INTRODUCTION

The first block of this course deals with the concepts of vector calculus. From your school courses you know that the value of a scalar quantity can be specified by a single real number and an appropriate unit. For example, length, mass and temperature are scalar quantities. However, there are many other quantities which are not specified by a single number. These belong to the class of vectors. You may know what a vector is. You may also know how to add, subtract and multiply vectors.

So we will begin the first unit on vector algebra by quickly recalling what you have learnt about vectors at school. Then we shall take a fresh look at vectors and vector algebra. At school you have learnt how to represent vectors geometrically. In Unit 1 you will learn how to express vectors in terms of their components with reference to a given coordinate system. You will also learn to add, subtract and multiply vectors in their component form. It is essential that you study vector algebra in this form as you will be using these results very often in physics courses.

Unit 2 is on vector differential calculus. Now you know that the temperature of the atmosphere varies as you climb up a hill. The gravitational force on an object due to another object varies with the distance between them. We shall begin Unit 2 by considering scalar and vector field functions, i.e., scalars and vectors which have different values at different points in a given region. Temperature is a scalar and the temperature distribution is an example of a scalar field function. Force is a vector and the gravitational force field is an example of a vector field function. In this unit you will also learn how to compute three kinds of derivatives of scalar and vector field functions: (1) the spatial rate of change of a scalar field called the gradient which results in a vector field, (2) the spatial rates of change of a vector field called divergence (which results in a scalar field) and (3) the curl (which gives a vector field). You will use the familiar Cartesian coordinate system for computing all these derivatives.

However, in physics we often come across situations in which the use of Cartesian coordinates results in a lot of mathematical complexity. Then for computational simplicity we use other coordinate systems. For example, in computing the flow of a fluid (water or LPG) along a pipe, or energy produced in a reactor the calculations become very simple if we use cylindrical polar coordinates. Similarly, the use of spherical polar coordinates simplifies the study of heat flow in a sphere or computation of potential at a point due to a uniformly charged sphere. So in Unit 3 we will discuss non-Cartesian coordinate systems. You will learn to express a vector as well as scalar and vector field functions in cylindrical and spherical polar coordinates. You will also learn how these coordinate systems are related to curvilinear coordinates.

In Unit 4 we shall take up the integration of scalar and vector fields. Many important results of physics, such as the moment of inertia of a body, the potential due to a charge distribution, and so on are obtained by performing multiple integration of scalar and vector fields. Moreover, many fundamental equations in physics, particularly those of electromagnetic theory and hydrodynamics are obtained by applying certain theorems. These are the Gauss’ divergence theorem, Stokes’ theorem and Green’s theorem. So in this unit you will also study these theorems and their applications. This block also contains an appendix on the proofs of these theorems. You may like to read it to enrich your knowledge.

You will not be examined on it. For ready reference, we have given the formulae appearing in this block at the end. In a sense, they constitute the gist of the material in this block.

As you study the material you will come across abbreviations in the text, e.g. Sec. 1.2, Fig. 1.1 and Eq. (1.1) etc. The abbreviation Sec. stands for section, Fig. for figure and Eq. for equation. Fig. x,y refers to the yth figure of Unit x, i.e. Fig. 1.1 is the first figure in Unit 1. Similarly, Sec. 3.3 is the third section in Unit 2 and Eq. (6.5) is the eighth equation in Unit 4.

In your study you will also find that the units are not of equal length and your study time for each unit will vary. Our average estimate of the study time for each unit is 6 h for Unit 1, 10 h for Unit 2, 8 h for Unit 3 and 11 h for Unit 4 giving a total 35 h for working through the text, solving SAQs and Terminal Questions.

We hope you enjoy studying the material and once again wish you success.
UNIT 1 VECTOR ALGEBRA

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1.1 INTRODUCTION

As you start reading this unit you may just wonder as to why you should study vectors. Of what use are they in physics? You perhaps know the answer to this question from your school courses. Vectors are used extensively in almost all branches of physics. In order to understand physics, you must know how to work with vectors: how to add, subtract or multiply them.

For example, you have studied about many physical quantities, such as the velocity of a body, its acceleration and the force acting on it. These are all vector quantities. You have determined the resultant of several forces acting on a body and found its acceleration from Newton's second law \( \mathbf{F} = \mathbf{ma} \). You have also studied about work which is expressed as the scalar product of force and displacement. And you also know that the angular momentum of a body is expressed as a vector product of its position vector and its linear momentum. What we are driving at here is that you are familiar with the addition and subtraction of vectors, and scalar and vector products of two vectors. So in Sec. 1.2 we will quickly recall what you have studied about vectors in your school courses.

In the remaining sections, we shall look afresh at vectors. We shall begin by defining vectors in a precise manner in Sec. 1.3. For this we shall first discuss vectors in their component form. Then in Sec. 1.4 we will repeat the vector algebra of Sec. 1.2 in detail, arriving at all the results in their component form. In Sec. 1.5, you will study products of more than two vectors. These will often be used in the undergraduate physics courses, particularly in the course Electric and Magnetic Phenomena. Lastly we will briefly discuss the two categories, namely, the polar vectors and axial vectors, into which all vectors can be grouped. In the next unit we will consider the differentiation of vectors which is again very useful in physics.

Study Guide: The average study time for this unit has been estimated at 6 hours. If you have passed your +2 examinations recently, you can quickly work through Sec. 1.2, study Example 1 carefully, and solve the SAQs given there in about half an hour's time or a little more. The rest of the time you can devote to Secs. 1.3 to 1.6. If, however, you have been out of touch with the +2 courses for quite some time then you will need to go through a +2 physics textbook to brush up your knowledge. The reference of this book is given at the end of Unit 4.

Objectives
After studying this unit you should be able to:
1. define a vector in terms of the transformation of its components from one Cartesian coordinate system to another
2. express a vector in terms of its components and the unit vectors in two (three) dimensional Cartesian coordinate systems.
In your school science courses you have studied about scalar and vector quantities. You have learnt about physical quantities like mass, length, time, area, frequency, volume, temperature, etc. You know that these are all scalar quantities. You have studied that a scalar quantity is completely specified by a single number (with a suitable choice of units). There are many more examples of scalar quantities in physics. For instance, the charge of an electron, resistance of a resistor, specific heat of water, etc. are all scalars.

You have also learnt about physical quantities like displacement, velocity, acceleration, momentum, force, etc. As you know these are all vector quantities. The definition of vectors that you have learnt in school is this: *Quantities which are specified by a magnitude and a direction in space are called vector quantities.*

In this section we will quickly recall the vector algebra that you have studied at school. But before we proceed further let us specify how we will represent vectors in this course.

**Notation**

In the printed text we will denote vectors by bold face letters, e.g. \( \mathbf{v}, \mathbf{a}, \mathbf{F} \), etc. In your written work, you should denote vectors by drawing arrows above them e.g. \( \vec{v}, \vec{F} \) or by drawing a line (straight or curvy) below them, e.g. \( v, F \). In print, we will denote the magnitude of a vector \( \mathbf{v} \) by \( |\mathbf{v}| \), called the modulus or magnitude of \( \mathbf{v} \), or by \( v \), a light letter in italics.

In diagrams vectors will be shown as arrows, i.e. straight lines with arrowheads on them as in Fig. 1.1. The length of the straight line gives the magnitude of the vector. The direction of the arrow specifies the direction of the vector. Now, what do we mean by a vector’s direction in space? It means that the vector has a certain orientation represented by the slope of the line. It also means that it has a *sense*, represented by an arrowhead. For example, in Fig. 1.1 we show an arrow representing vector \( \mathbf{a} \) lying in a plane. The orientation of \( \mathbf{a} \) is specified by a line making an angle of \( 45^\circ \) with north. Its sense is specified by the arrowhead pointing towards north-east as opposed to south-west. Near the arrow representing a vector, we usually put the symbol we use for that vector, e.g. \( \mathbf{a} \) in Fig. 1.1. In Fig. 1.1, point \( A \) is called the tail (or initial point) of the vector and \( B \) its head (or the terminal point). The line along which a vector is directed (e.g. \( \mathbf{AB} \) in Fig. 1.1) is called the line of action of that vector.

Let us now briefly recall some concepts related to vectors, which you have studied in school.

**Equality of vectors**

Two vectors are said to be equal if they have the same magnitude and the same direction. The four arrows in Fig. 1.2, represent equal vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) and \( \mathbf{d} \), even though they are drawn at different places on the page. Such vectors are also called **free vectors**. So, an *affine vector* means translated parallel to itself remains the same.

Sometimes the line of action of a vector remains fixed, e.g. the line of action of \( \mathbf{g} \) for a falling body. In such cases two vectors are equal only if they have the same magnitude, direction and the same line of action. Such vectors are often called **sliding vectors**, as the vectors \( \mathbf{c} \) and \( \mathbf{d} \) in Fig. 1.2.

Sometimes even the point of action of a vector is fixed. Such vectors are called bound vectors. For example, the force applied at any point of an elastic body is a bound vector. In this case two vectors are equal only if they are identical.

