\[
\begin{align*}
\text{or } & \int (0-0) \cdot \hat{k} 
\int E_x \cos (kx + ut) \hat{k} 
\int (0-0) \cdot \frac{1}{c} \int E_x \cos (kx + ut) \\
\text{or } & \frac{1}{c} \int E_x \cos (kx + ut) = 0
\end{align*}
\]

6. \( \nabla \cdot \vec{B} = 0 \)

From the vector identity Eq. (2.30), if \( \vec{A} \) is a vector field then

\( \nabla \cdot (\nabla \times \vec{A}) = 0 \)

This implies that \( \vec{B} = \nabla \times \vec{A} \).

\[
\vec{A} = [C \exp(-xy) \sin \alpha] \hat{\vec{k}}
\]

\[
\nabla \times \vec{A} = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
0 & 0 & C \exp(-xy) \sin \alpha
\end{vmatrix}
\]

\[
= \hat{i} \frac{\partial}{\partial y} [C \exp(-xy) \sin \alpha] - \hat{j} \frac{\partial}{\partial z} [C \exp(-xy) \sin \alpha]
\]

\[
= \hat{i} [-C \exp(-xy) \sin \alpha] - \hat{j} [C \exp(-xy) \cos \alpha - y C \exp(-xy) \sin \alpha]
\]

\[
= C \exp(-xy) \sin \alpha (-x \hat{i} + y \hat{j}) - \hat{j} C \exp(-xy) \cos \alpha
\]
3.1 INTRODUCTION

In the previous units we restricted ourselves completely to the Cartesian coordinate system. This system offers us unique advantage in that the unit vectors $i$, $j$ and $k$ are constant in direction as well as in magnitude. But it is not advisable to use Cartesian coordinates in all physical problems. For instance, when we deal with gravitational or electrostatic forces, use of Cartesian coordinates is usually not convenient. The same is true when we wish to study the velocity distribution of gas molecules, flow of a fluid along a pipe, flow of current in a cylindrical conductor, heat flow in a sphere or energy produced in a reactor. In all these situations, we use non-Cartesian coordinates—plane polar, cylindrical or spherical polar coordinates. The best choice for a coordinate system depends on the shape of the system under consideration. For systems with cylindrical or spherical symmetry, the use of these coordinates simplifies the algebra. So it is absolutely necessary for you to be familiar with all of them.

You will learn about all these coordinate systems in Sec. 3.2. In Sec. 3.3 you will learn to express a vector in different polar coordinates. The expressions for line, surface and volume elements are also developed here. The non-Cartesian coordinate systems are special cases of the general orthogonal curvilinear coordinates. In Sec. 3.4, we have introduced curvilinear coordinates by linking them with Cartesian coordinates. The expressions for gradient, divergence, curl, and Laplacian operators are derived first in curvilinear and then in polar coordinates. These expressions will be useful in Unit 4 where you learn to evaluate integrals involving various scalar and vector fields.

Objectives

- After going through this unit, you should be able to
  - relate plane polar, spherical polar and cylindrical polar coordinates to Cartesian coordinates
  - write a vector in terms of polar coordinates
  - differentiate between Cartesian coordinates and curvilinear coordinates
  - write line, surface and volume elements in terms of curvilinear and polar coordinates, and
  - derive expressions for gradient, divergence, curl and Laplacian operators in curvilinear and polar coordinates.

3.2 NON-CARTESIAN COORDINATE SYSTEMS

You are familiar with Cartesian coordinates from your physics and mathematics courses. On the basis of your knowledge, you would agree that Cartesian coordinates are the simplest and often natural. But there are many physical problems where it is advisable to use non-Cartesian coordinates. Some familiar examples include the motion of planets in the
Vector Calculus

solar system, motion of charged particles in electric and magnetic fields, flow of current in a conductor, energy produced in a power plant, temperature distribution in a circular plate and so on. Of the various non-Cartesian coordinate systems, the simplest is the plane polar coordinate system. It is very useful in the study of the motion of planets, oscillations of a simple pendulum and calculation of potential due to a dipole. Since you have learnt about it in your earlier courses (PHE-01, 02), we will discuss it briefly.

3.2.1 Plane Polar Coordinate System

Refer to Fig. 3.1a. It models a flat circular dinner plate. To specify any point P on its surface, you have to draw two fixed coordinate axes at right angles to each other passing through O, the centre of the plate. This is the familiar Cartesian coordinate system. You can locate the point of interest by giving its distance from the two axes. This is denoted as P (x2). You may now ask: Is it the only way to locate this point? The position of point P can also be uniquely defined by measuring its radial distance from the origin and the angle between the x-axis and the line joining the point to the origin, as shown in Fig. 3.1b. Point O is called the pole. The distance p is called the radius vector of the point P and \( \phi \) is its polar angle. These two coordinates taken together are called plane polar coordinates of P. Now you can specify the position of the point P as P (p, \( \phi \)). You would now like to know how x and y are related to p and \( \phi \).

To relate the x and y coordinates of P to its p and \( \phi \) coordinates, you can easily write referring to Fig. 3.1b

\[
\begin{align*}
  x &= p \cos \phi \quad (\infty < x < \infty) \\
  y &= p \sin \phi \quad (\infty < y < \infty)
\end{align*}
\]

(3.1)

It is quite easy to change from Cartesian coordinates to plane polar coordinates. By squaring these relations and adding, you will get

\[
p = \sqrt{x^2 + y^2}
\]

(3.2a)

and on dividing one by the other, you can write

\[
\phi = \tan^{-1} \left( \frac{y}{x} \right)
\]

(3.2b)

We take p to be positive when it is measured from the origin along the line OP. Similarly, \( \phi \) is taken to be positive in the anticlockwise direction from x-axis. The range of variation of these coordinates is given by

\[
0 < p < \infty, \quad 0 \leq \phi < 2\pi
\]

Now you may logically ask: Can we uniquely determine plane polar coordinates for all points? From Eq. (3.2) you will note that only for y = 0, \( \phi \) is undefined. This means that but for the origin, the polar coordinates are uniquely determined.

You will note that the transformation given by Eqs. (3.1) and (3.2) gives a one-to-one correspondence between (x, y) and (p, \( \phi \)) coordinates. This means that in two-dimensional (2-D) geometry, instead of labelling P by the Cartesian coordinates, we can equally well use plane polar coordinates.

Let us now consider the relation \( x^2 + y^2 = 1 \). You would recall that this equation defines a circle. So you can say that in plane polar coordinates, this relation defines a circle of unit radius about the origin. Similarly, \( x^2 + y^2 = 2 \) defines another circle of radius \( \sqrt{2} \) and so on. Thus for different values of p, you would obtain a family of concentric circles about the origin. Similarly, you may recall that the relation \( y = mx + c \) defines a straight line with \( m \) as its intercept on the y-axis. For \( c = 0 \), the straight line will pass through the origin. Does this not mean that \( y = \tan \phi \) for different values of \( \phi \) defines a family of concurrent lines passing through the origin? This is illustrated in Fig. 3.2.

We know that Cartesian coordinate system is orthogonal, i.e., the x and y-axes meet at right angles at any point in the plane and their directions are fixed. Can you say the same about plane polar coordinate system? To discover the answer to these questions, again refer to Fig. 3.2. You will note that each concurrent line cuts the concentric circles at one point only. When you draw a tangent at that point, you will note that the coordinate axes defining the directions of increasing p and increasing \( \phi \) are at right angles. That is, the polar coordinates form an orthogonal coordinate system in the plane. But for two different points \( P_1(p_1, \phi_1) \) and \( P_2(p_2, \phi_2) \)
Let us pause for a minute and summarise our findings about plane polar coordinates.

i) Any point can be represented uniquely by specifying its distance from the origin and the angle radius vector makes with the x-axis.

ii) The Cartesian coordinates are uniquely related to plane polar coordinates of a point $x = p \cos \phi$

$y = p \sin \phi$

(iii) The polar coordinates form an orthogonal coordinate system in a plane.

(iv) The directions of coordinate axes are not the same for all points.

SAQ 1

A simple pendulum is suspended from the ceiling of a lift. It is moving upward with acceleration $a$. The string of effective length of the pendulum makes angle $\phi$ with the vertical. Choose the point of suspension as origin, x-axis along the horizontal and y-axis vertically downwards. Express $x$ and $y$ in terms of plane polar coordinates at time $t$ after the lift takes off.

So far we have confined our discussion to plane polar coordinates. The space is three-dimensional (3-D) and we know of many 3-D systems. The ball hit by a cricketer, veins carrying blood into our heart, pipes carrying water and LPG; and a conductor carrying current are some of the more familiar examples. Some of these involve cylindrical geometry and for them the use of cylindrical polar coordinates appears natural. Let us now discuss these.

3.2.2 Cylindrical Coordinate System

You know that in Cartesian coordinate system, the position of a point $P$ in space is denoted by $P(x, y, z)$. We denote the cylindrical polar coordinates of this point by $P(p, \phi, z)$. By referring to Fig. 3.3, you will note that $p$ and $z$ respectively denote the distance of $P$ from the $z$-axis, and the $xy$ plane while $\phi$ is the angle which the line joining the origin with the projection of $P$ on the $xy$ plane makes with the x-axis. It is called the azimuthal angle.

From the preceding section you may recall that the equation $x^2 + y^2 = p^2$ with $p$ = constant defines a circle in a plane. This suggests that if you add a Cartesian z-axis to the plane polar coordinate system, so that $z$ denotes the perpendicular distance from the $xy$ plane, you would obtain a cylinder. That is, cylindrical polar coordinate system extends the plane polar coordinate system to three dimensions. So if a point has Cartesian coordinates $(x, y, z)$ and cylindrical polar coordinates $(p, \phi, z)$, by analogy with Eq. (3.1) or by referring directly to Fig. 3.3, you can write
Fig. 3.3: Representation of a point in Cartesian and cylindrical coordinates.

\[
\begin{align*}
x &= \rho \cos \phi \quad (0 < \rho < \infty) \\
y &= \rho \sin \phi \quad (0 < \rho < \infty) \\
z &= z \quad (0 < z < \infty)
\end{align*}
\]

As before, you can invert these relations to write

\[
\begin{align*}
\phi &= \tan^{-1}\left(\frac{y}{x}\right) \\
z &= z \\
(0 < \phi < \infty)
\end{align*}
\]

In case of plane polar coordinates, \( \phi \) is undefined at the origin. But in cylindrical coordinates \( \phi \) is undefined for all points on the z-axis \((x=0=y)\).

Fig. 3.4 (a) Contours of constant temperature (b) 2-D plot of countryside hills.

Refer to Fig. 3.4a, which shows contours of constant temperature over different parts of the country on a particular day. (You may have seen it over your TV sets during National News Telecast.) Fig. 3.4b shows a map of countryside containing hills. The map shows contours for area at a given height. These are 2-D representation of a 3-D view. Mathematically speaking, a surface is defined by a relation between the \((x, y, z)\) coordinates, once you fix one of these coordinates. For Cartesian coordinates, these are planes and illustrated in Fig. 3.5. You can visualize these by performing the following activity.

**Activity**

Take a 4" x 6" plane paper. Make a cut mark in the shape of plus (+) sign (size 2" x 3") at the centre using a razor blade. Now take another plane paper of size 2" x 4". Make a 4" cut mark at its centre lengthwise. Insert it in the first paper so that the cut is normal to its plane. Now take another paper of size 2" x 4" and insert it so that it is normal to both of them. These three papers define three plane surfaces for the case of cartesian coordinates. Identify them by comparing with Fig. 3.5. By looking at your activity you can see that the three planes intersect orthogonally at one point.
Fig. 3.5: Schematic representation of planes in Cartesian coordinate system.

Now refer to Fig. 3.6a, which shows a cylinder of radius $p$ and whose axis of symmetry is along the $z$-axis. You will note that for any point on the surface of cylinder, $p$ is constant. That is, $p =$ constant defines a circular cylindrical surface. (This also gives this coordinate system its name.) This is also called the $\phi$-$z$ surface. For different values of $p$, you will obtain co-axial right circular cylinders. Their common axis of symmetry is $z$-axis.

Fig. 3.6: Surfaces defined in cylindrical coordinate system.

SAQ 2

Draw $\phi$-$z$ surfaces for the cylindrical coordinate system.

Can you draw the surface defined by $\phi =$ constant and $z =$ constant? The $\rho$-$z$ surface defined by $\phi =$ constant is half-plane bounded on one edge by the $z$-axis (Fig. 3.6b). But the $\rho$-$\phi$ surface given by $z =$ constant is a plane parallel to the $xy$ plane (Fig. 3.6c) just as in the Cartesian coordinate system.

