Therefore, if we let $\delta=\varepsilon$, it follows that

$$\text{if } 0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta, \text{ then } |f(x,y)-L| = |x-x_0| = \sqrt{(x-x_0)^2}<\varepsilon \ (\because \delta=\varepsilon).$$

This proves that $\lim_{(x,y)\to(x_0,y_0)} x=x_0$. You can prove the second limit in a similar way.

The result of Eq. (A.1) and the following limit formulas for the sum, products and quotients of functions of several variables will enable you to determine the limits of a variety of functions. We state the formulas for functions of two variables. Similar formulas would apply to functions of three and more variables.

If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y)$ exist, then

i) $$\lim_{(x,y)\to(x_0,y_0)} (af+bg)(x,y) = a \lim_{(x,y)\to(x_0,y_0)} f(x,y) \pm b \lim_{(x,y)\to(x_0,y_0)} g(x,y) \qquad \text{(A.2)}$$

where a and b are constants.

ii) $$\lim_{(x,y)\to(x_0,y_0)} (fg)(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) \lim_{(x,y)\to(x_0,y_0)} g(x,y) \qquad \text{(A.3)}$$

iii) $$\lim_{(x,y)\to(x_0,y_0)} (f/g)(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) \Big/ \lim_{(x,y)\to(x_0,y_0)} g(x,y) \qquad \text{(A.4)}$$

In Example 1, we have determined the limits of some functions using these formulas.

UNIT 6 PARTIAL DIFFERENTIAL EQUATIONS IN PHYSICS

Structure

6.1 INTRODUCTION

In Unit 5 you have learnt the basic concepts of order, degree, linearity, and type of partial differential equations (PDEs). Such equations arise for systems whose behaviour is governed by more than one independent variable. We come across PDEs in such diverse fields as meteorology, structural engineering, fluid mechanics, elasticity, heat flow, pollutant and neutron diffusion, wave propagation, aerodynamics, electromagnetics and nuclear physics. Most applied problems in physics are formulated in terms of second-order PDEs. From PHE-02 course on Oscillations and waves, you are familiar with the wave equation which governs wave propagation—a phenomenon responsible for hearing, seeing, music and our communication with the world at large. In your course on electric and magnetic phenomena, you would have come across Laplace's and Poisson's equations. These equations can also be used to determine gravitational potential, steady- state temperature etc.

A particularly useful method employed frequently to solve several second-order partial differential equations is the method of separation of variables. Depending on the number of independent variables, this method facilitates to reduce a linear PDE to two or more ordinary differential equations, which you already know to solve. This methpd is illustrated in Sec. 6.2. Boundary value problems in physics invariably exhibit rectangular, spherical or cylindrical symmetry in one or more dimensions, In Sec. 6.3 we illustrate the above said method to obtain a unique solution, subject to the given initial and boundary conditions. Since the same PDE may apply to many problems, the method discussed here can be used to solve many more problems than are illustrated here.

The term separation of variables was used in Unit 1 of this course in a completely different context.

Objectives

After studying this unit you should be able to

- solve a given PDE using the method of separation of variables
- obtain a unique solution to a given physical problem.

6.2 THE METHOD OF SEPARATION OF VARIABLES

Linear second order PDEs form the backbone of theoretical physics, Apart from Laplace's equation and Poisson's equation, the most important of these are the Helmholtz equation, Telegraph equation, wave equation, Klein-Gordon equation, Schrodinger equation and Dirac's equation.

Nonlinear PDEs are encountered in the study of shock wave phenomenon, atmospheric physics and turbulence. Higher order PDEs occur in the study of viscous fluids and elasticity..

The first question that should logically come to your mind is: How-to solve a PDE? As a first strategy, we would like to reduce the given PDE to simpler differential equations containing fewer variables. (The process may be continued until a set of ordinary differential equations is obtained), Next, we put the ODEs so obtained in easily solvable form using methods discussed in Block 1 of this course. The simplest and most widely used method for reducing common and physically important PDEs is the **method** of **separation of** variables. Let us now learn how it works.,