**Addition of vectors**

As you know two vectors can be added graphically using either the triangle law or the parallelogram law. To find the sum \( \mathbf{a} + \mathbf{b} \) using the triangle law, you place the tail of \( \mathbf{b} \) at the head of \( \mathbf{a} \) (Fig. 1.3a). The vector \( \mathbf{c} \) from the tail of \( \mathbf{a} \) to the head of \( \mathbf{b} \) is the vector \( \mathbf{a} + \mathbf{b} \):

\[
\mathbf{c} = \mathbf{a} + \mathbf{b}
\]
Suppose the two vectors \( \mathbf{a} \) and \( \mathbf{b} \) you want to add have a common tail, say \( O \) (Fig. 1.3b). For example, you may want to add two forces acting simultaneously at a point. Then you could also use the parallelogram law to add \( \mathbf{a} \) and \( \mathbf{b} \) graphically by drawing a parallelogram with \( \mathbf{a} \) and \( \mathbf{b} \) as its sides. Then the vector \( \mathbf{c} \) represented by the diagonal of the parallelogram through the common tail is the sum of the vectors \( \mathbf{a} \) and \( \mathbf{b} \) (Fig. 1.3b). When you want to add more than two vectors, you can repeatedly apply any of these two laws.

You can see that vector addition defined according to Eq. (1.1) is not an algebraic sum. We cannot add the magnitudes of \( \mathbf{a} \) and \( \mathbf{b} \) to get the magnitude of \( \mathbf{c} \).

From the definition of vector addition it follows that

\[
\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \quad \text{(commutative law for addition)} \\
(a + b) + c = a + (b + c) \quad \text{(associative law for addition)}
\]

Thus the order in which you add vectors does not matter (Fig. 1.4).

In physics, quite often you come across forces or other vectors that are parallel and act at different points, so that their lines of action are different. You are required to find the resultant of such parallel vectors which do not act along the same line of action. Such vectors are parallel, \textit{noncollinear} vectors. For example, you may need to find the resultant of two weights hung on a beam (Fig. 1.5a). Of course, you might think that we can get their resultant by putting one of the vectors on top of another and adding them. But this will give us only the magnitude and direction of the resultant and not its line of action. So let us learn adding parallel, \textit{noncollinear} vectors in the following example in order to find the line of action.
Example 1: Graphical addition of parallel, noncollinear vectors

Consider two parallel vectors $\mathbf{a}$ and $\mathbf{b}$ which have different lines of action (Fig. 1.5b). Draw a line $\overline{AB}$ connecting their tails. Now draw two parallelograms, one parallelogram with sides $\mathbf{a}$ and $\frac{\mathbf{a} + \mathbf{b}}{2}$, the other with sides $\mathbf{b}$ and $-\frac{\mathbf{a} + \mathbf{b}}{2}$. Draw the diagonals $\mathbf{c}$ and $\mathbf{d}$ of these two parallelograms through $A$ and $B$, respectively. We can now verify that the sum of $\mathbf{a}$ and $\mathbf{b}$ is equal to the sum of $\mathbf{c}$ and $\mathbf{d}$. From Fig. 1.5b,

\begin{align*}
\mathbf{c} &= \mathbf{a} + \mathbf{b} = \mathbf{a} + \frac{\mathbf{a} + \mathbf{b}}{2} \\
\mathbf{d} &= \mathbf{b} - \mathbf{b} = -\frac{\mathbf{a} + \mathbf{b}}{2}
\end{align*}

\therefore \quad \mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d}

To find the vector $\mathbf{c} + \mathbf{d}$, you can slide $\mathbf{c}$ and $\mathbf{d}$ along their lines of action until their tails coincide at $E$. Their resultant can be found from the parallelogram law. It is the vector $\overrightarrow{EF}$ drawn from $E$. This is the correct sum of $\mathbf{a}$ and $\mathbf{b}$, parallel to them and equal in magnitude to $\mathbf{a} + \mathbf{b}$. It is correct in magnitude, direction and line of action. You can see that $E$ is the point of intersection of the vectors $\mathbf{c}$ and $\mathbf{d}$. When $E$ is extended, it meets $\overline{AB}$ at a point $P$ such that

$\mathbf{AP} = \mathbf{b} - \mathbf{b}$

We will now prove this result here. Using this method you can also add parallel vectors acting opposite to each other, for example, the forces on the two charges of an electric dipole placed in an electric field (Fig. 1.5c).

You may like to work out a simple SAQ before studying further. Don't be daunted by its length!

**SAQ 1**

a) Find the vectors in Fig. 1.6 equal to the vector $\mathbf{a}$ shown there.

![Fig. 1.6: (a) Finding equal vectors; (b) finding the resultant of forces.](image)

b) Four forces, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$, all in the same plane, are applied at a point (Fig. 1.6b). Find graphically the resultant $\mathbf{F}$ of these forces through the point $O$. Show that it is always the same, whatever be the order in which you add vectors. Draw at least two different sequences for adding these vectors.

c) Consider two displacements, one of magnitude 3 m and another of magnitude 4 m. Show graphically how the displacement vectors may be combined to get a resultant displacement of magnitude (i) 7 m, (ii) 1 m and (iii) 5 m.

The multiplication of a vector by a scalar follows logically from the definition of vector addition.

**Multiplication of a vector by a scalar**

What is the vector $\mathbf{a} + \mathbf{a} + \mathbf{a}$? From the methods of vector addition you can see that it is a vector three times as long as $\mathbf{a}$ and is in the same direction as $\mathbf{a}$. We can extend this to a
general definition of the product of a vector $a$ by a positive scalar $m$: The product $ma$ is a vector in the same direction as $a$, but its magnitude is $ma$, i.e., $m$ times the magnitude of $a$ (Fig. 1.7a). If $m < 0$, $ma$ is in a direction opposite to $a$, but its magnitude is $ma$. So for $m = -1$, the new vector $-a$ is a vector equal and opposite in direction (antiparallel) to $a$. In physics, the most common example of this kind is Newton's second law $F = ma$. Here force is expressed as a product of mass (scalar) and acceleration (vector).

Three more laws related to vectors follow from the discussion so far. These are stated below:

\[ m(na) = (mn)a \]
\[ (m+n)a = ma + na \]
\[ m(a+b) = ma + mb \]

where $m$ and $n$ are numbers.

Subtraction of vectors

Subtraction of a vector $b$ from $a$, i.e., the operation $a - b$ can be seen as adding the vector $(-b)$ to $a$, i.e.,

\[ a - b = a + (-b). \]

So to subtract $b$ from $a$ graphically, we multiply $b$ by $-1$ and add the new vector $-b$ to $a$ using either the triangle law or the parallelogram law (Fig. 1.7b).

![Figure 1.7](image-url)

(a) Multiplication of a vector by a scalar. (b) Graphical subtraction of a vector $b$ and $a$. (c) Null vector.

Null vector

Now suppose two equal and opposite forces are applied to a point (Fig. 1.7c). What is their resultant? It is $F + (-F) = F - F$. You can see that $F - F$ is a vector of zero magnitude. Can we define a direction for it? Obviously not. Such a vector is called a null vector, or a zero vector, i.e., it has zero magnitude and no direction is defined for it. It is denoted by $0$. We also get a zero vector when we multiply a vector by the scalar zero.

Unit vector

Consider the product of $a$ with the scalar $\frac{1}{a}$. You can see that the magnitude of the vector $\frac{a}{a}$ is 1. A vector of length or magnitude 1 is called a unit vector. Now since $\frac{1}{a}$ is a positive number, the direction of vector $\frac{a}{a}$ is along $a$. Hence $\frac{a}{a}$ is the unit vector in the direction of $a$.

It is denoted by $\hat{a}$ (Fig. 1.7d). Thus, we can also write

\[ a = \hat{a} \]

(1.5)

A unit vector is used to denote direction in space. So it is a handy tool to represent a vector. We can represent a vector in any direction as the product of its magnitude and the unit vector in that direction. By convention, unit vectors are taken to be dimensionless.

Multiplication of vectors

You have studied about two kinds of products of vectors, the scalar product (and the vector product. We shall discuss these products in greater detail in Sec. 1.4. Here we only recall their definitions and list some of their properties.

The scalar product of two non-zero vectors $a$ and $b$ (written as $a \cdot b$) is a scalar defined as

\[ a \cdot b = ab \cos \theta \]

(1.6a)
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where $\theta$ is the angle between $a$ and $b$ when they are drawn with a common tail (Fig. 1.8a).
The term $a \cdot b$ is pronounced as 'a dot b'. Therefore, $a \cdot b$ is also called the dot product. By 
convention, we take $\theta$ to be the angle smaller than or equal to $\pi$ so that $0 \leq \theta \leq \pi$.

If either $a$ or $b$ is a zero vector, or $a$ is perpendicular to $b$ then from Eq. (1.6a):
\[ a \cdot b = 0. \]  
(1.6b)
The scalar product defined in this way has several properties, which we are stating briefly.