SAQ 3

Figs. (3.6a-c) depict surfaces defined by $p =$ constant, $\phi =$ constant and $z =$ constant separately. To ensure that you have understood this concept, we would like you to draw these surfaces on one figure.

In the above paragraphs you have learnt that once one coordinate out of the three $(x, y, z)$ or $(p, \phi, z)$ defining a coordinate system is fixed, the relation between them manifests as a surface. What will happen if you fix two of these? You will observe that intersection of $y =$ constant and $z =$ constant surfaces defines the $x$-axis. Similarly, intersection of surfaces given by $z =$ constant and $x =$ constant defines the $y$-axis. What does the intersection of $x =$ constant, and $y =$ constant define? It defines the $z$-axis. In the case of cylindrical coordinates, intersection of surfaces given by $\phi =$ constant and $z =$ constant defines the $\rho$-curve. It is a ray perpendicular to the $z$-axis. What is the shape of (a) $S$-curve given by
$z = \text{constant, } p = \text{constant} \text{ and } (b) \ z$-curve given by $p = \text{constant, } \phi = \text{constant}\text{? While a curve is a horizontal circle centred on the } z\text{-axis, the } z\text{-curve is a line parallel to the } z\text{-axis. These are illustrated in Fig. 3.7.}

![Diagram of coordinate systems](image)

Fig. 3.7: $p, \phi, \text{ and } z\text{-curves}

By now you must have realised that the choice of a coordinate system is determined by the geometry of the system under consideration. Suppose you wish to study the flow of heat in the interior of a sphere or calculate potential at a point due to a uniformly charged sphere. Will the use of Cartesian or cylindrical coordinates look natural? Spherical symmetry suggests that it will be more convenient to define a polar angle. This brings us to spherical polar coordinates. Let us now discuss it.

### 3.2.3 Spherical Polar Coordinate System

In spherical polar coordinate system, the position of a point is specified by the radial distance $r$, the polar angle $\theta$ and the azimuthal angle $\phi$ as shown in Fig. 3.8. You can see that while $\theta$ is measured in the clockwise direction from the $z$-axis, $\phi$ is measured in the anticlockwise direction from the $x$-axis.

![Diagram of spherical polar coordinates](image)

Fig. 3.8: Representation of a point in Cartesian and spherical polar coordinate system.

Referring to this figure you can see that the projection of OP onto the $xy$ plane, $OP' = r \sin \theta$ while its projection on $z$-axis is $r \cos \theta$. The components of $OP'$ along $x$ and $y$-axes are $OP' \cos \phi$ and $OP' \sin \phi$. Hence, for a point $P$ having Cartesian coordinates $(x, y, z)$ and spherical polar coordinates $(r, \theta, \phi)$, you can write

$$
\begin{align*}
x &= OP' \cos \phi = r \sin \theta \cos \phi \\
y &= OP' \sin \phi = r \sin \theta \sin \phi \\
z &= OP \cos \theta = r \cos \theta
\end{align*}
$$

(3.5)

By inverting the expressions for $x, y$ and $z$, you can obtain spherical polar coordinates. For this you should compute squares in each case and add the result. Then use of the identity $\sin^2 \theta + \cos^2 \theta = 1$ simplifies the resultant expression. The results are
The surfaces defined by \( r = \text{constant}, \theta = \text{constant} \) and \( \phi = \text{constant} \) are illustrated in Fig. 3.9. It is instructive to note that in spherical coordinates, \( \phi = \text{constant} \) is the half plane as in cylindrical coordinates.

\[
\begin{align*}
&\text{Sphere} \quad r = \text{constant} \\
&\text{Cone} \quad \theta = \text{constant} \\
&\text{Plane} \quad \phi = \text{constant}
\end{align*}
\]

**Fig. 3.9:** Surfaces defined in spherical polar coordinate system.

**SAQ 4**

Draw a picture of the solid defined by the set of points \((r, \theta, \phi)\) such that

\[
0 \leq r \leq a, \quad 0 \leq \theta \leq \frac{\pi}{4}, \quad 0 \leq \phi \leq 2\pi
\]

Before you proceed, we would like you to go through the following example.

**Example 1**

The Cartesian coordinates of a point are \((2, -2, 3)\) in arbitrary units. Compute its (i) cylindrical coordinates, and (ii) spherical polar coordinates.

**Solution**

i) Cylindrical coordinates

From Eq. (3.4), we have

\[
\begin{align*}
\rho &= \sqrt{x^2 + y^2} = \sqrt{2^2 + (-2)^2} = 2\sqrt{2} \\
\phi &= \tan^{-1} \left( \frac{y}{x} \right) = \tan^{-1} \left( \frac{-2}{2} \right) = \tan^{-1} (-1) \approx \frac{3\pi}{4} \\
\end{align*}
\]

But \(x\) is positive and \(y\) is negative. So \(3\pi/4\) is inadmissible. Hence \(\phi = 7\pi/4\), and \(z = 3\).

ii) Spherical polar coordinates

From Eq. (3.6), we have

\[
\begin{align*}
\rho &= \sqrt{x^2 + y^2 + z^2} = \sqrt{2^2 + (-2)^2 + 3^2} = \sqrt{17} \\
\theta &= \tan^{-1} \left( \frac{\sqrt{x^2 + y^2}}{z} \right) = \tan^{-1} \left( \frac{2\sqrt{2}}{3} \right) \\
\phi &= \tan^{-1} \left( \frac{y}{x} \right) = \tan^{-1} (-1) \approx 7\pi/4
\end{align*}
\]
3.3 EXPRESSING A VECTOR IN POLAR COORDINATES

From Unit 1 you would recall that we can resolve a vector \( \mathbf{A} \) in Cartesian coordinates as

\[
\mathbf{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}
\]

(3.7)

where \( \hat{i} \) and \( \hat{k} \) are unit vectors along the \( x, y, \) and \( z \)-axes, respectively. This means that to express \( \mathbf{A} \) in polar coordinates, we must first relate \( \hat{i}, \hat{j} \) with the unit vectors associated with the system of interest. Let us first consider the cylindrical coordinate system.

Cylindrical coordinate system

For cylindrical coordinate system, we define the unit vectors \( \hat{r}, \hat{\phi}, \hat{z} \) at a given point \( P \) as follows: Let \( R \) be the point on the \( x \)-axis with the same \( z \)-coordinate as \( P \). Then we define \( \hat{r} \) to be the unit vector at \( P \) which is normal to the cylindrical surface \( \rho = \text{constant through } P \). So \( \hat{r} \) will be in the direction of \( \rho \), i.e., along the direction of increasing \( \rho \), as shown in Fig. 3.10a. Similarly, we define \( \hat{\phi} \) to be the unit vector normal to the half plane \( \phi = \text{constant through } P \) in the direction of increasing \( \phi \). The unit vector \( \hat{z} \) is defined normal to the plane \( \phi = \text{constant} \) through \( P \) in the direction of increasing \( z \). Now refer to Fig. 3.10b. You will note that \( \hat{r} \) makes an angle \( \phi \) with the \( x \)-axis, \( \frac{\pi}{2} \) with the \( y \)-axis and \( \frac{3\pi}{2} \) with the \( z \)-axis. So the components of \( \hat{r} \) along the directions of \( \hat{i}, \hat{j} \) and \( \hat{k} \) are, respectively

\[
\hat{r} = \cos \phi \hat{i} + \sin \phi \hat{j}
\]

and

\[
\hat{z} = \cos \frac{\phi}{2} \hat{k}
\]

(3.8)

On combining these results, you can write

\[
\hat{r} = \cos \phi \hat{i} + \sin \phi \hat{j}
\]

(3.9a)

Again, by looking at Fig. 3.10b, you will note that \( \hat{\phi} \) makes an angle \( \frac{\pi}{2} + \phi \) with the \( x \)-axis, \( 4 \) with the \( y \)-axis and \( \frac{3\pi}{2} \) with the \( z \)-axis. Can you now compute the components of \( \hat{\phi} \) along the directions of \( \hat{i}, \hat{j}, \hat{k} \) following the procedures outlined for \( \hat{r} \)? The result is

\[
\hat{\phi} = -\sin \phi \hat{i} + \cos \phi \hat{j} + \hat{k}
\]

(3.9b)

Since \( \hat{\phi} \) is parallel to \( z \)-axis, its projection along the \( x \) and \( y \) axes will be zero. Hence,

\[
\hat{\phi} = \hat{k}
\]

(3.9c)

These relations cap readily be inverted. To this end, you have to multiply Eq. (3.9a) by \( \cos \phi \) and Eq. (3.9b) by \( \sin \phi \). Add the resulting expressions and use the identity

\[
\cos^2 \phi + \sin^2 \phi = 1
\]

On simplification you will get

\[
\hat{i} = \cos \phi \hat{r} - \sin \phi \hat{\phi}
\]

(3.10a)

You can similarly write

\[
\hat{j} = \sin \phi \hat{r} + \cos \phi \hat{\phi}
\]

(3.10b)

and

\[
\hat{k} = \hat{z}
\]

(3.10c)

We know that directions of the Cartesian unit vectors (\( \hat{i}, \hat{j}, \hat{k} \)) are uniquely fixed. Is the same
true for unit vectors $\hat{e}_p$, $\hat{e}_\theta$, and $\hat{e}_\phi$. The directions of these unit vectors vary from point to point. But computing their dot and cross products, you can easily check that these vectors are normal to each other and form a right-handed system.

We will now like you to study the following example carefully.

Example 2
In Cartesian coordinates, the position vector is given by

$$ \mathbf{r} = x \hat{i} + y \hat{j} + z \hat{k} $$

Express $\mathbf{r}$ in terms of cylindrical coordinates $(p, \phi, z)$ and the associated unit vectors $\hat{e}_p$, $\hat{e}_\phi$, $\hat{e}_z$.

Solution
The position vector is given by

$$ \mathbf{r} = x \hat{i} + y \hat{j} + z \hat{k} $$

On substituting for $x$, $y$, and $z$ from Eq. (3.3) and $(\hat{i}, \hat{j}, \hat{k})$ from Eqs. (3.10a, b, c), we get

$$ \mathbf{r} = p \cos \phi \hat{e}_p - p \sin \phi \hat{e}_\phi + \hat{e}_z $$

On collecting the coefficients of $\hat{e}_p$, $\hat{e}_\phi$, and $\hat{e}_z$, we get

$$ \mathbf{r} = p \hat{e}_p + z \hat{e}_z $$

In this example, you have seen that the position vector can be resolved into components parallel to the unit vectors $\hat{e}_p$, $\hat{e}_\phi$, and $\hat{e}_z$. In fact you can do so for any vector and write

$$ \mathbf{A} = A_p \hat{e}_p + A_\phi \hat{e}_\phi + A_z \hat{e}_z $$

(3.11)

where

$$ A_p = A \cos \phi + A_\phi \sin \phi $$

(3.11a)

and

$$ A_\phi = -A \sin \phi + A_\phi \cos \phi $$

(3.11b)

Solution
From Example 2 we recall that position vector of a particle moving through space is given by

$$ \mathbf{r}(t) = p \hat{e}_p + z \hat{e}_z $$

On substituting for $\hat{e}_p$ from Eq. (3.9a), we have

$$ \mathbf{r}(t) = (\rho \cos \phi \hat{e}_p + \rho \sin \phi \hat{e}_\phi) + \hat{e}_z $$

(i)

Since $\hat{e}_z = \hat{k}$.

To compute the velocity of this particle, we have to keep in mind that $\rho$ as well as $\phi$ change with time as $r$ changes. So on differentiating (i) with respect to time, we get

$$ \mathbf{v} = \dot{\mathbf{r}} = \dot{\rho} \hat{e}_p + \rho \dot{\phi} \hat{e}_\phi + \dot{z} \hat{e}_z $$

(ii)

where dot over $\rho$, $\phi$, and $z$ denotes their respective first time derivatives.

On comparing this expression with the equation

$$ \mathbf{v} = v_\rho \hat{e}_p + v_\phi \hat{e}_\phi + v_z \hat{e}_z $$

we find that

$$ v_\rho = \dot{\rho}, \quad v_\phi = \rho \dot{\phi}, \quad \text{and} \quad v_z = \dot{z} $$

(iii)

Spherical Coordinate System

For spherical polar coordinates, we define the unit vectors as follows: At the point
P (r, \theta, \phi), the unit vector \( \hat{e}_r \) is normal to the surface \( r = \text{constant} \), \( \hat{e}_\theta \) is normal to the surface defined by \( \theta = \text{constant} \) and \( \hat{e}_\phi \) is normal to the surface defined by \( \phi = \text{constant} \) through the given point. Their directions are along increasing \( r, \theta \) and \( \phi \) respectively as shown in Fig. 3.11a. By considering the components of the unit vectors \( \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi \) in the directions of unit vectors \( \hat{i}, \hat{j}, \hat{k} \) (Fig. 3.11b) we would like you to show that

\[
\begin{align*}
\hat{e}_r &= \sin \phi \cos \phi + \sin \phi \sin \phi \hat{k} \\
\hat{e}_\theta &= \cos \phi \cos \phi - \cos \phi \sin \phi \hat{k} \\
\hat{e}_\phi &= -\sin \phi \hat{k}
\end{align*}
\]

Conversely

\[
\begin{align*}
\hat{i} &= \sin \phi \cos \phi + \cos \phi \cos \phi \hat{k} \\
\hat{j} &= \sin \phi \sin \phi + \cos \phi \sin \phi \hat{k} \\
\hat{k} &= \cos \phi \hat{k}
\end{align*}
\]

Equations (3.12)

In case you cannot establish these results now, you will get another opportunity in the next section. So you should not feel depressed.

**Example 4.**

A particle is moving in space. Express its position vector and components of its velocity in spherical coordinates.

**Solution**

The position vector of a particle moving in space is written as

\[
r = x \hat{i} + y \hat{j} + z \hat{k}
\]

Substituting for \( x, y, z \) from Eq. (3.5) and \( \hat{i}, \hat{j}, \hat{k} \) from Eq. (3.13) you will get

\[
r = r \sin \phi \cos \phi \hat{i} + r \sin \phi \sin \phi \hat{j} + r \cos \phi \hat{k}
\]

On collecting the coefficients of \( \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi \) this expression gives us the required result:

\[
r = \hat{r}, (r \sin^2 \phi \cos \phi + r \sin \phi \sin \phi + r \cos \phi) \\
+ \hat{\theta} (r \sin \phi \cos \phi + r \sin \phi \sin \phi + r \cos \phi) \\
+ \hat{\phi} (r \sin \phi \cos \phi + r \sin \phi \sin \phi + r \cos \phi) \\
\]
In spherical polar coordinates, the position vector of a particle moving in space is given by

\[ \mathbf{r} = r \hat{\mathbf{r}} = 0 \sin^2 \phi \mathbf{\hat{r}} + \sin \phi \mathbf{\hat{\psi}} \]

Differentiating it with respect to time, you will get

\[ \mathbf{v} = \mathbf{\hat{r}} \cdot \mathbf{\hat{r}} + r \dot{\theta} \mathbf{\hat{\phi}} + r \sin \phi \mathbf{\hat{\theta}} \]

You must have realised that while evaluating the velocity components in non-Cartesian coordinates, we first expressed polar unit vectors in terms of Cartesian unit vectors. These expressions were differentiated and the result was re-expressed in terms of \((\mathbf{\hat{r}}, \mathbf{\hat{\phi}}, \mathbf{\hat{\theta}})\) and \((\mathbf{\hat{e}}_r, \mathbf{\hat{e}}_\phi, \mathbf{\hat{e}}_\theta)\). This involves straightforward algebra and we would like you to understand it rather well. You are therefore advised to carefully go through the following example where we have computed acceleration in cylindrical coordinates.

Example 5

Compute the components of acceleration in cylindrical coordinates for a particle moving in space.

**Solution**

From Example 3 you would recall that

\[ \mathbf{v} = \rho \mathbf{\hat{r}} + \rho \dot{\phi} \mathbf{\hat{\phi}} + z \mathbf{\hat{z}} \]  

On differentiating this with respect to time, we get

\[ \mathbf{a} = \mathbf{\hat{r}} \cdot \mathbf{\hat{r}} + \rho \ddot{\rho} \mathbf{\hat{r}} + \rho \dot{\phi} \mathbf{\hat{\phi}} + \rho \dot{\phi} \mathbf{\hat{r}} + \rho \ddot{\phi} \mathbf{\hat{\phi}} + \rho \dot{\theta} \mathbf{\hat{z}} \]  

To evaluate time derivatives of \( \mathbf{\hat{r}}, \mathbf{\hat{\phi}}, \mathbf{\hat{z}} \), we recall that

\[ \dot{\mathbf{\hat{r}}} = \mathbf{\hat{\theta}} \sin \phi + \mathbf{\hat{\phi}} \cos \phi \]

\[ \dot{\mathbf{\hat{\phi}}} = -\mathbf{\hat{r}} \sin \phi - \mathbf{\hat{z}} \cos \phi \]

\[ \dot{\mathbf{\hat{z}}} = \mathbf{\hat{\phi}} \]

Hence

\[ \dot{\mathbf{\hat{r}}} = (-\mathbf{\hat{r}} \sin \phi + \mathbf{\hat{\phi}} \cos \phi) \mathbf{\hat{\phi}} = \mathbf{\hat{e}}_\phi \mathbf{\hat{\phi}} \]

\[ \dot{\mathbf{\hat{\phi}}} = (-\mathbf{\hat{r}} \cos \phi - \mathbf{\hat{z}} \sin \phi) = -\mathbf{\hat{e}}_\phi \mathbf{\hat{r}} \]

\[ \dot{\mathbf{\hat{z}}} = \mathbf{\hat{\phi}} \]

Using these results in (ii), we get

\[ \mathbf{a} = \left[ \ddot{\rho} - \rho (\dot{\phi})^2 \right] \mathbf{\hat{r}} + \left[ \rho \ddot{\phi} + 2 \dot{\rho} \dot{\phi} \right] \mathbf{\hat{\phi}} + \rho \sin \phi \mathbf{\hat{z}} \]

\[ \mathbf{a} = \left[ \ddot{\rho} - \rho (\dot{\phi})^2 \right] \mathbf{\hat{r}} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho^2 \dot{\phi} \right) \mathbf{\hat{\phi}} + \rho \sin \phi \mathbf{\hat{z}} \]  

So the cylindrical components of acceleration of a particle moving in space are

\[ a_r = \ddot{\rho} - \rho (\dot{\phi})^2, \quad a_\phi = \rho \ddot{\phi}, \quad a_z = \rho \sin \phi \]  

A special case of cylindrical coordinates is plane polar coordinate system in which the \( z \)-coordinate is not present. So expressions for velocity and acceleration in the plane polar coordinate system are respectively given by (i) and (iii) with \( z = 0 \).
In the preceding examples you have learnt to express the position vector as well as its first and second time derivatives (velocity and acceleration) in polar coordinates. Now consider a variable force $F(t)$ acting on a body. Suppose it displaces the body through $dI$. (In mathematical language, $dI$ is a differential element of vector length.) Then, the work done by this force on the body is given by $W = F \cdot dI$. Similarly, the magnetic flux coming out of a surface of area $dA$ is given by the relation $\Phi = B \cdot dA$, where $n$ is unit normal in the outward direction. You must, therefore, know to express differential elements of vector length, in terms of non-Cartesian polar coordinates. Another important quantity is differential volume element, which is a scalar. You will use these expressions extensively in the next unit. So we would like you to master the basic technique rather than remembering these formulae by heart.

### 3.3.1 Differential Element of Vector Length

From Example 4 you would recall that in spherical polar coordinates, the position vector of a point moving through space is given by

$$ \mathbf{r} = r \hat{e}_r $$

On substituting for $\hat{e}_r$ from Eq. (3.12), we have

$$ \mathbf{r} = r \left( \hat{e}_r \cos \theta + \hat{e}_\theta \sin \theta \sin \phi + \hat{e}_\phi \cos \theta \right) $$

To computed $r$, you would recall that $r$, $\theta$ and $\phi$ will change as $r$ changes. Hence, you can write

$$ dr = dr \left( \hat{e}_r \cos \theta + \hat{e}_\theta \sin \theta \sin \phi + \hat{e}_\phi \cos \theta \right) $$

In spherical coordinates, small line elements along $\phi$, $\theta$ and $\hat{e}_\phi$ are given by $d\phi$, $d\theta$ and $r \sin \theta d\phi$, respectively. On collecting coefficients of $dr$, $r d\theta$ and $r \sin \theta d\phi$, this expression becomes

$$ dr = \left( \hat{e}_r \cos \theta + \hat{e}_\theta \sin \theta \sin \phi + \hat{e}_\phi \cos \theta \right) \cdot dr $$

Combining this with Eq. (3.12), we find that differential element of vector length in spherical polar coordinates can be written as

$$ dr = \hat{e}_r dr + \hat{e}_\theta r d\theta + \hat{e}_\phi r \sin \theta d\phi $$

(SAQ 5)

**Spend 5 min**

In cylindrical polar coordinates, the position vector of a point moving through space is given by

$$ \mathbf{r} = \rho \hat{e}_\rho + \zeta \hat{e}_\zeta $$

Show that a small change $d\mathbf{r}$ in the position vector can be written as

$$ d\mathbf{r} = \hat{e}_\rho d\rho + \hat{e}_\phi \rho d\phi + \hat{e}_\zeta d\zeta $$

(SAQ 5)

where $d\rho$, $\rho d\phi$ and $d\zeta$ denote small line element in the increasing directions of $\rho$, $\phi$ and $\zeta$, respectively.

### 3.3.2 Differential Element of Vector Area

You now know to express the line element in terms of polar coordinates. On re-examining Eq. (3.14) you can say that in spherical coordinates $d\mathbf{r}$, $r d\theta$ and $r \sin \theta d\phi$ are respectively, analogous to $d\mathbf{x}$, $d\mathbf{y}$ and $d\mathbf{z}$ in Cartesian coordinate system. Similarly from Eq. (3.15) you can identify that in cylindrical coordinates, $d\rho$, $\rho d\phi$ and $d\zeta$ are analogous to $d\mathbf{x}$, $d\mathbf{y}$ and $d\mathbf{z}$, respectively. You can use this analogy to obtain expressions for components of differential element of vector area. In Cartesian coordinate system, the area of the face normal to $x$-axis is given by

$$ dA_x = dy dz $$

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You can use the same definition for non-Cartesian coordinates as well. In terms of cylindrical coordinates, you can write
\[ dA = (dr)_\theta \times (dr)_r = \rho d\phi \, dz \] (3)

Similarly
\[ dA_\rho = (dr)_\rho \times (dr)_\phi = d\rho \, dz \] (3.16b)

and
\[ dA_\phi = \rho \, d\phi \, d\rho \] (3.16c)

On combining Eqs. (3.16a, b, c), you can express the differential element of vector area in cylindrical coordinates as
\[ dA = dA_\rho \hat{e}_\rho + dA_\phi \hat{e}_\phi + dA_\theta \hat{e}_\theta \] (3.17)

You can follow the same procedure and show that differential element of vector area in spherical coordinates can be expressed as
\[ dA = r^2 \sin \theta \, d\theta \, d\phi \, d\rho + r \, d\theta \, d\phi \, dr \hat{e}_r \] (3.18)

SAQ 6
Show that the volume element in cylindrical and spherical polar coordinates can be written as
\[ dV = \rho \, d\phi \, d\rho \, dz \] (3.18a)

and
\[ dV = r^2 \sin \theta \, d\theta \, d\phi \, dr \] (3.18b)

These are illustrated in Fig. 3.12.

Let us pause for a minute and ask: What have we achieved so far? In this section you have learnt to express a vector in terms of polar coordinates. In physics, \( V \) is a very special vector operator. In Unit 2 you learnt that it can operate on real-valued functions, i.e., scalar fields.
While operating on a scalar field, it gives us a vector field. For instance, the negative gradient of electrostatic potential defines electric field and gradient of temperature is a measure of the heat flux vector. So whenever you wish to calculate electric field due to a uniformly charged conducting sphere or a cylindrical transmission cable, you must know to express the gradient operator in terms of polar coordinates. Let us now learn to do so.