Properties of the dot product

1) $a \cdot b$ is a scalar 
2) $a \cdot b = b \cdot a$, i.e., the dot product is commutative. 
3) $a \cdot (b + c) = a \cdot b + a \cdot c$, i.e., the dot product is associative over 
addition. 
4) $(m \cdot a) \cdot b = m (a \cdot b) = a \cdot (m \cdot b)$ 
5) If $a \cdot b = 0$, and $a$ and $b$ are not zero vectors then $a$ is perpendicular to $b$. 
6) $|a| = \sqrt{a \cdot a}$ 
7) $a \cdot a \geq 0$ for any non-zero vector $a$. 
8) $a \cdot a = 0$ only if $a = 0$. 

\[ a \times b \]  
(1.8a)
The vector or cross product of $a$ and $b$, written as $a \times b$, is defined to be the vector 
\[ c = a \times b = (a \cdot b \sin \theta) \hat{c} \]
(1.8a)
where $\theta$ is the angle between $a$ and $b$ (Fig. 1.8b). The expression $a \times b$ is pronounced as 'a cross 
b'. The magnitude of $c$ is $|a \sin \theta|$, where $\theta$ is the angle smaller than or equal to $\pi$, i.e., $0 \leq \theta \leq \pi$.
Here $\hat{c}$ is a unit vector perpendicular to $a$ and $b$. The sense of $\hat{c}$ is given by the right-hand rule: 
Rotate the fingers of your right hand so that the fingertips point along the direction of rotation of 
a into b through $\theta$ (Fig. 1.8c). The thumb gives the direction of $\hat{c}$ (Fig. 1.9). Defined in this way $a$, $b$ 
and $c$ are said to form a right-handed triple or a right-handed triad.

\[ a \times b \]  
(1.8a)

Fig. 1.9: The right-hand rule for finding the directions of $a \times b$. $a$, $b$ and $c$ are said to form a right-handed 
triple. The direction of $c$ is also the direction in which a right-handed screw would move if it is rotated from $a$ 
towards $b$. 


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Note that in the definition of the cross product the order of a and b is very important. Thus, \( b \times a \) is not the same vector as \( a \times b \) (Fig. 1.8c). In fact, you can use the right-hand rule to show that

\[
a \times b = -b \times a
\]

(1.8b)

So the vector product is not commutative. We will now state without proof some more properties of the vector product.

### Properties of the vector product

1) \( a \times b \) is a vector

2) \( a \times b = -b \times a \) (1.9a)

3) If \( a \) and \( b \) are non-zero vectors, and \( a \times b = 0 \), then \( a \) is parallel to \( b \).

4) \( a \times a = 0 \), for any vector \( a \) (1.9b)

The properties (3) and (4) follow directly from the definition because in both cases \( \theta \) is zero.

5) \( a \times (b + c) = (a \times b) + (a \times c) \) (1.9c)

6) \( (a + b) \times c = (a \times c) + (b \times c) \) (1.9d)

That is, the vector product is distributive over addition. Notice that the order in which these vectors appear remains the same.

7) \( (ma) \times b = m (a \times b) = a \times (mb) \) (1.9e)

You may like to end this section by working out another SAQ.

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**SAQ 2**

a) Let \( V \) be the wind velocity of 50 \( \text{km h}^{-1} \) from north-east. Write down the vector representing a wind velocity of (i) 75 \( \text{km h}^{-1} \) from north-east, (ii) 100 \( \text{km h}^{-1} \) from south-west, in terms of \( V \).

b) Let \( \hat{i} \) and \( \hat{j} \) denote unit vectors in the directions of east and north, respectively. Specify the following vectors in terms of \( \hat{i} \) and \( \hat{j} \): (i) The displacements of persons going from New Delhi to Kanyakumari (about 2300 \( \text{km} \) due south) and from New Delhi to Dibrugarh (about 1700 \( \text{km} \) due east).

c) Draw a diagram to show that

\[
a - (b - c) = (a - b) + c
\]

So far we have quickly recalled vectors as you have studied them at school. Let us now look afresh at vectors, in the form in which you will be using them in your undergraduate physics courses.

### 1.3 A NEW LOOK AT VECTORS

So far you have represented vectors geometrically, i.e., without referring them to any coordinate system. This kind of a representation is fairly easy to visualise in two-dimensions. However, our world happens to be three-dimensional. Visualising vectors geometrically in three dimensions is complicated. Therefore, we need an alternative way to represent vectors, which is termed the analytical (or algebraic) approach. Let us learn how to express vectors analytically, i.e., in terms of their components. You will also learn a new, precise way of defining vectors which will be of use in advanced courses.

#### 1.3.1 Vector Components Relative to a Coordinate System

You are familiar with the two-dimensional Cartesian coordinate system (Fig. 1.10). Here, Ox and Oy are two mutually perpendicular (or orthogonal) axes. Let us consider a vector \( a \) in the \( xy \)-plane. Let us draw perpendiculars from the ends of \( a \) on the Ox and Oy-axes. Then the projections \( a_x \) and \( a_y \), so formed on the Ox and Oy axes are called, respectively, the \( x \) and \( y \) components (or Cartesian components) of \( a \). From Fig. 1.10, you can see that
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Fig. 1.10: Cartesian components \( a, a, \) of a vector in two dimensions. \((x, y,),(x, y)\) are the coordinates of the tail and head of \( a \), respectively. \( \alpha \) and \( \beta \) are the angles which \( a \) makes with the \( x \) and \( y \)-axes, respectively.

\[ a = r \cos \alpha \]
\[ a = r \sin \alpha \]
\[ a = \sqrt{a_x^2 + a_y^2} \]
\[ \tan \alpha = \frac{a_x}{a_y} \]

The meaning of the symbols in Eqs. (1.10a to d) is explained in the caption of Fig. 1.10. Read it carefully and verify Eqs. (1.10a to d) quickly before studying further. The angle \( \alpha \) specifies the direction of \( a \). The cosines of the angles \( \alpha \) and \( \beta \), which \( a \) makes with the \( x \) and \( y \)-axes, respectively, are called the direction cosines of \( a \) with respect to this coordinate system. These are denoted by \( l \) and \( m \). Thus

\[ l = \cos \alpha = \frac{a_x}{a} \]
\[ m = \cos \beta = \frac{a_y}{a} \]

Now let \( i \) and \( j \) be the unit vectors in the positive \( x \) and positive \( y \)-directions, respectively. Whereas \( a_x \) and \( a_y \) are the scalar components of \( a \) in the \((x, y)\) plane, \( a_i \) and \( a_j \) are its vector components. You can see from Fig. 1.10 that \( a \) is the sum of its vector components:

\[ a = a_i i + a_j j \]

Eq (1.11) represents a two-dimensional vector in terms of its components and the unit vectors along the two coordinate axes \( OX \) and \( OY \). We can extend these concepts to vectors in three dimensions.

Vector components in three dimensions

You have studied the three-dimensional Cartesian coordinate system in your school courses. In this system we have a set of three orthogonal (mutually perpendicular) axes. So a vector \( a \) has three components \( a_x, a_y, a_z \) along the \( x, y \) and \( z \)-axes (Fig. 1.11). In terms of the coordinates of the tail \((x, y, z)\) and head \((x, y, z)\) of \( a \), these are

\[ a_x = x_2 - x_1 \]
\[ a_y = y_2 - y_1 \]
\[ a_z = z_2 - z_1 \]

The magnitude of \( a \) is the distance between the two points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\):

\[ a = \sqrt{a_x^2 + a_y^2 + a_z^2} \]

How do we specify the direction of \( a \)? This is given by the angles \( \alpha, \beta, \) and \( \gamma \) between \( a \) and the positive \( x, y \) and \( z \)-axes, respectively. Generally we use the cosines of these angles. These are

\[ \cos \alpha = \frac{a_x}{a} \]
\[ \cos \beta = \frac{a_y}{a} \]
\[ \cos \gamma = \frac{a_z}{a} \]
205 \cos \hat{i}, \cos \hat{j}, \cos \hat{k} are called the direction cosines of \( \mathbf{a} \) in a three-dimensional orthogonal system. We denote \( \cos \hat{i}, \cos \hat{j}, \cos \hat{k} \) by \( l, m, \) and \( n, \) respectively.

Let us now extend Eq. (1.11) to three dimensions. Let \( \hat{i}, \hat{j}, \) and \( \hat{k} \) be the respective unit vectors in the positive \( x, y \) and \( z \) directions. Here \( \hat{i}, \hat{j}, \hat{k} \) form a right-handed triple and the corresponding Cartesian coordinate system \((x, y, z)\) is a right-handed system. Then \( a\hat{i}, a\hat{j}, a\hat{k} \) are the vector components of \( \mathbf{a} \) in the \( x, y \) and \( z \) directions, respectively, and

\[ \mathbf{a} = a\hat{i} + a\hat{j} + a\hat{k} \]  
\[(1.13)\]

So far you have studied how to represent vectors analytically, i.e., in terms of their components. You may like to work out on SAQ to consolidate your understanding and then study further.

**SAQ 3**

a) What are the components of the vectors \( \mathbf{b} \) and \( \mathbf{c} \) shown in Fig. 1.6a? Determine their magnitudes and directions.

b) Draw on Fig. 1.6a, a vector \( \mathbf{p} \) having components \( p_x = 3, \) \( p_y = 2 \) and a vector \( \mathbf{q} \) of magnitude 4 and direction given by \( \theta = 60^\circ \).

c) Verify Eq. (1.13) graphically for a vector \( \mathbf{b} \) with its tail at the origin.

Let us now address the question of a precise definition of vectors. To define a vector precisely we have to consider how a vector transforms when we change (or transform) the coordinate system in which it is represented.

### 1.3.2 Transformation of Coordinate Systems and Vector Components

Let us start with a two-dimensional vector. Consider the displacement \( \mathbf{d} \) of a particle of mass \( m \) with respect to the origin of a two-dimensional \((x, y)\) Cartesian coordinate system (Fig. 1.12).