### 3.3.3 Gradient of a Scalar Field

To express gradient of a scalar field \( f \) in cylindrical coordinates, let us first write \( \nabla f \) in terms of its components:

\[
\nabla f = \hat{e}_\rho \frac{\partial f}{\partial \rho} + \hat{e}_\phi \frac{\partial f}{\partial \phi} + \hat{e}_z \frac{\partial f}{\partial z}
\]

(3.19)

where \( \hat{e}_\rho, \hat{e}_\phi \) and \( \hat{e}_z \) are unknown functions and we have to determine them. To this end, let us compute the dot product of \( \hat{e}_\rho \) and \( \nabla f \). Since \( \hat{e}_\rho, \hat{e}_\phi \) and \( \hat{e}_z \) are orthogonal, we get

\[
\hat{e}_\rho \cdot \nabla f = \frac{\partial f}{\partial \rho}
\]

That is, \( \frac{\partial f}{\partial \rho} \) is the spatial rate of change off in the direction of unit vector \( \hat{e}_\rho \). At the point \( P(\rho, \phi, z) \), \( \frac{\partial f}{\partial \rho} \) signifies the rate of change off in the direction of increasing \( \rho \) (with \( \phi \) and \( z \) held constant). Refer to Fig. 3.13a. You will observe that the distance between the points \( (\rho, \phi, z) \) and \( (\rho + \Delta \rho, \phi, z) \) is \( \Delta \rho \). So when these points are close together, you can write

\[
\lim_{\Delta \rho \to 0} \frac{f(\rho + \Delta \rho, \phi, z) - f(\rho, \phi, z)}{\Delta \rho} = \frac{\partial f}{\partial \rho}
\]

where \( \frac{\partial f}{\partial \rho} \) denotes partial derivative with respect to \( \rho \).

Thus, the component of \( \nabla f \) in the \( \hat{e}_\rho \) direction is

\[
\frac{\partial f}{\partial \rho} \hat{e}_\rho
\]

(3.20a)

Similarly, \( \frac{\partial f}{\partial \phi} \hat{e}_\phi \) is the component of \( \nabla f \) in the \( \hat{e}_\phi \) direction, i.e., \( \hat{e}_\phi \cdot \nabla f = \frac{\partial f}{\partial \phi} \hat{e}_\phi \). But \( \hat{e}_\phi \cdot \nabla f \) denotes the rate of change off in the direction of the unit vector \( \hat{e}_\phi \), which is equal to

\[
\lim_{\Delta \phi \to 0} \frac{f(\rho, \phi + \Delta \phi, z) - f(\rho, \phi, z)}{\Delta \phi} = \frac{\partial f}{\partial \phi}
\]

This form of the term in the denominator of this limit arises because the distance between the points \( (\rho, \phi, z) \) and \( (\rho, \phi + \Delta \phi, z) \) is \( \rho \Delta \phi \) (Fig. 3.13b). Hence

\[
\frac{\partial f}{\partial \phi} \hat{e}_\phi
\]

(3.20b)

Finally, you can write the component of \( \nabla f \) in the \( \hat{e}_z \) direction, \( \frac{\partial f}{\partial z} \hat{e}_z \) (Fig. 3.13c):

\[
\frac{\partial f}{\partial z} \hat{e}_z
\]

(3.20c)

On combining Eqs. (3.19) and (3.20a-c), you will obtain

\[
\nabla f = \frac{\partial f}{\partial \rho} \hat{e}_\rho + \frac{\partial f}{\partial \phi} \hat{e}_\phi + \frac{\partial f}{\partial z} \hat{e}_z
\]

(3.21)

By the same method you can compute the expression for the gradient of a scalar in spherical coordinates. To ensure that you have understood the procedure, we would like you to do it and check your answer with that given in Sec. 3.5.
You can similarly work out expressions for divergence, curl, and the Laplace operator. However, mathematical steps get involved. Normally we circumvent this inconvenience by working in terms of curvilinear coordinates. This is a general coordinate system and can be specialised to any system of interest. Let us, therefore, now proceed to understand the nature of curvilinear coordinate system.

### 3.4 CURVILINEAR COORDINATE SYSTEM

We know that in the Cartesian coordinate system in 3-D space, the position of a point P is denoted by \((x, y, z)\). These can be related to other coordinates through a set of functional relations. You now know these relations for cylindrical and spherical coordinates (Eqs. (3.3) and (3.5), respectively). Let us now generalise these relations by introducing three new quantities \(u_1, u_2, u_3\) by writing the functional relationship as

\[
x = x(u_1, u_2, u_3) \\
y = y(u_1, u_2, u_3) \\
z = z(u_1, u_2, u_3) \tag{3.22}
\]

You can invert these relations to express \(u_1, u_2, u_3\) in terms of \(x, y, z\):

\[
u_1 = u_1(x, y, z) \\
u_2 = u_2(x, y, z) \\
u_3 = u_3(x, y, z) \tag{3.23}
\]

You will note that the transformations given by Eqs. (3.22) and (3.23) give a one-to-one correspondence between \((x, y, z)\) and \((u_1, u_2, u_3)\). That is, you can label a given point either by the coordinates \((x, y, z)\) or \((u_1, u_2, u_3)\). The values \((u_1, u_2, u_3)\) corresponding to a given point \(P(x, y, z)\) are called the curvilinear coordinates of \(P\) and Eq. (3.23) defines a curvilinear coordinate system. Geometrically, the transformation from \((x, y, z)\) to \((u_1, u_2, u_3)\) may be viewed as a mapping from the Cartesian space to curvilinear space. Let us choose \(u_1 = C_1\) (a constant). Then, from Sec. 3.2 we would recall that

\[
u_1(x, y, z) = C_1 \tag{3.24a}
\]

represents a surface, say \(S_1\), in \(u\)-space, as shown in Fig. 3.14. Similarly, if you choose \(u_2 = C_2\) and \(u_3 = C_3\), the equations

\[
u_2(x, y, z) = C_2 \tag{3.24b}
\]

and

\[
u_3(x, y, z) = C_3 \tag{3.24c}
\]

represent surfaces \(S_2\) and \(S_3\), say. These surfaces are also shown in Fig. 3.14. They intersect at point \(P\) whose Cartesian coordinates \((x, y, z)\) can be obtained by solving Eqs. (3.24a, b, c).
From Sec. 3.2 you would recall that intersection of two surfaces at a point defines a coordinate curve through that point. By referring to Fig. 3.14, you will note that intersection of surfaces $S_1$ and $S_2$ defines the $u_1$-coordinate curve. Similarly, the intersection of $S_2$ and $S_3$ surfaces defines the $u_2$-coordinate curve and so on. You will note that, unlike in the case of Cartesian coordinate system, the $u_i$ ($i = 1, 2, 3$) curves are not straight lines. For this reason, the directions of curvilinear coordinate axes are determined by drawing tangents to the coordinate curve at a point. Moreover, they vary from one point to another point in space, which means that you cannot define one coordinate system for every point in $u$-space. That is, for curvilinear coordinate system, you have to define unit vectors for each point separately.

A vector $F$ may be written in terms of its components $F_1, F_2, F_3$ as

$$F = F_1 \hat{e}_1 + F_2 \hat{e}_2 + F_3 \hat{e}_3$$

where $\hat{e}_1, \hat{e}_2,$ and $\hat{e}_3$ are respective unit vectors along tangents to $(u_1, u_2, u_3)$ coordinate lines at the given point.

So far our discussion of curvilinear coordinate system has been somewhat abstract. To be precise, we should know the orthogonality condition for curvilinear coordinates. We should be able to specify the nature of unit vectors. This involves the calculation of partial derivatives. Let us first understand what a partial derivative is and how to calculate it.

**Partial Differentiation**

Let us re-examine Fig. 3.1a. Here $P(x, y)$ is a point in 2-D Cartesian space. Starting from the origin $O$, you can reach $P$ along two different routes. You can move along $x$-axis, i.e., keep $y$ fixed. Then keeping $x$ fixed you can move parallel to $y$-axis. Alternatively, you can first go along $y$-axis and then keeping $y$ fixed, move parallel to $x$-axis. We use a similar technique to compute changes in a function which depends on two (or more) variables. From PHE-06 course on Thermodynamics and Statistical Mechanics you may recall many such situations. The temperature of a gas is a function of its volume and pressure. The internal energy of a paramagnetic salt depends on temperature and magnetic field strength, and so on. In all such cases we express the change in the dependent variable as a sum of two terms. Each term is a product of the rate of change of dependent variable with respect to one variable, keeping other constant, times the change in that variable. For instance, to calculate the change in temperature of the gas, we first keep volume constant and let pressure vary. Then

$$\frac{dT}{dp} \Delta p$$

denotes the product of the rate of change of temperature with pressure at constant $V$ and the change $\Delta p$ in pressure. Next, we keep $p$ fixed and calculate the change in temperature due to changes in volume and multiply the result with $\frac{dT}{dV}$ to obtain

$$\frac{dT}{dV} \Delta V.$$ Hence, the **net** change in temperature is given by

$$dT = \left(\frac{dT}{dp}\right)_V \Delta p + \left(\frac{dT}{dV}\right)_p \Delta V.$$ (3.25)

The subscripts $V$ and $p$ denote that these variables have been kept fixed.

You may now consider a function of three variables $X = X(x, y, z)$. By induction, a change in $X$ can be written as

$$dX = \left(\frac{dX}{dp}\right)_v \Delta p + \left(\frac{dX}{dV}\right)_p \Delta V + \left(\frac{dX}{dz}\right)_T \Delta T.$$ (3.26)

Let us now proceed to discover the orthogonality condition for the curvilinear coordinate system.

**3.4.1 Orthogonality Condition**

Consider a point $P(x, y, z)$ in Cartesian space. If its position vector is $\mathbf{r}$, you would recall from Unit 1 that

$$d\mathbf{r} = \hat{i} \, dx + \hat{j} \, dy + \hat{k} \, dz$$
and square of the distance between two neighbouring points along a curve is given by
\[(ds)^2 = dr^2 + (dx)^2 + (dy)^2 + (dz)^2\] (3.27)

Using Eq. (3.26) you can express \(dx, dy, dz\) in terms of \(u_1, u_2, u_3\) through the relations

\[
dx = \frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2 + \frac{\partial x}{\partial u_3} du_3
\]

\[
dy = \frac{\partial y}{\partial u_1} du_1 + \frac{\partial y}{\partial u_2} du_2 + \frac{\partial y}{\partial u_3} du_3
\]

and

\[
dz = \frac{\partial z}{\partial u_1} du_1 + \frac{\partial z}{\partial u_2} du_2 + \frac{\partial z}{\partial u_3} du_3
\]

For convenience, we have dropped the subscripts \(u_1, u_2, u_3\).

On substituting for \((dx)^2, (dy)^2\) and \((dz)^2\) in Eq. (3.27) and collecting the coefficients of \(du_i du_j\) \((i, j = 1, 2, 3)\) you can rewrite Eq. (3.27) as

\[
(ds)^2 = g_{11} (du_1)^2 + 2 g_{12} du_1 du_2 + g_{22} (du_2)^2
\]

\[
+ 2 g_{13} du_1 du_3 + g_{33} (du_3)^2 + 2 g_{23} du_2 du_3
\] (3.28)

where

\[
g_{ii} = \frac{\partial x}{\partial u_i} \frac{\partial x}{\partial u_i} + \frac{\partial y}{\partial u_i} \frac{\partial y}{\partial u_i} + \frac{\partial z}{\partial u_i} \frac{\partial z}{\partial u_i}
\] (3.29)

are referred to as the metric coefficients of the curvilinear coordinate system. In the following example we have illustrated the calculation of metric coefficients for cylindrical coordinate system.

**Example 6**

In cylindrical coordinates, \(u_1 = \rho, u_2 = \theta, u_3 = z\) and

\[x = \rho \cos \theta\]

\[y = \rho \sin \theta\]

\[z = z\]

Compute \(g_{ij}\)'s and the arc length.

**Solution**

From Eq. (3.29), we have

\[g_{11} = \left(\frac{\partial x}{\partial \rho}\right)^2 + \left(\frac{\partial y}{\partial \rho}\right)^2 + \left(\frac{\partial z}{\partial \rho}\right)^2\]

\[= \left(\frac{\partial x}{\partial \rho}\right)^2 + \left(\frac{\partial y}{\partial \rho}\right)^2 + \left(\frac{\partial z}{\partial \rho}\right)^2\]

\[= \cos^2 \theta + \sin^2 \theta + 0\]

\[= 1\]

\[g_{12} = \frac{\partial x}{\partial \rho} \frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \rho} \frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \rho} \frac{\partial z}{\partial \theta}\]

\[= \frac{\partial x}{\partial \rho} \frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \rho} \frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \rho} \frac{\partial z}{\partial \theta}\]

\[= \cos \theta (-\rho \sin \theta) + \sin \theta (\rho \cos \theta) + 0\]

\[= 0\]

\[g_{22} = \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta} \frac{\partial z}{\partial \theta}\]

\[= \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta} \frac{\partial z}{\partial \theta}\]

\[= \cos \theta (-\rho \cos \theta) + \sin \theta (-\rho \sin \theta) + 0\]

\[= \cos^2 \theta + \sin^2 \theta + 0\]

\[= 1\]

\[g_{33} = \frac{\partial x}{\partial z} \frac{\partial x}{\partial z} + \frac{\partial y}{\partial z} \frac{\partial y}{\partial z} + \frac{\partial z}{\partial z} \frac{\partial z}{\partial z}\]

\[= \frac{\partial x}{\partial z} \frac{\partial x}{\partial z} + \frac{\partial y}{\partial z} \frac{\partial y}{\partial z} + \frac{\partial z}{\partial z} \frac{\partial z}{\partial z}\]

\[= 1\]

\[\Rightarrow (ds)^2 = \rho^2 (d\rho)^2 + \rho^2 \sin^2 \theta (d\theta)^2 + (dz)^2\]
Similarly, you will find that $g_{33}=g_{33}=0$. A coordinate system is said to be orthogonal if $g_{ij}\neq0,\forall i\neq j$. In cylindrical coordinates, the expression for square of the arc element is given by

$$(dx)^2 = (dy)^2 + r^2(\sin\theta)^2 + (dz)^2$$

In plane polar coordinates, the third term in this expression would drop out.

<table>
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<th>SAQ 7</th>
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In spherical polar coordinates, $u_i = r, du_i = 0, u_3 = \phi$. Using Eq. (3.28), show that the expression for square of the arc element is given by

$$(dx)^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2(\theta) (d\phi)^2$$

In Example 6 you have seen that the condition for a curvilinear coordinate system to be orthogonal is

$$g_{ij} = 0 \text{ for } i \neq j \quad (3.30)$$

So for an orthogonal curvilinear coordinate system, Eq. (3.28) for square of arc length takes a very compact form:

$$dr\,dt = (dx)^2 = g_{11} (du_1)^2 + g_{22} (du_2)^2 + g_{33} (du_3)^2$$

$$= (h_1\, du_1)^2 + (h_2\, du_2)^2 + (h_3\, du_3)^2$$

where we have put $g_{ii} = h_i^2$ (i = 1, 2, 3). This means that for cylindrical polar-coordinate system

$$(du_1)^2 = (dr)^2 + r^2 (d\theta)^2$$

and

$$(du_2)^2 = r^2 (d\phi)^2$$

Since you have worked out SAQ 7, you can show that for spherical polar coordinate system

$$(du_1)^2 = (dr)^2 + r^2 \sin^2\theta (d\theta)^2$$

and

$$(du_2)^2 = r^2 \sin^2\theta (d\phi)^2$$

Let us pause for a minute and ask: What physical meaning can we assign to the coefficients $h_1, h_2, h_3$? To answer this question, we note that when an element of arc $dx$ is directed along the $u_i$ coordinate line, $du_i = du_i = 0$. Then Eq. (3.31) gives

$$(dx)^2 = (du_1)^2 + (du_2)^2 + (du_3)^2$$

Similarly, you will find that $g_{33} = g_{33} = 0$. A coordinate system is said to be orthogonal if $g_{ij} \neq 0, \forall i \neq j$. In cylindrical coordinates, the expression for square of the arc element is given by

$$(dx)^2 = (dy)^2 + r^2(\sin\theta)^2 + (dz)^2$$

In plane polar coordinates, the third term in this expression would drop out.
so that
\[ ds_1 = h_1 du_1 \]  
(3.34a)

That is, the length of the arc \( ds \) along the \( u_1 \)-coordinate line is the product of \( dh_1 \) and the differential of \( u_1 \). Similarly, you can show that the components of length of arc \( ds \) along the \( u_2 \) and \( u_3 \) coordinate lines are
\[ ds_2 = h_2 du_2 \]  
(3.34b)
\[ ds_3 = h_3 du_3 \]  
(3.34c)

Since \( ds_1 \) and \( ds_2 \) are real, you can say that \( h_i \) (\( i = 1, 2, 3 \)) are positive quantities.

Since we are considering an orthogonal coordinate system, you should compute the dot product \( dr \cdot dr \). Since we are considering an orthogonal coordinate system,

\[ dr = \sum_{i=1}^{3} \frac{\partial r}{\partial u_i} du_i \]  
(3.35)

You will recognise that the symbol \( \frac{\partial r}{\partial u_i} \) denotes the derivative of \( r \) with respect to a particular variable \( u_i = (i = 1, 2, 3) \). This implies that if we fix \( u_2 \) and \( u_3 \), \( r \) becomes a function of \( u_1 \) alone. That is, the terminus of \( r \) will move along the \( u_1 \) coordinate line in the \( u \)-coordinate system (Unit 1). So the vector
\[ \frac{\partial r}{\partial u_1} \]  
(3.36)

will be tangent to the coordinate line \( u_1 \) at point \( P \). Similarly, we can say that vector
\[ \frac{\partial r}{\partial u_2} \]  
(3.37)

are, respectively, tangent to \( u_2 \) and \( u_3 \)-coordinate lines, as shown in Fig. 3.15. If we denote these vectors by \( a_i \), i.e.

\[ a_i = \frac{\partial r}{\partial u_i} \]  
(3.38)

Eq. (3.35) can be rewritten as

\[ dr = \sum_{i=1}^{3} a_i du_i \]  
(3.39)

The vectors \( a_i \) are called base vectors in the curvilinear coordinate system. Can you draw an analogy between the base vectors \( a_1, a_2, a_3 \) and the unit vectors \( \hat{i}, \hat{j}, \hat{k} \)? The two sets of vectors are analogous in that they can resolve any vector \( F \) into its components \( F_x, F_y, F_z \) as shown in Fig. 3.15. However, \( a_1, a_2, a_3 \) are not unique; these are defined at a point and vary from point to point. Are they unit vectors? To discover the answer to this question, you should compute the dot product \( dr \cdot dr \). Since we are considering an orthogonal coordinate system,
Then, on comparing the coefficients of \((du_1)^2, (du_2)^2\) and \((du_3)^2\) with Eq. (3.31), you will get \(|a_i| = h_i (i = 1, 2, 3)\). So the base vector \(a_i\) can be written as

\[a_i = h_i \hat{e}_i (i = 1, 2, 3)\]  

where \(\hat{e}_i\) is a unit vector along \(a_i\)-coordinate line.

Using this result in Eq. (3.37) we find that

\[dr = \sum_{i=1}^{3} h_i du_i \hat{e}_i\]  

(3.39)

For a given \(i, \hat{e}_i\) can be expressed as

\[\hat{e}_i = \frac{1}{h_i} \frac{\partial r}{\partial u_i}\]  

(3.40)

Before proceeding further, you must study the following example carefully where we have illustrated the method to evaluate the unit vectors for cylindrical and spherical polar coordinates.

Example 7

Starting from Eq. (3.40) evaluate \(\hat{e}_i\)'s for cylindrical and spherical polar coordinate systems.

Solution

The cylindrical polar coordinate system is defined by \(u_1 = \rho, u_2 = \phi, u_3 = z, h_1 = 1, h_2 = \rho, h_3 = 1\). So from Eq. (3.40) we can write

\[\hat{e}_1 = \hat{e}_2 = \frac{1}{h_1} \frac{\partial \rho}{\partial \rho} = \frac{\partial \rho}{\partial \rho} \left( \frac{\rho \cos \phi}{\rho \sin \phi} + \frac{\rho \sin \phi}{\rho \cos \phi} \right) = \hat{c} \cos \phi + \hat{c} \sin \phi\]

\[\hat{e}_3 = \frac{1}{h_3} \frac{\partial z}{\partial \phi} = \hat{c}\]  

(i)

\[\hat{e}_2 = \frac{1}{h_2} \frac{\partial r}{\partial \phi} = \frac{\rho \cos \phi + \rho \sin \phi}{\rho} = \hat{c} \cos \phi + \hat{c} \sin \phi\]

(ii)

and

\[\hat{e}_3 = \frac{1}{h_3} \frac{\partial r}{\partial z} = \hat{c}\]  

(iii)

You will note that (i), (ii), (iii) are identical with Eqs. (3.9 a, b, c).

The spherical polar coordinate system is defined by \(u_1 = r, u_2 = \theta, u_3 = \phi, h_1 = 1, h_2 = r, h_3 = \sin \theta\). Again referring to Eq. (3.40), we can write

\[\hat{e}_1 = \frac{1}{h_1} \frac{\partial \rho}{\partial \rho} = \frac{\partial \rho}{\partial \rho} \left( \frac{\rho \sin \theta \cos \phi + \rho \sin \theta \sin \phi}{\rho} \right) = \hat{c} \sin \theta \cos \phi + \hat{c} \sin \theta \sin \phi + \hat{c} \cos \theta\]

\[\hat{c} \sin \theta \cos \phi + \hat{c} \sin \theta \sin \phi + \hat{c} \cos \theta\]

\[\hat{e}_2 = \frac{1}{h_2} \frac{\partial \rho}{\partial \theta} = \frac{\partial \rho}{\partial \theta} \left( \frac{\cos \phi}{\sin \theta} \right) = \hat{c} \cos \theta \cos \phi + \hat{c} \cos \theta \sin \phi - \hat{c} \sin \theta\]

\[= \hat{c} \cos \theta \cos \phi + \hat{c} \cos \theta \sin \phi - \hat{c} \sin \theta\]

and

\[\hat{e}_3 = \frac{1}{h_3} \frac{\partial \rho}{\partial \theta} = \frac{1}{\sin \theta} \frac{\partial \rho}{\partial \phi} = \frac{\partial \rho}{\partial \phi} \left( \frac{\rho \sin \theta \cos \phi}{\rho \sin \theta \cos \phi} \right) = \hat{c} \sin \theta \cos \phi + \hat{c} \sin \theta \sin \phi + \hat{c} r \cos \theta\]

\[= \hat{c} \sin \theta \cos \phi + \hat{c} \sin \theta \sin \phi + \hat{c} r \cos \theta\]  

You will note that (i), (ii), (iii) are identical with Eqs. (3.9 a, b, c).
3.4.3 Vector Differential Operators

Now that we have introduced the curvilinear coordinate system, we will develop expressions for vector differential operators—the divergence, the curl and the Laplacian. Let us begin with the computation of divergence of a vector field.

**Divergence**

Consider a vector field \( \mathbf{F} \). We can express it in terms of its components \( F_1, F_2, F_3 \) along \( \mathbf{e}_1, \mathbf{e}_2, \) and \( \mathbf{e}_3 \) as:

\[
\mathbf{F}(u_1, u_2, u_3) = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2 + F_3 \mathbf{e}_3
\]

In summation notation, it can be written in a compact form:

\[
\nabla \cdot \mathbf{F} = \sum_{i=1}^{3} \frac{\partial F_i}{\partial u_i}
\]

We have preferred to use this notation as it makes expressions look more elegant. The divergence of \( \mathbf{F} \) is then given by:

\[
\nabla \cdot \mathbf{F} = \sum_{i=1}^{3} \left( \frac{\partial F_i}{\partial u_i} \right)
\]

From Unit 2, you would recall that divergence of the product of a scalar and a vector is the sum of the scalar times the divergence of the vector and the dot product of the vector with gradient of the scalar. So you can write

\[
\nabla \cdot (\mathbf{a} \mathbf{F}) = \mathbf{a} \cdot \nabla \mathbf{F} + \mathbf{F} \cdot \nabla \mathbf{a}
\]

Thus, to evaluate \( \nabla \cdot \mathbf{F} \), you must know \( \nabla \mathbf{e}_i \) and \( \nabla F_i \). From Unit 2 you would recall that \( \nabla F_i \) is a vector having the magnitude and direction of the maximum space rate of change of \( F_i \). So the component of \( \nabla F_i \) in the direction normal to the surface defined by \( u_i \) = constant is given by

\[
\nabla F_i \parallel = \frac{\partial F_i}{\partial u_i}
\]

where \( d s_i \) is a differential length in the direction of increasing \( u_i \). This direction is indicated by the unit vector \( \mathbf{e}_i \). Using Eq. (3.42) we can rewrite it as

\[
\nabla F_i \parallel = \frac{1}{h_i} \frac{\partial F_i}{\partial u_i}
\]

where \( h_i \) is scale factor.