![Fig. 1.12: A vector \( \mathbf{d} \) represented in two different Cartesian coordinate systems. Remember that both these systems are orthogonal.](image)

In terms of its components in the \( xy \) coordinate system we can write \( \mathbf{d} \) as

\[ \mathbf{d} = d_x\hat{i} + d_y\hat{j} \]  
\[(1.14a)\]

Now let us rotate the coordinate axes by an angle \( \alpha \) so that we have a different Cartesian coordinate system \((x', y')\), i.e., we have transformed the \((xy)\) system into the \((x'y')\) system. What quantities have undergone a change? First consider the mass of the particle. Whether you measure the mass in the \((xy)\) or in the \((x'y')\) coordinate system it remains the same. This gives us an important result about scalars:

Scalar quantities remain invariant (unchanged) under any transformation of coordinate systems.

What about the displacement vector \( \mathbf{d} \)? Obviously, its components \( d'_x, d'_y \) in the \((x', y')\) system are different from \( d_x, d_y \). If \( \hat{i}', \hat{j}' \) denote unit vectors along \( x', y' \) axes, then we have
Vector Calculus

The question is: How have the components of \( d \) transformed with the transformation of the coordinate system? In other words, what are the relations between \( d_x, d_y, d_z \)?

You can see from Fig. 1.12 that

\[
d' = d \cos \theta, \quad d_y = d \sin \theta
\]

and

\[
d'_x = d \cos \theta' - d \cos (\theta - \alpha) = d (\cos \theta \cos \alpha + \sin \theta \sin \alpha)
\]

\[
d'_y = d \sin \theta \cos (\theta - \alpha) = d (\cos \theta \sin \alpha - \cos \alpha \sin \theta)
\]

We can express these results in a general notation as

\[
d'_x = c_{11} d_x + c_{12} d_y
\]

\[
d'_y = c_{21} d_x + c_{22} d_y
\]

From your school mathematics courses you have studied about matrices. You can see that Eqs. (1.15a, b, c, and d) can also be expressed in terms of matrices and their product as follows:

\[
\begin{pmatrix}
d'_x \\
d'_y \\
\end{pmatrix} =
\begin{pmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{pmatrix}
\begin{pmatrix}
d_x \\
d_y
\end{pmatrix}
\]

Eqs. (1.15a, b, c, and d) specify how the components of the displacement vector transform under a transformation of the coordinate system. In this specific example we have transformed the \((x, y)\) coordinate system into \((x', y')\) system by simply rotating the axes, keeping their origins common. With these ideas we can **define** a two-dimensional vector precisely.

A two-dimensional vector \( \mathbf{a} \) is defined as a set of numbers (components) in every orthogonal coordinate system, such that if \( a_x, a_y \) are its components in one system and \( a'_x, a'_y \) are the components in another system, the two sets of components are related by equations similar to Eqs. (1.15a, b, c, and d), i.e.

\[
a'_x = c_{11} a_x + c_{12} a_y
\]

\[
a'_y = c_{21} a_x + c_{22} a_y
\]

or by the converse relations

\[
a_x = c_{11} a'_x + c_{12} a'_y
\]

\[
a_y = c_{21} a'_x + c_{22} a'_y
\]

where \( c_{11} = \cos \alpha, c_{12} = \sin \alpha, c_{21} = -\sin \alpha, c_{22} = \cos \alpha\).

In terms of matrices these relations are

\[
\begin{pmatrix}
a'_x \\
a'_y
\end{pmatrix} =
\begin{pmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{pmatrix}
\begin{pmatrix}
a_x \\
a_y
\end{pmatrix}
\]

where \( c_{ij} \)'s are given by Eq. (1.16d).

As you know the direction of \( \mathbf{a} \) is specified by the respective direction cosines in the two systems. What about its magnitude? It can be shown that the magnitude of the vector \( \mathbf{a} \) remains the same under a coordinate transformation, i.e.

\[
|\mathbf{a'}| = |\mathbf{a}|
\]

We leave its proof as an exercise for you. Try the following SAQ.
We can readily extend these results to three dimensions.

1.33 The Precise Definition of Three-dimensional Vectors

Here we will use a general method to arrive at the precise definition. Let \((x,y,z)\) and \((x',y',z')\) be any two systems of Cartesian coordinates. Let the representation of a vector \(a\) in the two systems, respectively, be

\[
a = a_1^i + a_2^j + a_3^k \quad (1.18a)
\]

and

\[
a = a_1'^i + a_2'^j + a_3'^k \quad (1.18b)
\]

Here \(\hat{i}, \hat{j}, \hat{k}\) and \(\hat{i}', \hat{j}', \hat{k}'\) are the unit vectors in the positive \(x, y, z\) and \(x', y', z'\) directions, respectively. Then using Eqs. (1.7b,c) and (1.18a,b) we obtain

\[
\hat{i}' \cdot a = a_1^i \cdot \hat{i}' + a_2^j \cdot \hat{j}' + a_3^k \cdot \hat{k}' \quad (1.19a)
\]

and also

\[
\hat{i}' \cdot a = a_1'^i \cdot \hat{i}' + a_2'^j \cdot \hat{j}' + a_3'^k \cdot \hat{k}' \quad (1.19b)
\]

Again using the definition of scalar product in Eq. (1.19b), you can see that

\[
\hat{i}' \cdot a = a_1^i, \quad \hat{j}' \cdot a = a_2^j, \quad \hat{k}' \cdot a = a_3^k
\]

Thus we can write from Eqs. (1.19a and b) that

\[
a_1 = a_1^i, \quad a_2 = a_2^j, \quad a_3 = a_3^k \quad (1.20a)
\]

Similarly, you can take the scalar products \(\hat{i} \cdot a\) and \(\hat{k} \cdot a\) to show that

\[
a_1 = a_1^i + a_1^j + a_3^k \quad (1.20b)
\]

and

\[
a_1 = a_1^i \times a_1^j \quad (1.20c)
\]

SAQ 5

Verify Eqs. (1.20b) and (1.20c).

To write Eqs. (1.20a to c) in a compact form we adopt the following notation

\[
\begin{align*}
\hat{i}' & = c_{11} \hat{i} + c_{12} \hat{j} + c_{13} \hat{k} \\
\hat{j}' & = c_{21} \hat{i} + c_{22} \hat{j} + c_{23} \hat{k} \\
\hat{k}' & = c_{31} \hat{i} + c_{32} \hat{j} + c_{33} \hat{k}
\end{align*}
\]

(1.21)

These coefficients are nothing but the cosines of the angles between the respective positive axes. For example, \(c_{12}\) is the cosine of the angle between positive \(x'\) and \(y\)-axes, and so on.

Then we can write

\[
\begin{align*}
a_1 & = c_{11} a_1 + c_{12} a_2 + c_{13} a_3 \\
a_2 & = c_{21} a_1 + c_{22} a_2 + c_{23} a_3 \\
a_3 & = c_{31} a_1 + c_{32} a_2 + c_{33} a_3
\end{align*}
\]

(1.22)

In the matrix form:

\[
\begin{pmatrix}
a_1 \\ a_2 \\ a_3
\end{pmatrix} =
\begin{pmatrix}
c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33}
\end{pmatrix}
\begin{pmatrix}
a_1 \\ a_2 \\ a_3
\end{pmatrix}
\]

(1.22d)

You can similarly derive the inverse relations.

\[
\begin{align*}
a_1 & = c_{11} a_1 + c_{12} a_2 + c_{13} a_3 \\
a_2 & = c_{21} a_1 + c_{22} a_2 + c_{23} a_3 \\
a_3 & = c_{31} a_1 + c_{32} a_2 + c_{33} a_3
\end{align*}
\]

(1.23a)

1.33 The Precise Definition of Three-dimensional Vectors

Here we will use a general method to arrive at the precise definition. Let \((x,y,z)\) and \((x',y',z')\) be any two systems of Cartesian coordinates. Let the representation of a vector \(a\) in the two systems, respectively, be

\[
a = a_1^i + a_2^j + a_3^k \quad (1.18a)
\]

and

\[
a = a_1'^i + a_2'^j + a_3'^k \quad (1.18b)
\]

Here \(\hat{i}, \hat{j}, \hat{k}\) and \(\hat{i}', \hat{j}', \hat{k}'\) are the unit vectors in the positive \(x, y, z\) and \(x', y', z'\) directions, respectively. Then using Eqs. (1.7b,c) and (1.18a,b) we obtain

\[
\hat{i}' \cdot a = a_1^i \cdot \hat{i}' + a_2^j \cdot \hat{j}' + a_3^k \cdot \hat{k}' \quad (1.19a)
\]

and also

\[
\hat{i}' \cdot a = a_1'^i \cdot \hat{i}' + a_2'^j \cdot \hat{j}' + a_3'^k \cdot \hat{k}' \quad (1.19b)
\]

Again using the definition of scalar product in Eq. (1.19b), you can see that

\[
\hat{i}' \cdot a = a_1^i, \quad \hat{j}' \cdot a = a_2^j, \quad \hat{k}' \cdot a = a_3^k
\]

Thus we can write from Eqs. (1.19a and b) that

\[
a_1 = a_1^i, \quad a_2 = a_2^j, \quad a_3 = a_3^k \quad (1.20a)
\]

Similarly, you can take the scalar products \(\hat{j}' \cdot a\) and \(\hat{k}' \cdot a\) to show that

\[
a_2 = a_1^i + a_1^j + a_3^k \quad (1.20b)
\]

and

\[
a_3 = a_1^i \times a_1^j \quad (1.20c)
\]

SAQ 5

Verify Eqs. (1.20b) and (1.20c).