By repeating Eq. (3.42) for \( u_2 \) and \( u_3 \) and adding the result vectorially, we find that

\[
\nabla F = \mathbf{e}_1 \frac{\partial F_1}{\partial u_1} + \mathbf{e}_2 \frac{\partial F_2}{\partial u_2} + \mathbf{e}_3 \frac{\partial F_3}{\partial u_3}
\]

\[
= \sum_{j=1}^{3} \mathbf{e}_j \frac{\partial F_j}{\partial u_j}
\]

This shows that in curvilinear coordinates, the del operator can be represented as

\[
\nabla = \sum_{j=1}^{3} \frac{\partial}{\partial u_j} \mathbf{e}_j
\]

Let us now proceed to evaluate \( \nabla \cdot \mathbf{F} \). Since \( \mathbf{e}_j \)'s are orthogonal, we can write

\[
\mathbf{e}_1 \cdot \mathbf{e}_2 = 0
\]

\[
\mathbf{e}_2 \cdot \mathbf{e}_3 = 0
\]

and

\[
\mathbf{e}_3 \cdot \mathbf{e}_1 = 0
\]
This means that evaluation of $\nabla \cdot \mathbf{\hat{e}}_i$ requires calculation of

$$\nabla \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = \mathbf{e}_2 \cdot ((\nabla \times \mathbf{e}_3) - (\nabla \times \mathbf{e}_3))$$  \hspace{1cm} (3.45)

To evaluate the terms on the left hand side of Eq. (3.45) you would recall from Unit 2 that if $\psi$ is a scalar, then

$$\nabla \times (\nabla \psi) = 0$$

Let us take $\psi = u_i$. Then, we have

$$\nabla \times (\nabla u_i) = 0$$

Using Eq. (3.44), we can write

$$\nabla u_i = \frac{\hat{e}_i}{h_i}$$

so that

$$\nabla \times (\nabla u_i) = 0 - \frac{\hat{e}_i}{h_i}$$

Using the identity $\nabla \times (\alpha \mathbf{A}) = \alpha (\nabla \times \mathbf{A}) - \mathbf{A} \times \nabla \alpha$, where $\alpha$ is a scalar, you can write

$$\nabla \times (\nabla u_i) = 0 = \frac{1}{h_i} (\nabla \times \hat{e}_i) - \hat{e}_i \times \nabla \left( \frac{1}{h_i} \right)$$

so that

$$\nabla \times \hat{e}_i = -\frac{1}{h_i} \hat{e}_i \times \nabla h_i$$

Hence

$$\nabla \times \hat{e}_i = -\frac{1}{h_2} \hat{e}_i \times \nabla h_2$$

On combining this result with Eq. (3.44), you will get

$$\nabla \times \hat{e}_i = -\frac{1}{h_2} \hat{e}_i \times \left( \frac{\hat{e}_i}{h_1} \frac{\partial}{\partial u_1} + \frac{\hat{e}_i}{h_2} \frac{\partial}{\partial u_2} + \frac{\hat{e}_i}{h_3} \frac{\partial}{\partial u_3} \right) h_2$$

Using the orthogonality condition of unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$, you can simplify this expression to

$$\nabla \times \hat{e}_i = -\frac{1}{h_2} \hat{e}_i \times \left( \frac{\hat{e}_i}{h_1} \frac{\partial}{\partial u_1} + \frac{\hat{e}_i}{h_2} \frac{\partial}{\partial u_2} + \frac{\hat{e}_i}{h_3} \frac{\partial}{\partial u_3} \right) h_2$$  \hspace{1cm} (3.47a)

You may note that though $\hat{e}_2$ is a unit vector, its curl does not necessarily vanish.

**SAQ 8**

Starting from Eq. (3.46) prove that

$$\nabla \times \hat{e}_1 = \frac{h_2}{h_1 h_3} \left( \frac{\hat{e}_1}{h_1} \frac{\partial}{\partial u_1} - \frac{\hat{e}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3} \frac{\partial}{\partial u_3} \right) h_2$$  \hspace{1cm} (3.47b)

On inserting these results in Eq. (3.45), you would obtain

$$\nabla \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = \frac{h_2}{h_1 h_3} \left( \frac{\hat{e}_1}{h_1} \frac{\partial}{\partial u_1} - \frac{\hat{e}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3} \frac{\partial}{\partial u_3} \right) h_2$$

Since $\hat{e}_1 \cdot \hat{e}_1 = 1 = \hat{e}_2 \cdot \hat{e}_2$ and $\hat{e}_3 \cdot \hat{e}_3 = 1 = \hat{e}_3$, the expression for $\nabla \cdot \mathbf{\hat{e}}_i$ takes a very compact form:

$$\nabla \cdot \mathbf{\hat{e}}_i$$

We can generalise this result as

\[ \nabla \cdot \mathbf{\hat{e}}_i = \frac{1}{h_j h_k} \frac{\partial (h_j h_k)}{\partial u_i} \quad i, j, k = 1, 2, 3, \quad i \neq j \neq k \tag{3.48} \]

This result shows that like \( \nabla \times \mathbf{\hat{e}}_i \), \( \nabla \cdot \mathbf{\hat{e}}_i \) is also positive definite.

On substituting for \( V F_i \) from Eq. (3.43) and \( \nabla \cdot \mathbf{\hat{e}}_i \) from Eq. (3.48) in Eq. (3.41), we get

\[ \nabla \cdot F = \frac{1}{h_j h_k h_3} \sum_{i=1}^{3} \frac{\partial (h_j h_k)}{\partial u_i} \sum_{i=1}^{3} \mathbf{\hat{e}}_i (h_j h_k) \frac{\partial F_i}{\partial u_i} \]

\[ \nabla \cdot F = \frac{1}{h_j h_k h_3} \sum_{i=1}^{3} \frac{\partial (h_j h_k)}{\partial u_i} \sum_{i=1}^{3} \mathbf{\hat{e}}_i (h_j h_k) \frac{\partial F_i}{\partial u_i} \tag{3.49} \]

**SAQ 9**

Show that in Cartesian coordinates, cylindrical coordinates and spherical polar coordinates, divergence of a vector can be expressed as

\[ \nabla \cdot r = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \]

\[ \nabla \cdot r = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho F_\rho) + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_\phi}{\partial \phi} \]

and

\[ \nabla \cdot r = \frac{1}{r^2 \sin \theta} \left( \frac{\partial}{\partial r} (r^2 F_r) + r \frac{\partial}{\partial \theta} (\sin \theta F_\theta) + r \frac{\partial F_\phi}{\partial \phi} \right) \]

**Example 7**

A particle is moving in space. Show that divergence of its position vector is invariant under coordinate transformation.

**Solution**

To show this, let us compute \( \nabla \cdot \mathbf{r} \) for Cartesian, cylindrical and spherical polar coordinates.

In Cartesian coordinates, \( \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \) so that

\[ \nabla \cdot \mathbf{r} = \frac{\partial}{\partial x} (\mathbf{i} \cdot \mathbf{r}) + \frac{\partial}{\partial y} (\mathbf{j} \cdot \mathbf{r}) + \frac{\partial}{\partial z} (\mathbf{k} \cdot \mathbf{r}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \]

In cylindrical coordinates

\[ \mathbf{r} = \rho \mathbf{\hat{e}}_{\rho} + z \mathbf{\hat{e}}_z \]

and

\[ \nabla \cdot \mathbf{r} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \mathbf{\hat{e}}_{\rho}) + \frac{1}{\rho} \frac{\partial}{\partial \phi} (\mathbf{\hat{e}}_\phi) + \frac{\partial}{\partial z} (\mathbf{\hat{e}}_z) \]

In this case \( \mathbf{\hat{e}}_{\rho} = \mathbf{\hat{e}}_{\rho} \) and \( \mathbf{\hat{e}}_\phi = \mathbf{\hat{e}}_\phi \). Hence,
Similarly, in spherical polar coordinates

\[ \mathbf{\nabla} \cdot \mathbf{r} = r \hat{r}_r \]

and

\[ \mathbf{\nabla} \times \mathbf{r} = \frac{1}{r^2 \sin \theta} \left[ \sin \theta \frac{\partial}{\partial r} (r^2 \phi) + r \frac{\partial}{\partial \theta} (\sin \theta \phi) + r \frac{\partial}{\partial \phi} \phi \right] \]

Here, \( r = r_\theta = r_\phi = 0 \).

Hence

\[ \mathbf{\nabla} \cdot \mathbf{r} = \frac{1}{r^2 \sin \theta} \left[ \sin \theta \frac{\partial}{\partial r} (r^2 \phi) \right] = \frac{1}{r^2 \sin \theta} \left[ 2r \sin \theta \right] = 2 \]

Since the value of \( \mathbf{\nabla} \cdot \mathbf{r} \) comes out to be the same in all coordinate systems, we say that it is invariant.

Following the procedure outlined in arriving at Eq. (3.49), we would like you to show that

\[ \mathbf{\nabla} \times \mathbf{F} = \mathbf{V} \times \left( \sum_{i=1}^{n} F_i \hat{e}_i \right) \]

where \( \mathbf{V} = \mathbf{V} \times \mathbf{e}_i \) and \( \mathbf{F}_i = \mathbf{V} \times (\mathbf{F}_i - \mathbf{V} \times \mathbf{F}_i) \).

In cylindrical and spherical polar coordinates, Eq. (3.50) takes the form

\[ \mathbf{\nabla} \times \mathbf{F} = \frac{1}{\rho} \begin{vmatrix} \hat{e}_\rho & \hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_\rho & F_\theta & F_z \end{vmatrix} \]

(3.50)

**Hint:** Write \( \mathbf{V} \times \mathbf{F} = \mathbf{V} \times \left( \sum_{i=1}^{n} F_i \hat{e}_i \right) \) and substitute

for \( \mathbf{V} \times \mathbf{\hat{e}_i} \) and \( \nabla F_i \).

Example 8

The magnetic potential of a single current loop in the \( xy \)-plane is given by

\[ \mathbf{V} = \mathbf{V} \times \left[ \mathbf{\nabla} \times \mathbf{e}_n \mathbf{n} \right] \]

Express it in spherical polar coordinates.
Solution

\[ \nabla \times \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & \frac{1}{r \sin \theta} \frac{\partial}{\partial r} (r \sin \theta A_\phi) & 0 \end{vmatrix} \]

Taking the curl again, we obtain

\[ \nabla \times \frac{1}{r^2 \sin \theta} \left[ \hat{r} \frac{\partial}{\partial \theta} (r \sin \theta A_\phi) - \hat{\theta} \frac{\partial}{\partial r} (r \sin \theta A_\phi) \right] \]

By expanding the determinant, we have

\[ \nabla \times \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & \frac{1}{r \sin \theta} \frac{\partial}{\partial r} (r \sin \theta A_\phi) & 0 \end{vmatrix} \]

LaplaceOperator

Like the del operator, the Laplace operator also finds applications in fluid mechanics, electromagnetism, elasticity, propagation of waves and quantum mechanics. It is therefore important to give expressions for \( \nabla^2 \) in curvilinear, spherical and cylindrical coordinates.

You will use these in the next unit as well as in Block 2 of FHE-05 course. We know that

\[ \nabla^2 f = \nabla \cdot \nabla f \]

where \( f \) is a scalar.

Using Eq. (3.43), you can write

\[ \nabla^2 f = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( \frac{1}{h_i} \frac{\partial f}{\partial x_i} \right) \]

Inserting the result contained in Eq. (3.49), this expression becomes

\[ \nabla^2 f = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \left( \frac{\partial}{\partial x_i} \left( h_i \frac{\partial f}{\partial x_i} \right) \right) \]

In the expanded form, you can write

\[ \nabla^2 f = \frac{1}{h_1 h_2 h_3} \left( \frac{\partial}{\partial x_1} \left( h_1 \frac{\partial f}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( h_2 \frac{\partial f}{\partial x_2} \right) \right) + \frac{\partial}{\partial x_3} \left( \frac{\partial f}{\partial x_3} \right) \]  

(3.52)

SAQ 10

Express Eq. (3.52) in cylindrical and spherical polar coordinates. Spend 10 min

Let us now summarise the contents of this unit.