To write Eqs. (1.20a to c) in a compact form we adopt the following notation

\[
\begin{align*}
\hat{i}' & = c_{11} \hat{i} + c_{12} \hat{j} + c_{13} \hat{k} \\
\hat{j}' & = c_{21} \hat{i} + c_{22} \hat{j} + c_{23} \hat{k} \\
\hat{k}' & = c_{31} \hat{i} + c_{32} \hat{j} + c_{33} \hat{k}
\end{align*}
\]

(1.21)

These coefficients are nothing but the cosines of the angles between the respective positive axes. For example, \(c_{12}\) is the cosine of the angle between positive \(x'\) and \(y\)-axes, and so on.

Then we can write

\[
\begin{align*}
a_1 & = c_{11} a_1 + c_{12} a_2 + c_{13} a_3 \\
a_2 & = c_{21} a_1 + c_{22} a_2 + c_{23} a_3 \\
a_3 & = c_{31} a_1 + c_{32} a_2 + c_{33} a_3
\end{align*}
\]

(1.22)

In the matrix form:

\[
\begin{pmatrix}
a_1 \\ a_2 \\ a_3
\end{pmatrix} =
\begin{pmatrix}
c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33}
\end{pmatrix}
\begin{pmatrix}
a_1 \\ a_2 \\ a_3
\end{pmatrix}
\]

(1.22d)

You can similarly derive the inverse relations.

\[
\begin{align*}
a_1 & = c_{11} a_1 + c_{12} a_2 + c_{13} a_3 \\
a_2 & = c_{21} a_1 + c_{22} a_2 + c_{23} a_3 \\
a_3 & = c_{31} a_1 + c_{32} a_2 + c_{33} a_3
\end{align*}
\]

(1.23)
Recall that cosine of angle \( a \) with cosine of angle \( b \) is equal to cosine of angle \( a \) times cosine of angle \( b \) plus sine of angle \( a \) times sine of angle \( b \). In other words, what are the relations between \( d_x', d_y', d_z' \)? You can see from Fig. 1.12 that

\[
d_x' = d_x \cos \theta, \quad d_y' = d_y \sin \theta
\]

and

\[
d_x'' = d_x \cos \gamma - d_y \sin \gamma \cos (\theta - \alpha) = d \cos \theta \cos \alpha + \sin \theta \sin \alpha
\]

\[
d_y'' = d_y \sin \gamma \cos (\theta - \alpha) = d \sin \theta \cos \alpha
\]

We can express these results in a general notation as

\[
d_x'' = c_{11} d_x + c_{12} d_y
\]

\[
d_y'' = c_{21} d_x + c_{22} d_y
\]

In your school mathematics course you have studied about matrices. You can see that Eqs. (1.15a, b, c) and (1.16a, b, c) can also be expressed in terms of matrices and their products as follows:

\[
\begin{bmatrix} d_x' \\ d_y' \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} d_x \\ d_y \end{bmatrix}
\]

Eqs. (1.15a, b, c) specify how the components of the displacement vector transform under a transformation of the coordinate system. In this specific example we have transformed the \((x, y)\) coordinate system into \((x', y')\) system by simply rotating the axes, keeping their origins common. With these ideas we can define a two-dimensional vector precisely.

A two-dimensional vector \( \mathbf{a} \) is defined as a set of numbers (components) in every orthogonal coordinate system, such that if \( a_x, a_y \) are its components in one system and \( a'_x, a'_y \) are the components in another system, the two sets of components are related by equations similar to Eqs. (1.15a, b, c), i.e.,

\[
a'_x = c_{11} a_x + c_{12} a_y
\]

\[
a'_y = c_{21} a_x + c_{22} a_y
\]

or by the converse relations

\[
a'_x = c_{11} a'_x + c_{12} a'_y
\]

\[
a'_y = c_{21} a'_x + c_{22} a'_y
\]

In terms of matrices these relations are

\[
\begin{bmatrix} d'_x \\ d'_y \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} d_x \\ d_y \end{bmatrix}
\]

where \( c_{ij}' \) are given by Eq. (1.16e).

As you know the direction of \( \mathbf{a} \) is specified by the respective direction cosines in the two systems. What about its magnitude? It can be shown that the magnitude of the vector \( \mathbf{a} \) remains the same under a coordinate transformation, i.e.,

\[
a = \sqrt{a_x^2 + a_y^2} = \sqrt{a_x'^2 + a_y'^2}
\]

We leave its proof as an exercise for you. Try the following SAQ.
We can readily extend these results to three dimensions.

1.33 The Precise Definition of Three-dimensional Vectors

Here we will use a general method to arrive at the precise definition. Let \((x', y', z')\) and \((x, y, z)\) be any two systems of Cartesian coordinates. Let the representation of a vector \(\mathbf{a}\) in the two systems, respectively, be

\[
\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \quad (1.18a)
\]

and

\[
\hat{\mathbf{a}} = a'_1 \hat{\mathbf{i}} + a'_2 \hat{\mathbf{j}} + a'_3 \hat{\mathbf{k}} \quad (1.18b)
\]

Here \(\mathbf{i}, \mathbf{j}, \mathbf{k}\) and \(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\) are the unit vectors in the positive \(x, y, z\) and \(x', y', z'\) directions, respectively. Then using Eqs. (1.7b, c) and (1.18a, b) we obtain

\[
\hat{\mathbf{i}} \cdot \mathbf{a} = a_1 \hat{\mathbf{i}} \cdot \mathbf{i} + a_2 \hat{\mathbf{i}} \cdot \mathbf{j} + a_3 \hat{\mathbf{i}} \cdot \mathbf{k} \quad (1.19a)
\]

and also

\[
\hat{\mathbf{i}} \cdot \hat{\mathbf{a}} = a'_1 \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} + a'_2 \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} + a'_3 \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} \quad (1.19b)
\]

Again using the definition of scalar product in Eq. (1.19b), you can see that \(\hat{\mathbf{i}} \cdot \mathbf{a} = a'_1\), since \(\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = 1, \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 1.1 \cos 90^\circ = 0\) and \(\hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = 1.1 \cos 90^\circ = 0\). Thus we can write from Eqs. (1.19a and b) that

\[
a_i = a_1 \hat{\mathbf{i}} \cdot \mathbf{i} + a_2 \hat{\mathbf{i}} \cdot \mathbf{j} + a_3 \hat{\mathbf{i}} \cdot \mathbf{k} \quad (1.20a)
\]

Similarly, you can take the scalar products \(\hat{\mathbf{j}} \cdot \mathbf{a}\) and \(\hat{\mathbf{k}} \cdot \mathbf{a}\) to show that

\[
a_j = a_1 \hat{\mathbf{j}} \cdot \mathbf{i} + a_2 \hat{\mathbf{j}} \cdot \mathbf{j} + a_3 \hat{\mathbf{j}} \cdot \mathbf{k} \quad (1.20b)
\]

and

\[
a_k = a_1 \hat{\mathbf{k}} \cdot \mathbf{i} + a_2 \hat{\mathbf{k}} \cdot \mathbf{j} + a_3 \hat{\mathbf{k}} \cdot \mathbf{k} \quad (1.20c)
\]
You can easily verify by using the definition of scalar product that Eqs. (1.1) and (1.2) can be written as

\[ c = \frac{1}{2} \left( a_1^2 + b_1^2 - a_2^2 - b_2^2 \right) \]

\[ c_1 = \frac{1}{2} \left( a_1^2 + b_1^2 - a_2^2 - b_2^2 \right) \]

\[ c_2 = \frac{1}{2} \left( a_1^2 + b_1^2 - a_2^2 - b_2^2 \right) \]

\[ c_3 = \frac{1}{2} \left( a_1^2 + b_1^2 - a_2^2 - b_2^2 \right) \]

Now we can give a precise definition of a three-dimensional vector:

A three-dimensional vector is defined as a set of three numbers (or components) referred to an orthogonal coordinate system. If \( a_1, a_2, a_3 \) are the components of \( a \) in one system and \( a'_1, a'_2, a'_3 \) are its components in another system, these two sets of components are related by Eqs. (1.22) and (1.23). A physical quantity is called a vector if it transforms under a change of coordinate system in accordance with Eqs. (1.21), (1.22) and (1.23).

While the components of a vector are different in different coordinate systems, its magnitude remains the same in all systems:

\[ a = a' = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{a'_1^2 + a'_2^2 + a'_3^2} \]

Thus the magnitude of a vector remains invariant under any transformation of coordinate systems.

You may like to end this section with an SAQ.

**SAQ 6**

Verify Eqs. (1.22a, b and c).

In this section you have learnt the precise analytical definition of a vector in terms of the transformation of its components, under the transformation of coordinate systems. Let us now reconsider the basic vector algebra using vectors in their component form. As these analytical methods are used often in physics it is essential for you to learn them.

### 1.4 Analytical Approach to Vector Algebra

In Sec. 1.2 you have learnt how to add, subtract and multiply vectors using the graphical method. In this section you will be learning the same algebra using the analytical method, i.e., using vectors in component form. In our subsequent discussion we shall restrict ourselves to a single coordinate system. Thus, if two vectors \( a \) and \( b \) are equal, then their corresponding components must be equal. So given two vectors

\[ a = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}, \quad b = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k} \]

(1.25)

Before going further, we would like you to recall that the unit vectors \( \hat{i}, \hat{j}, \hat{k} \) are dimensionless. Now in physics, vectors represent physical quantities and have certain dimensions associated with them. Thus, the magnitude of a vector or its components have dimensions. For example, a displacement \( d \) of 8 m along the \( x \)-axis is written as

\[ d = 8 \text{ m} \]

and \( \hat{i} \) is dimensionless.

Let us now learn adding and subtracting vectors analytically.

#### 1.4.1 Vector Addition and Subtraction

Suppose we have to add the two vectors \( a \) and \( b \) of Eq. (1.25). Let the resultant \( c \) of the vectors \( a \) and \( b \) have components \( c_1, c_2, c_3 \). Then

\[ c = a + b \]

(1.27a)

or

\[ c_1 = a_1 + b_1 \]

\[ c_2 = a_2 + b_2 \]

\[ c_3 = a_3 + b_3 \]

(1.27b)

From Eq. (1.26) it follows that

\[ c_1 = a_1 + b_1 \]

\[ c_2 = a_2 + b_2 \]

\[ c_3 = a_3 + b_3 \]

(1.27c)
The same procedure applies to subtraction. So the general rule is: To add or subtract vectors, add or subtract their like components. This rule can be applied to add more than two vectors. You can also extend it to multiplication of a vector by a scalar, e.g.

\[ m \mathbf{a} = m(a, \mathbf{i} + a, \mathbf{j} + a, \mathbf{k}) = ma, \mathbf{i} + ma, \mathbf{j} + ma, \mathbf{k} \]  

(1.28)

Now that you know how to analytically add and subtract vectors, you may like to solve an SAQ to test your knowledge.

**SAQ 7.**

A radar station detects an aeroplane approaching from the east. At first sighting, the aeroplane is 500m away at 30° above the horizon. The aeroplane is tracked for another 120° in the vertical plane containing the east-west direction. At final sighting it is 1000m away from the radar (Fig. 1.13). Find the displacement of the aeroplane in component form during the period of observation.

![Fig. 1.13](image)

We will now discuss the products of two vectors in their component form and also take up some of their physical applications.

### 1.4.2 Scalar Product in Component Form

You already know the definition of the scalar product (Eq. (1.6a)) and its properties. You have also worked out the scalar products of unit vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) etc. in Sec. 1.3.3. Let us repeat these results here. From Eqs. (1.6a, b) and (1.7a) it follows that

\[ \mathbf{i} \cdot \mathbf{i} = 1, \quad \mathbf{j} \cdot \mathbf{j} = 1, \quad \mathbf{k} \cdot \mathbf{k} = 1 \]  

(1.29a)

\[ \mathbf{i} \cdot \mathbf{j} = 0, \quad \mathbf{j} \cdot \mathbf{k} = 0, \quad \mathbf{k} \cdot \mathbf{i} = 0 \]  

(1.29b)

You can now use Eqs. (1.29a) and (1.29b) to express \( \mathbf{a} \cdot \mathbf{b} \) in its component form.

**SAQ 8**

Show that for two vectors \( \mathbf{a} \) and \( \mathbf{b} \) given as

\[ \mathbf{a} = a, \mathbf{i} + a, \mathbf{j} + a, \mathbf{k} \]

\[ \mathbf{b} = b, \mathbf{i} + b, \mathbf{j} + b, \mathbf{k} \]

\[ \mathbf{a} \cdot \mathbf{b} = a, b, + a, b, + a, b, \]  

(1.30)

The scalar product finds many uses in physics. Let us now consider some of these applications. You can quickly go through this part.

**Projection of a vector along another vector**

You know that \( a, \ a, \ a, \) in Eq. (1.25) are components of \( \mathbf{a} \) along the \( \mathbf{x}, \mathbf{y}, \mathbf{z} \) axes. We can also define the component or projection of \( \mathbf{a} \) along any other vector \( \mathbf{b} \) using the concept of scalar product.

Suppose \( \mathbf{a} \) and \( \mathbf{b} \) are non-zero vectors and the angle between them is \( \theta \). Then the real number

\[ p = |\mathbf{a}| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \]  

(1.31a)

is called the projection or component of \( \mathbf{a} \) in the direction of \( \mathbf{b} \) (Fig. 1.14).
If \( a = 0 \), then \( \theta \) is undefined and we set \( p = 0 \). Similarly, the projection of \( \mathbf{b} \) on \( \mathbf{a} \) is the real number

\[
q = |\mathbf{b}| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}
\]

We may also interpret \( \mathbf{a} \cdot \mathbf{b} = (-\mathbf{a} \cdot \cos \theta) \) as the magnitude of \( \mathbf{a} \) multiplied by the projection of \( \mathbf{b} \) in the direction of \( \mathbf{a} \).

**Angle between two vectors**

For two non-zero vectors \( \mathbf{a} \) and \( \mathbf{b} \) with an angle \( \theta \) between them, we have from Eq. (1.6a) that

\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}
\]

Using the property of the dot product we can also write the angle \( \theta \) as

\[
\theta = \cos^{-1} \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \right)
\]

**Work expressed as a scalar product**

You are familiar with the example of work, expressed as a scalar product. Let a force \( \mathbf{F} \) be applied on an object at an angle \( \theta \) to the direction of displacement \( \mathbf{d} \). Then

\[
W = (\mathbf{F} \cos \theta) \cdot \mathbf{d} = \mathbf{F} \cdot \mathbf{d}
\]

Many other physical quantities can be expressed as the dot product of two vectors.

**Flux expressed as a scalar product**

A useful physical concept that can be expressed as a scalar product is that of flux. Let us take a simple example first for intuitive understanding. See Fig. 1.15.

**Fig. 1.15:** The flux of rain through a planar surface

Let us represent the collection area of the bucket-top by a vector \( \mathbf{A} \), its magnitude equal to its area \( A \) and direction perpendicular to the surface. Let \( \mathbf{R} \) be another vector such that its direction is along the direction of rainfall. Let its magnitude be equal to the rate at which...
rain falls on a unit area of the surface perpendicular to \( R \). Then you can see that the amount of water collected by the bucket per unit time is \( \mathbf{A} (R \cos \theta) \), where \( \theta \) is the angle between \( \mathbf{A} \) and \( R \). This is nothing but the flux of rain through the top of the bucket. So it is given by \( \mathbf{A} \cdot R \).

The idea of flux is used very much in electromagnetism. The electric flux and magnetic flux can be expressed as scalar products. In Example 2, we shall see how it is done.

**Example 2: Magnetic and Electric Flux**

Let us first consider the magnetic flux. In your school laboratories you must have sketched lines of force around bar magnets to show that a magnetic field exists around them. So you are familiar with the concept of magnetic field. Let us now consider a surface of area \( A \) in a region where a magnetic field exists. The area could be a rectangle (as shown in Fig. 1.16(a)), a circle, or any other shape. Let the magnetic field \( \mathbf{B} \) be perpendicular to the surface, as in Fig. 1.16(a).

![Fig. 1.16: A uniform magnetic field \( \mathbf{B} \) (indicated by the parallel field lines) passing through a surface of area \( A \) which is (a) perpendicular to \( \mathbf{B} \), (b) not perpendicular to \( \mathbf{B} \). The dashed surface of area \( A \) in (b) is the projection of \( A \) perpendicular to field \( \mathbf{B} \).]

Then the magnetic flux \( \Phi_B \) through this surface is defined as the product of \( \mathbf{B} \) and \( A \), i.e.,

\[
\Phi_B = B \cdot A
\]

If the magnetic field \( \mathbf{B} \) makes an angle \( \theta \) with the normal to the surface (Fig. 1.16(b)) then \( \Phi_B \) is defined as

\[
\Phi_B = B_A \cdot A \cos \theta \quad \text{(B uniform)}
\]

where \( A_\perp \) is the projection of the area \( A \) on a surface perpendicular to \( \mathbf{B} \).

We can again represent the area \( A \) of a surface by a vector \( \mathbf{A} \). The magnitude of \( \mathbf{A} \) is \( A \) and its direction is perpendicular to the surface as shown in Fig. 1.16(b). The angle \( \theta \) is also the angle between \( \mathbf{B} \) and \( \mathbf{A} \) so we can write

\[
\Phi_B = B \cdot A \quad \text{(B uniform)} \quad (1.34a)
\]

We can also interpret \( \Phi_B \) in terms of the field lines associated with the magnetic field. The field lines can always be drawn so that \( \mathbf{B} \) is proportional to the number of field lines passing through a unit area perpendicular to the field. Now, if \( N \) is the number of field lines through \( A \), then

\[
B = \frac{N}{A_\perp} \quad \text{i.e., } N = B A_\perp = \Phi_B
\]

Thus, the magnetic flux through an area is proportional to the number of magnetic field lines passing through the area. Of course, the number of field lines cannot be counted. It is mentioned here only for the sake of visualising the concept of flux.

In analogy to Eq. (1.34a) we define the electric flux due to a uniform electric field \( \mathbf{E} \) through a surface of area \( A \) as

\[
\Phi_E = E \cdot A \quad (1.34b)
\]

\( \Phi_E \) through an area is also proportional to the number of electric field lines passing through the area.
You may now like to solve an SAQ based on the scalar product.