3.5 SUMMARY

- The cylindrical coordinates \((p, \phi, z)\), of a point \( P \) in space are related to its Cartesian coordinates \((x, y, z)\) by
  \[ x = p \cos \phi, \quad y = p \sin \phi, \quad z = z \]
  The plane polar coordinates of the projection of \( P \) on to the \( xy \) plane are \( p \) and \( \phi \).
- The spherical polar coordinates \((r, \theta, \phi)\) of a point \( P \) are related to its Cartesian coordinates \((x, y, z)\) by
  \[ x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta \]
The generalised curvilinear coordinates are related to Cartesian coordinates through the relation
\[ u_i = u_i(x, y, z) \]

The square of length of an arc and differential element of vector area in generalized curvilinear coordinates are given by
\[ (ds)^2 = h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_3^2 (du_3)^2 \]

\[ 
\begin{align*}
\Delta A &= h_1 h_2 h_3 \left| \begin{array}{ccc}
\frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\
\frac{\partial u_1}{\partial x} & \frac{\partial u_2}{\partial x} & \frac{\partial u_3}{\partial x} \\
\frac{\partial u_1}{\partial y} & \frac{\partial u_2}{\partial y} & \frac{\partial u_3}{\partial y}
\end{array} \right| \\
&= \sqrt{\begin{vmatrix}
h_1^2 & h_1 h_2 & h_1 h_3 \\
h_2 h_1 & h_2^2 & h_2 h_3 \\
h_3 h_1 & h_3 h_2 & h_3^2
\end{vmatrix}}
\end{align*}
\]

In curvilinear coordinates, the grad, div, curl and the Laplacian operator can be expressed as
\[ \nabla f = \frac{1}{h} \frac{\partial f}{\partial u_i} e_i + \frac{1}{h} \frac{\partial f}{\partial u_j} e_j + \frac{1}{h} \frac{\partial f}{\partial u_k} e_k \]
\[ \nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \sum_{i \neq j} \frac{\partial}{\partial u_i} \left( h_i h_j F_k \right) \\
\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix}
h_1 & h_2 & h_3 \\
h_2 & h_3 & h_1 \\
h_3 & h_1 & h_2
\end{vmatrix}
\]

and
\[ \nabla^2 f = \frac{1}{h_1 h_2 h_3} \sum_{i \neq j} \frac{\partial}{\partial u_i} \left( h_i h_j \frac{\partial f}{\partial u_k} \right) \\
\]
3.6 TERMINAL QUESTIONS

1. Show that
\[
\begin{align*}
\frac{d\hat{e}_x}{dt} &= \hat{e}_y \\
\frac{d\hat{e}_y}{dt} &= -\hat{e}_x \\
\frac{d\hat{e}_z}{dt} &= \cos\theta \hat{e}_x + \sin\theta \hat{e}_y \\
\frac{d\hat{e}_\theta}{dt} &= -\theta \hat{e}_r + \hat{e}_\phi \\
\frac{d\hat{e}_\phi}{dt} &= -\sin\theta \hat{e}_\theta - \cos\theta \hat{e}_r \\
\frac{d\hat{e}_r}{dt} &= \sin\theta \hat{e}_\phi - \cos\theta \hat{e}_\theta \\
\end{align*}
\]
and
\[
\frac{d\hat{e}_t}{dt} = -\sin\theta \hat{e}_\phi - \cos\theta \hat{e}_\theta
\]

2. A particle is moving through space. Express its acceleration in spherical polar coordinates.

3. Verify Eq. (3.50)

4. A rigid body is rotating about a fixed axis with a constant angular velocity \( \omega \). Take \( \omega \) to be along the z-axis. Using spherical polar coordinates, calculate i) \( \mathbf{v} = \mathbf{\omega} \times \mathbf{r} \) and ii) \( \mathbf{V} \times \mathbf{V} \).

5. A central force field is given by
\[
\mathbf{F} = \frac{1}{r^2} \mathbf{r} + \mathbf{\alpha}
\]
Calculate \( \mathbf{V} \times \mathbf{F} \).

3.7 SOLUTIONS AND ANSWERS

Self Assessment Questions

1. Refer to Fig. 3.16. You will note that
\[
\begin{align*}
x &= f \sin \phi \\
y &= f \cos \phi - \frac{1}{2} \varphi^2
\end{align*}
\]

2. The \( \phi \)-surfaces for the cylindrical coordinate system are coaxial right circular cylinders having the z-axis for their common axis. This is shown in the figure 3.17.

3. Fig. (3.50)

4. The solid is a conical section of a sphere. It is illustrated in Fig. 3.19.

5. \( \mathbf{r} = \rho \hat{e}_\rho + z \hat{e}_z \)

Insert the value of \( \hat{e}_\rho \). This gives
\[
\mathbf{r} = \rho \left( \hat{e}_\rho \cos \phi + \hat{e}_z \sin \phi \right) + \hat{e}_z
\]

A small change in \( \mathbf{r} \) gives rise to changes in \( \rho, \phi, \) and \( z \). So we can write
\[
\begin{align*}
d\mathbf{r} &= d\rho \hat{e}_\rho + \rho d\phi \hat{e}_\rho + d\phi \hat{e}_\phi + dz \hat{e}_z \\
&= d\rho \hat{e}_\rho + \rho d\phi \hat{e}_\rho + \phi d\phi \hat{e}_\phi + dz \hat{e}_z
\end{align*}
\]

6. In Cartesian coordinates, an element of volume is defined as
\[
dV = dx \, dy \, dz
\]
We use the same definition in non-Cartesian coordinates.
In cylindrical coordinates
\[ dV = (dr)(d\theta)(dz) \]
\[ = (d\rho)(d\phi)(dz) \]
In spherical polar coordinates
\[ dV = (dr)(d\theta)(d\phi) \]
\[ = (dr)(d\theta)(r\sin\theta)(d\phi) \]
\[ = r^2 \sin\theta \, d\theta \, d\phi \]

7. From Eq. (3.29) we know that metric coefficients are given by

\[ g_{11} = \left( \frac{\partial x}{\partial r} \right)^2 = \frac{1}{r^2} \]
\[ g_{12} = \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial r} = \frac{1}{r} \cos \phi \]
\[ g_{13} = \frac{\partial x}{\partial \phi} = 0 \]
\[ g_{22} = \left( \frac{\partial y}{\partial \theta} \right)^2 + \left( \frac{\partial y}{\partial \phi} \right)^2 = \frac{1}{r^2} \sin^2 \phi \]
\[ g_{33} = \left( \frac{\partial z}{\partial \theta} \right)^2 + \left( \frac{\partial z}{\partial \phi} \right)^2 = \sin^2 \theta \]

\[ \alpha = \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi} \frac{\partial y}{\partial \theta} = \frac{1}{r} \sin \phi \cos \phi \]
\[ - \frac{1}{r} \sin \phi \sin \phi = 0 \]

Similarly, you can show that
\[ g_{13} = g_{31} = 0 \]

Hence,
\[ d\tau = g_{11}(dx)^2 + g_{22}(dy)^2 + g_{33}(dz)^2 \]
\[ = r^2 \sin^2 \theta \, d\theta^2 + r^2 \cos^2 \theta \, d\phi^2 \]

8. We know that
\[ \nabla \times \mathbf{E} = -\frac{d}{dt} \mathbf{B} \times \nabla \mathbf{h} \]
Substituting for $V_{h_3}$, we find that

$$V \times \hat{e}_3 = \frac{1}{h_3} \hat{e}_3 \times \left( \frac{\partial}{\partial u_1} \hat{e}_1 + \frac{\partial}{\partial u_2} \hat{e}_2 + \frac{\partial}{\partial u_3} \hat{e}_3 \right) h_3$$

and

$$= \frac{1}{h_3} \left( \frac{\partial}{\partial u_1} \hat{e}_3 \times \hat{e}_1 + \frac{\partial}{\partial u_2} \hat{e}_3 \times \hat{e}_2 + \frac{\partial}{\partial u_3} \hat{e}_3 \times \hat{e}_3 \right) h_3$$

9. In curvilinear coordinates

$$V \cdot F = \frac{1}{h_1 h_2 h_3} \left( \frac{\partial}{\partial u_1} (h_1 h_2 F_1) + \frac{\partial}{\partial u_2} (h_2 h_3 F_2) + \frac{\partial}{\partial u_3} (h_1 h_3 F_3) \right)$$

In Cartesian coordinates, $h_1 = h_3 = 1$. Thus, $u_1 = x$, $u_2 = y$, $u_3 = z$, $F_1 = F_x$, $F_2 = F_y$, and $F_3 = F_z$. Hence,

$$V \cdot F = \frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial z} F_z$$

In cylindrical coordinates, $h_2 = h_3 = 1$. Thus, $u_1 = \rho$, $u_2 = \phi$, $u_3 = z$, $F_1 = F_\rho$, $F_2 = F_\phi$, and $F_3 = F_z$. Hence,

$$V \cdot F = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho F_\rho) + \frac{\partial}{\partial \phi} F_\phi + \frac{\partial}{\partial z} F_z$$

In spherical coordinates, $h_1 = h_3 = 1$, $h_2 = h_0 = \rho$. Thus, $u_1 = \rho$, $u_2 = \phi$, $u_3 = \theta$, $F_1 = F_\rho$, $F_2 = F_\phi$, and $F_3 = F_\theta$. Hence,

$$V \cdot F = \frac{1}{\rho^2 \sin \theta} \left( \frac{\partial}{\partial \rho} (\rho^2 F_\rho) + \frac{\partial}{\partial \phi} (\rho \sin \theta F_\phi) + \frac{\partial}{\partial \theta} (\rho \sin \theta F_\theta) \right)$$

10. $V^2 F = \frac{1}{h_1 h_2 h_3} \left( \frac{\partial}{\partial u_1} (h_1 h_2 F_{u_1}) + \frac{\partial}{\partial u_2} (h_2 h_3 F_{u_2}) + \frac{\partial}{\partial u_3} (h_1 h_3 F_{u_3}) \right)$

For cylindrical coordinates, $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$. Thus, $u_1 = \rho$, $u_2 = \phi$, and $u_3 = z$. Hence,

$$V^2 F = \frac{1}{\rho} \left( \frac{\partial}{\partial \rho} \left( \rho F_\rho \right) + \frac{\partial}{\partial \phi} \left( \rho F_\phi \right) + \frac{\partial}{\partial z} \left( \rho F_z \right) \right)$$

Similarly, for spherical coordinates, $u_1 = \rho$, $u_2 = \phi$, $u_3 = \theta$, $h_1 = 1$, $h_2 = \rho$, and $h_3 = \theta$. Hence,

$$V^2 F = \frac{1}{\rho^2 \sin \theta} \left( \frac{\partial}{\partial \rho} \left( \rho^2 F_\rho \right) + \frac{\partial}{\partial \phi} \left( \rho^2 F_\phi \right) + \frac{\partial}{\partial \theta} \left( \rho^2 \sin \theta F_\theta \right) \right)$$

Terminal Questions

1. We know that

$$\hat{e}_\rho = \cos \theta F + \sin \theta \hat{e}_\theta$$

and

$$\hat{e}_\theta = -\sin \rho \hat{e}_\rho + \cos \rho \hat{e}_\theta$$

(i)

(ii)
Then
\[ \frac{d}{dt} A = \left[ -\sin \theta \left( \hat{e}_i - \cos \phi \hat{e}_j + \sin \phi \hat{e}_k \right) \right] \hat{e}_0 = \dot{\hat{e}}_0 \]
and,
\[ \frac{d}{dt} \hat{e}_0 = \left[ -\cos \theta \cos \phi \hat{e}_i - \sin \theta \sin \phi \hat{e}_j + \cos \phi \hat{e}_k \right] \hat{e}_0 = \dot{\hat{e}}_0 \]

(iii)
\[ \hat{e}_0 = \cos \theta \cos \phi \hat{e}_i + \sin \theta \sin \phi \hat{e}_j - \cos \phi \hat{e}_k \]

(iv)
\[ \hat{e}_0 = -\sin \theta \hat{e}_i + \cos \theta \hat{e}_j \]

and
\[ \hat{e}_0 = -\sin \theta \hat{e}_i + \cos \theta \hat{e}_j \]

Hence
\[ \frac{d}{dt} \hat{e}_0 = \cos \theta \cos \phi \hat{e}_i - \sin \theta \sin \phi \hat{e}_j - \cos \phi \hat{e}_k \]

On adding these we find that
\[ \sin \theta \hat{e}_i + \cos \theta \hat{e}_k = \cos \theta \left( \hat{e}_i + \sin \phi \hat{e}_j - \cos \phi \hat{e}_k \right) \]

Hence
\[ \frac{d}{dt} \hat{e}_0 = -\left( \sin \theta \hat{e}_i + \cos \theta \hat{e}_k \right) \hat{e}_0 \]