**SAQ 9**

a) When will the work done on a particle by a force \( \mathbf{F} \) be (i) zero, even when it undergoes a finite displacement, and (ii) maximum?

b) Vectors \( \mathbf{a} \) and \( \mathbf{b} \) are given by

\[
\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} \quad \text{and} \quad \mathbf{b} = -2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}
\]

i) Find the magnitudes of \( \mathbf{a} \) and \( \mathbf{b} \) and the angle between them.

ii) Which of the following vectors is perpendicular to \( \mathbf{a} \)?
\[
\mathbf{c} = -\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}, \quad \mathbf{d} = -3\mathbf{i} + \mathbf{k}, \quad \mathbf{e} = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}
\]

iii) Find the projection of the vector \( \mathbf{a} \) onto \( \mathbf{b} \).

You have just studied the scalar product of two vectors and some of its applications. Let us now discuss the vector product.

### 1.4.3 Vector Product in Component Form

Let us express the vector product in component form with respect to a Cartesian coordinate system. But before we do so we would like to note that there are two types of such systems. Depending on the orientation of axes, they are termed as right-handed or left-handed.

By definition, a Cartesian coordinate system is called right-handed if the unit vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) in the directions of positive \( x, y, z \)-axes form a right-handed triple (recall Sec. 1.2 and see Figs. 1.9 and 1.17a).

A Cartesian coordinate system is called left-handed if \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) form a left-handed triple, i.e., if you spread your left hand so that your thumb is along \( \mathbf{i} \) and index finger is along \( \mathbf{j} \), then your middle finger will be along \( \mathbf{k} \). Such a system is shown in Fig. 1.17b. In physics we usually use right-handed systems.

Now let two vectors \( \mathbf{a} \) and \( \mathbf{b} \) be given as

\[
\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}
\]

where \( a_1, a_2, a_3 \) and \( b_1, b_2, b_3 \) are their components with respect to a right-handed Cartesian coordinate system. Then

\[
\mathbf{a} \times \mathbf{b} = (a_1b_2 - a_2b_1)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_2b_3 - a_3b_2)\mathbf{k}
\]

Using properties 5 and 7 of the vector product (Eqs. (1.9c) and (1.9e)) we can write

\[
\mathbf{a} \times \mathbf{b} = a_1b_2\mathbf{i} \times \mathbf{i} + a_1b_3\mathbf{i} \times \mathbf{j} + a_2b_3\mathbf{j} \times \mathbf{k} + a_2b_1\mathbf{j} \times \mathbf{i} + a_3b_1\mathbf{k} \times \mathbf{i} + a_3b_2\mathbf{k} \times \mathbf{j}
\]

So you have to know the cross product of the unit vectors \( \mathbf{i} \times \mathbf{j} \) with themselves and each other to determine \( \mathbf{a} \times \mathbf{b} \). Let us consider the products \( \mathbf{i} \times \mathbf{i} \) and \( \mathbf{j} \times \mathbf{j} \). From the definition of cross product,

\[
\mathbf{i} \times \mathbf{i} = (1.1 \sin 0°) = 0 \quad \text{(1.35a)}
\]

So

\[
\mathbf{i} \times \mathbf{i} = 0.
\]

And

\[
\mathbf{j} \times \mathbf{j} = (1.1 \sin 90°) = 1 \quad \text{(1.35b)}
\]

According to the right-hand rule the direction of \( \mathbf{i} \times \mathbf{j} \) is along \( \mathbf{k} \) (Fig. 1.17a) so that

\[
\mathbf{i} \times \mathbf{j} = \mathbf{k}
\]

You may like to find the rest of the cross products yourself.
SAQ 10

Show that

\[ \mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}. \]  
(1.35)

b) Fill up the following blank spaces

\[ \mathbf{i} \times \mathbf{k} = \mathbf{j}, \quad \mathbf{k} \times \mathbf{j} = \mathbf{i}, \quad \mathbf{j} \times \mathbf{i} = \mathbf{k}. \]  
(1.35a)

From SAQ 10 you can see a cyclic pattern in the cross-products \( \mathbf{i} \times \mathbf{j} \mathbf{k}, \mathbf{j} \times \mathbf{k} \mathbf{i}, \mathbf{k} \times \mathbf{i} \mathbf{j} \), etc.

(Fig. 1.18). It is a good way to remember these cross-products. If you go around the circle clockwise all vector products are positive, i.e. \( \mathbf{i} \times \mathbf{j} = \mathbf{k} \) and so on. If you go in an anti-clockwise direction the cross products are negative, i.e. \( \mathbf{i} \times \mathbf{j} = -\mathbf{k} \) and so on. Using the results of SAQ 10 you can write the vector product in its component form:

\[ \mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2) \mathbf{i} + (a_3 b_1 - a_1 b_3) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k}, \]  
(1.36a)

Since you have studied determinants in school you can readily see that

\[ \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \]  
(1.36b)

We now consider some applications of the vector product, which are useful in physics.

**Applications of the vector product**

The most familiar example of a vector product in physics is that of torque (Fig. 1.19). You may already have studied this concept. The torque due to a force \( \mathbf{F} \) which acts on a particle at position \( \mathbf{r} \) is defined by

\[ \mathbf{T} = \mathbf{r} \times \mathbf{F}. \]  
(1.37)

This simple equation [Eq. (1.37)] contains many ideas. You may know that torque is a measure of the ability of an applied force to produce a twist, or to rotate a body. Note that a large force applied parallel to \( \mathbf{r} \) would produce no twist, it would only pull.

Only \( F \sin \theta \), i.e., the component of \( \mathbf{F} \) perpendicular to \( \mathbf{r} \), produces torque. The direction of torque is along the axis of rotation. These things are precisely what Eq. (1.37) tells us. For example, since \( \mathbf{r} \times \mathbf{F} \) is a zero vector, a force along \( \mathbf{r} \) yields zero torque. The direction of \( \mathbf{T} \) is given by the right-hand rule (Fig. 1.19).

In Example 2 we had represented area as a vector. With the help of the vector product, we can see how this is possible.

**Example 3 : Area as a vector**

Usually you think of area as a scalar quantity. However, in many applications in physics (e.g. fluid mechanics or in electromagnetics) we also want to know the orientation of the area. Suppose we want to calculate the rate at which water in a stream flows through a wire loop of a given area. This rate will obviously be different if we place the loop parallel to perpendicular to the flow. When the loop is parallel, the flow through it is zero. So let us now see how the vector product can be used to specify the direction of an area.

Consider the area \( A \) of the parallelogram formed by the vectors \( \mathbf{c} \) and \( \mathbf{d} \) (Fig. 1.20). It is given by

\[ A = \text{base} \times \text{height} = c \sin \theta = d \sin \theta \]  
(1.38)

\[ A = \mathbf{c} \times \mathbf{d}. \]  
(1.39)

\[ A = \mathbf{c} \times \mathbf{d} = (c_2 d_3 - c_3 d_2) \mathbf{i} + (c_3 d_1 - c_1 d_3) \mathbf{j} + (c_1 d_2 - c_2 d_1) \mathbf{k}. \]  
(1.39a)

\[ A = \begin{vmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}. \]  
(1.39b)

**Fig. 1.19 : Torque \( \mathbf{T} \) due to a force \( \mathbf{F} \).**

(b) shows the top view of (a).

**Fig. 1.20 : Area \( A \) as a vector.**
You can remember \( A \times B \) easily if you keep in mind the cyclic pattern: 
\[
\begin{align*}
(\mathbf{a}, \mathbf{b}, \mathbf{c}) &= (\mathbf{a}, \mathbf{b}, \mathbf{c}) \\
&= (\mathbf{b}, \mathbf{c}, \mathbf{a}) \\
&= (\mathbf{c}, \mathbf{a}, \mathbf{b})
\end{align*}
\]
(1.38)

By this definition the direction of area is perpendicular to the plane of the area and is given by the right-hand rule. Thus, \( A \) is parallel to a normal to the surface (Fig. 1.20). The sense of \( A \) is arbitrary because we could also have defined \( A = -d \times c \). However, once we choose the sense, it is unique.

To get the remaining terms replace each factor by the factor following it in a cycle. So to get the second term from the first term replace \( \mathbf{a} \) by \( \mathbf{b} \), \( \mathbf{b} \) by \( \mathbf{c} \) and \( \mathbf{c} \) by \( \mathbf{a} \).

Thus you get the second term:
\[
\frac{1}{2} (\mathbf{a} \times \mathbf{b})
\]
Similarly, to get the third term replace \( \mathbf{a} \) by \( \mathbf{c} \), \( \mathbf{b} \) by \( \mathbf{a} \) and \( \mathbf{c} \) by \( \mathbf{b} \).

And you have
\[
\frac{1}{2} (\mathbf{c} \times \mathbf{a})
\]
Thus
\[
\frac{1}{2} (\mathbf{a} \times \mathbf{c})
\]
So if we think of \( A \) as a vector then we can write
\[
\mathbf{A} = \mathbf{a} \times \mathbf{d}
\]
(1.38)

The force on a charged particle moving with velocity \( \mathbf{v} \) in a magnetic field \( \mathbf{B} \) is given by F = q \( \mathbf{v} \times \mathbf{B} \). The force on an element \( \mathbf{dl} \) of a current-carrying conductor in a magnetic field \( \mathbf{B} \) is:
\[
\mathbf{f} = (\mathbf{d} \times \mathbf{B})
\]
(1.39)
where \( I \) is the current through the conductor.