2. We know that
\[ \mathbf{v} = \hat{e}_i \hat{e}_0 + \hat{e}_j \hat{e}_0 + \hat{e}_k \hat{e}_0 \]
Differentiate with respect to time. The result is
\[ \mathbf{v} = \dot{\hat{e}}_0 \hat{e}_i + \dot{\hat{e}}_0 \hat{e}_j + \dot{\hat{e}}_0 \hat{e}_k \]

In terminal question 1 you have evaluated \( \hat{e}_0 \) and \( \hat{e}_0 \). Inserting these results, you will get
\[ \mathbf{v} = \mathbf{v} \]
\[ \dot{\hat{e}}_0 \hat{e}_i + \dot{\hat{e}}_0 \hat{e}_j + \dot{\hat{e}}_0 \hat{e}_k \]
On collecting the coefficients of \( \hat{e}_0 \) and \( \hat{e}_1 \), you will get:

\[
\mathbf{a} = \mathbf{e}_0 - 2 \mathbf{r} \left( \mathbf{r} \mathbf{e}_0 + \mathbf{r} \mathbf{e}_1 \right) - \mathbf{r} \left( \mathbf{r} ^2 \mathbf{e}_0 - \mathbf{r} \mathbf{e}_1 \right) - \mathbf{r} \left( \mathbf{r} \mathbf{e}_0 - \mathbf{r} \mathbf{e}_1 \right)
\]

Hence

\[
\mathbf{a} = \mathbf{e}_0 - 2 \mathbf{r} \left( \mathbf{r} \mathbf{e}_0 + \mathbf{r} \mathbf{e}_1 \right) - \mathbf{r} \left( \mathbf{r} \mathbf{e}_0 - \mathbf{r} \mathbf{e}_1 \right)
\]

\[
a_0 = 2 \mathbf{r} \mathbf{e}_0 + \mathbf{r} \mathbf{e}_1
\]

\[
a_1 = \mathbf{r} \mathbf{e}_0 - \mathbf{r} \mathbf{e}_1
\]

3. \( \mathbf{V} \times \mathbf{F} = \mathbf{V} \times (\mathbf{F}_1 \hat{e}_1 + \mathbf{F}_2 \hat{e}_2 + \mathbf{F}_3 \hat{e}_3)
\]

\[
= \nabla \times (\mathbf{F}_1 \hat{e}_1) + \nabla \times (\mathbf{F}_2 \hat{e}_2) + \nabla \times (\mathbf{F}_3 \hat{e}_3)
\]

\[
\mathbf{V} \times (\mathbf{F}_1 \hat{e}_1) = \mathbf{V} \times (\nabla \times \mathbf{e}_1) - \mathbf{e}_1 \times \nabla \mathbf{F}_1
\]

From Eq. (3.46)

\[
\mathbf{V} \times \hat{e}_1 = \frac{1}{h_1} \hat{e}_1 \times \nabla h_1
\]

\[
= \frac{1}{h_1} \hat{e}_1 \times \left( \frac{\partial}{\partial x_1} \hat{e}_1 \right) \left( \frac{\partial}{\partial x_2} \hat{e}_2 \right)
\]

\[
= \frac{1}{h_1} \left( \frac{\partial}{\partial x_1} \hat{e}_1 \right) \left( \frac{\partial}{\partial x_2} \hat{e}_2 \right)
\]

Similarly

\[
\mathbf{V} \times (\mathbf{F}_2 \hat{e}_2) = \mathbf{V} \times (\nabla \times \mathbf{e}_2) - \mathbf{e}_2 \times \nabla \mathbf{F}_2
\]

\[
\mathbf{V} \times (\mathbf{F}_3 \hat{e}_3) = \mathbf{V} \times (\nabla \times \mathbf{e}_3) - \mathbf{e}_3 \times \nabla \mathbf{F}_3
\]

Hence

\[
\mathbf{V} \times (\mathbf{F}_1 \hat{e}_1) = \frac{\partial}{\partial x_1} \hat{e}_1 \left( \frac{\partial}{\partial x_2} \hat{e}_2 \right) \left( \frac{\partial}{\partial x_3} \hat{e}_3 \right) - \frac{\partial}{\partial x_2} \hat{e}_2 \left( \frac{\partial}{\partial x_3} \hat{e}_3 \right) \left( \frac{\partial}{\partial x_1} \hat{e}_1 \right)
\]

\[
\mathbf{V} \times (\mathbf{F}_2 \hat{e}_2) = \frac{\partial}{\partial x_2} \hat{e}_2 \left( \frac{\partial}{\partial x_3} \hat{e}_3 \right) \left( \frac{\partial}{\partial x_1} \hat{e}_1 \right) - \frac{\partial}{\partial x_3} \hat{e}_3 \left( \frac{\partial}{\partial x_1} \hat{e}_1 \right) \left( \frac{\partial}{\partial x_2} \hat{e}_2 \right)
\]

\[
\mathbf{V} \times (\mathbf{F}_3 \hat{e}_3) = \frac{\partial}{\partial x_3} \hat{e}_3 \left( \frac{\partial}{\partial x_1} \hat{e}_1 \right) \left( \frac{\partial}{\partial x_2} \hat{e}_2 \right) - \frac{\partial}{\partial x_1} \hat{e}_1 \left( \frac{\partial}{\partial x_2} \hat{e}_2 \right) \left( \frac{\partial}{\partial x_3} \hat{e}_3 \right)
\]

We can write analogous expressions for other terms:

\[
\mathbf{V} \times (\mathbf{F}_2 \hat{e}_2) = \frac{\partial}{\partial x_2} \hat{e}_2 \left( \frac{\partial}{\partial x_3} \hat{e}_3 \right) \left( \frac{\partial}{\partial x_1} \hat{e}_1 \right) - \frac{\partial}{\partial x_3} \hat{e}_3 \left( \frac{\partial}{\partial x_1} \hat{e}_1 \right) \left( \frac{\partial}{\partial x_2} \hat{e}_2 \right)
\]

and

\[
\mathbf{V} \times (\mathbf{F}_3 \hat{e}_3) = \frac{\partial}{\partial x_3} \hat{e}_3 \left( \frac{\partial}{\partial x_1} \hat{e}_1 \right) \left( \frac{\partial}{\partial x_2} \hat{e}_2 \right) - \frac{\partial}{\partial x_1} \hat{e}_1 \left( \frac{\partial}{\partial x_2} \hat{e}_2 \right) \left( \frac{\partial}{\partial x_3} \hat{e}_3 \right)
\]
On adding these results and collecting the coefficients of $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$, we get

$$\nabla \times F = \frac{\hat{e}_1}{h_2 h_3} \left[ \frac{\partial}{\partial x_2} (F_3 h_2) - \frac{\partial}{\partial x_3} (F_2 h_3) \right]$$

$$+ \frac{\hat{e}_2}{h_3 h_1} \left[ \frac{\partial}{\partial x_3} (F_1 h_1) - \frac{\partial}{\partial x_2} (F_3 h_3) \right]$$

$$+ \frac{\hat{e}_3}{h_1 h_2} \left[ \frac{\partial}{\partial x_1} (F_3 h_2) - \frac{\partial}{\partial x_3} (F_2 h_3) \right]$$

This can be written as

$$\nabla \times F = \frac{1}{h_1 h_2 h_3} \begin{vmatrix}
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\
F_1 & F_2 & F_3 \\
\hat{e}_1 & \hat{e}_2 & \hat{e}_3
\end{vmatrix}$$

4. i) In spherical polar coordinates

$$\mathbf{r} = r \hat{e}_r$$

and

$$\mathbf{\omega} = \omega (\cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta)$$

Hence

$$\mathbf{v} = \omega \times \mathbf{r} = \omega r (\cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta) \times \hat{e}_r$$

$$= -r \sin \theta (\hat{e}_r \times \hat{e}_\theta)$$

$$= r \sin \theta \hat{e}_\phi$$

ii) $\nabla \times \mathbf{v} = \frac{1}{r^2 \sin \theta} \begin{vmatrix}
\frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & r \sin \theta \frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & r \sin \theta \frac{\partial}{\partial \theta} \\
0 & 0 & r \sin \theta
\end{vmatrix}$

$$= \frac{1}{r^2 \sin \theta} \left[ (\hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi) \begin{array}{c}
0 \\
0 \\
r \sin \theta
\end{array} \begin{array}{c}
\cos \theta \sin \phi \\
\cos \phi \\
\sin \theta \sin \phi
\end{array} \right]$$

$$= 2 \hat{e}_\phi \cos \theta - 2r \hat{e}_\theta \sin \theta$$

$$= 2 \hat{e}_\phi (\cos \theta \hat{e}_r \sin \theta - \hat{e}_\theta \sin \theta) + 2 \hat{e}_\theta = 2 \mathbf{\omega}$$

5. $\mathbf{F} = \frac{\hat{e}_r}{r^3} + \frac{r \hat{e}_\phi}{r^3 \sin \theta}$

Hence

$$\nabla \times \mathbf{F} = \frac{1}{r^2 \sin \theta} \begin{vmatrix}
\hat{e}_r & \hat{e}_\phi & r \sin \theta \hat{e}_\theta \\
\frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & r \sin \theta \frac{\partial}{\partial \theta} \\
\frac{2 \rho \cos \theta}{\rho^3} & \frac{\rho}{\rho^2 \sin \theta} & 0
\end{vmatrix}$$

$$= \frac{1}{r^2 \sin \theta} \left[ (\hat{e}_r, \hat{e}_\phi, \hat{e}_\theta) \begin{array}{c}
\cos \theta \\
\cos \phi \\
\sin \theta \sin \phi
\end{array} \begin{array}{c}
\frac{\partial}{\partial \rho} \\
\frac{\partial}{\partial \phi} \\
-r \sin \theta \frac{\partial}{\partial \theta}
\end{array} \right]$$

$$= \frac{\hat{e}_\phi}{r^2 \sin \theta} \begin{array}{c}
\cos \phi \\
\sin \phi
\end{array} \begin{array}{c}
\rho \sin \theta \\
-\rho \cos \theta
\end{array}$$

$$= \frac{\hat{e}_\phi}{r^2} \left( \frac{2 \rho - 3 \rho \cos \theta}{r^2} \right)$$
UNIT 4 INTEGRATION OF SCALAR AND VECTOR FIELDS

Structure

4.1 Introduction

4.2 Integration of a Vector with Respect to a Scalar

Objectives

4.3 Multiple Integrals

4.4 Line Integral of a Field

4.5 Surface Integral of a Field

4.6 Volume Integral of a Field

4.7 Vector Integral Theorems

4.8 Summary

4.9 Terminal Questions

4.10 Solutions and Answers

4.1 INTRODUCTION

In the previous unit you have studied about different coordinate systems. In the process you have learnt to represent the gradient of a scalar field and the divergence and curl of a vector field in several coordinate systems. In Unit 2 you have studied about differentiation of vectors. In this unit you will learn about integration of scalar and vector fields.

These integrals find many applications in physics. For example, the work done by a force or the magnetic field due to a current-carrying conductor can be expressed as a line integral. The flux of a magnetic field can be represented as a surface integral. Line, surface and volume integrals find application in determining the potential due to a continuous distribution of matter or charges. We shall discuss some of these examples here. These integrals form the foundations of many important equations in physics like the Maxwell’s equation of Electromagnetism, the equation of continuity and so on.

In this unit, first we shall discuss the integration of a vector with respect to a scalar. You will find that this will basically be an extension of the idea of ordinary integral. This finds application in determining the trajectory of a particle if its equation of motion is known. We shall also discuss the integration of scalar and vector products of vectors with respect to a scalar.

Next we shall discuss the integration of scalar and vector fields with respect to coordinates. As you know a scalar or a vector field may be a function of one or more coordinates. Their integrals which we shall come across will necessarily not be in terms of single variable. So we shall learn about double and triple integrals. You will see that the three kinds of field integrals, i.e., the line, surface and volume integrals are respectively the extensions of ordinary, double and triple integrals.

Finally, we shall discuss how one kind of a field integral can be transformed into another type. In the process you will learn to apply the vector integral theorems — namely the Gauss’ divergence theorem, Stokes’ theorem and Green’s theorem. Here, we shall only state these theorems without proof. However, if you are interested in knowing the proofs you may go through the Appendix. These theorems provide us with very elegant methods for arriving at several fundamental equations of physics.

In the next block, we shall take up Statistics and Probability and their applications in physics.