You may like to work out an SAQ to test whether you have grasped the concept of vector product.

**SAQ II**

a) Find \( x \) and \( y \) such that the vectors \( \mathbf{B} = x \mathbf{i} + 3 \mathbf{j} \) and \( \mathbf{C} = 2 \mathbf{i} + y \mathbf{j} \) are each parallel to \( \mathbf{A} = 5 \mathbf{i} + 6 \mathbf{j} \).

b) Consider a force \( \mathbf{F} = (-3 \mathbf{i} + 4 \mathbf{j} + 5 \mathbf{k}) \) newton, acting at a point \( P (7 \mathbf{i} + 3 \mathbf{j} + 4 \mathbf{k}) \) m. What is the torque in Nm about the origin?

In this section you have studied addition and subtraction of vectors, and products of two vectors using the analytical method. In physics repeated products of more than two vectors occur very often. So let us now discuss such products of vectors.

### 1.5 Multiple Products of Vectors

Some of the commonly occurring multiple products in physics applications are the scalar and vector products of three vectors. You know that if \( \mathbf{b} \) and \( \mathbf{c} \) are two vectors, then \( \mathbf{b} \times \mathbf{c} \) is a vector. Now if \( \mathbf{a} \) is a third vector then the scalar product of \( \mathbf{a} \) with \( \mathbf{b} \times \mathbf{c} \) is a scalar. This is called the scalar triple product. The cross product of \( \mathbf{a} \) with \( \mathbf{b} \times \mathbf{c} \) is called the vector triple product since it yields a vector. Let us first consider the scalar triple product.

**1.5.1 Scalar Triple Product**

The scalar triple product of three vectors \( \mathbf{a}, \mathbf{b}, \text{ and } \mathbf{c} \) is defined as
\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = |\mathbf{a}| |\mathbf{b} \times \mathbf{c}| \cos \beta
\]
where \( \beta \) is the angle between \( \mathbf{a} \) and the vector \( \mathbf{b} \times \mathbf{c} \). You can see that \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \) is a scalar quantity. The scalar triple product \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \) can be interpreted as the component of \( \mathbf{a} \) along \( \mathbf{b} \times \mathbf{c} \). The absolute value of the scalar product has a geometrical meaning too. Let us see what it is.

**Geometrical Interpretation of the Scalar Triple Product**

Consider the parallelepiped of Fig. 1.21. Let its adjacent sides be given by \( \mathbf{a}, \mathbf{b}, \text{ and } \mathbf{c} \). Let us find out its volume. The volume \( V \) of the parallelepiped is
\[
V = \text{area of the base} \times \text{altitude}
\]

The area of the base of the parallelepiped is \( |\mathbf{b} \times \mathbf{c}| \) as you may recall from Example 3. What is the altitude of the parallelepiped? From Fig. 1.21 you can see that it is \( |\mathbf{a} \cos \beta| \). Thus
\[
V = |\mathbf{a}| |\mathbf{b} \times \mathbf{c}| \cos \beta
\]
(1.40)

Now volume cannot be negative. That is why using Eq. (1.40a) we put it equal to the
absolute value of the scalar triple product. Thus,

$$V = |a \cdot (b \times c)|$$  \hspace{1cm} (1.40b)

Therefore, the absolute value of the scalar triple product is equal to the volume of the parallelepiped with \(a\), \(b\) and \(c\) as adjacent sides.

Scalar triple product in component form

You may now like to express the scalar triple product in its component form. Try the following SAQ for this purpose.

**SAQ 12**

Let \(a = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}\), \(b = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}\), and \(c = c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k}\) and \(a \cdot (b \times c) = \) ... 

Fill up the following blank spaces and express \(a \cdot (b \times c)\) in terms of the components of \(a\), \(b\) and \(c\).

a) \(b \times c = \) ...

b) \(a \cdot (b \times c) = \) ...

(1.40c)

Again we can express the result of Eq. (1.40c) in the form of a determinant, as follows:

We can write Eq. (1.40c) as

$$a \cdot (b \times c) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

or

$$a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(1.40d)

Defined in this manner the scalar triple product has some properties which we will now state.

**Properties of the scalar triple product**

You may know that the interchange of two rows reverses the sign of the determinant. Using this property of the determinant in Eq. (1.40d) we can write

$$a \cdot (b \times c) = -b \cdot (c \times a) = -a \cdot (c \times b)$$ etc.

(1.41a)

Interchanging the rows twice we get

$$a \cdot (b \times c) = b \cdot (c \times a) = (a \times b)$$

(1.41b)

The geometrical significance of Eq. (1.41b) is that these three products represent the same volume. Since dot product is commutative we have from Eq. (1.41b)

$$a \cdot (b \times c) = (a \times b) \cdot c$$

(1.41c)

Again for any constant \(k\)

$$[k \cdot a \cdot (b \times c)] = k[a \cdot (b \times c)]$$

(1.41d)

If \(a \cdot (b \times c) = 0\), then the volume of the parallelepiped formed with sides \(a\), \(b\) and \(c\) is zero.

So \(a\), \(b\) and \(c\) are coplanar, i.e., they lie in the same plane.

Since \(a \times b\) is perpendicular to \(a\), we have

$$a \cdot (a \times b) = 0$$

(1.41e)

i.e., if any two vectors in the scalar triple product are equal, it becomes zero.

Let us now consider an application of the scalar triple product.

**Example 4**

In Sec. 1.4.3 you have studied that the force acting on a charged particle \(q\) which moves with a velocity \(v\) in the magnetic field \(B\) is

$$\mathbf{F} = q \mathbf{v} \times \mathbf{B}$$

Using the concept of scalar triple product we can show that this force cannot bring about a
change in the charge’s energy. You would know, of course, that the change in energy is equal to the work done. Recall from Eq. (1.33) of Sec. 1.4.2 that the work done on the charge for an infinitesimal displacement \( dr \) is

\[
F \cdot dr \equiv q(\mathbf{v} \times \mathbf{B}) \cdot dr
\]

If the displacement \( dr \) occurs in time \( dt \), then \( dr = v \cdot dt \)

Using Eq. (1.41) we have that \( (\mathbf{v} \times \mathbf{B}) \cdot v = 0 \)

Since the work done is zero, the force \( F \) does not bring about any change in the energy of the charge.

---

**SAQ 13**

The volume of a tetrahedron is one-sixth of the volume of a parallelepiped. The three sides of a tetrahedron are given to be

\[
\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}, \quad \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \text{and} \quad \mathbf{c} = \mathbf{i} + \mathbf{j} - \mathbf{k}
\]

Find the volume of the tetrahedron.

Having studied scalar triple product, let us now know about the vector triple product.

### 1.5.2 Vector Triple Product

The vector triple product of three vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) is the vector product of \( \mathbf{a} \) and \( \mathbf{b} \times \mathbf{c} \). It is denoted by \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \). Let us find a simpler expression for the vector triple product.

Let \( \mathbf{d} = \mathbf{b} \times \mathbf{c} \). Then from Eq. (1.36a),

\[
\mathbf{d} = d_1 \mathbf{i} + d_2 \mathbf{j} + d_3 \mathbf{k}
\]

with

\[
d_1 = b_2 c_3 - b_3 c_2, \quad d_2 = b_3 c_1 - b_1 c_3, \quad \text{and} \quad d_3 = b_1 c_2 - b_2 c_1.
\]

Again

\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \times \mathbf{d}
\]

or

\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (a_2 d_3 - a_3 d_2) \mathbf{i} + (a_3 d_1 - a_1 d_3) \mathbf{j} + (a_1 d_2 - a_2 d_1) \mathbf{k} \tag{1.42a}
\]

We will now simplify one of the three components of \( \mathbf{a} \times \mathbf{d} \). Let us take its \( x \) component which is

\[
a_2 d_3 - a_3 d_2 = a_2 (b_1 c_3 - b_2 c_1) - a_3 (b_3 c_1 - b_1 c_3)
\]

\[
= (a_2 b_1 c_3 - a_3 b_1 c_3) - (a_2 b_3 c_1 - a_3 b_3 c_1)
\]

\[
= (a_2 b_1 + a_3 b_3) (c_1 - c_3) - (a_1 b_2 + a_3 b_1) (c_2 - c_3)
\]

(adding and subtracting \( a_2 b_3 c_1 \))

\[
= (a_2 - a_1) b_1 (c_1 - c_3)
\]

Now you work out the following SAQ to simplify the \( y \) and \( z \) components of \( \mathbf{a} \times \mathbf{d} \), as we have done for the \( x \) component.

**SAQ 14**

Determine the \( y \) and \( z \) components of \( \mathbf{a} \times \mathbf{d} \) where \( \mathbf{a} \) and \( \mathbf{d} \) are as given above.

Thus, using the results of SAQ 14 and simplifying Eq. (1.42a) we get

\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) (\mathbf{i} \times \mathbf{b} + \mathbf{j} \times \mathbf{k}) - (\mathbf{a} \cdot \mathbf{b}) (\mathbf{i} \times \mathbf{j} + \mathbf{j} \times \mathbf{i} + \mathbf{k} \times \mathbf{i})
\]

or

\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} \tag{1.42b}
\]

From this expression of \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \) you can see that \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \) is not the same vector as
So far you have studied examples of a variety of physical quantities that are vectors. These are essentially of two kinds. The direction of some of the vector quantities is clearly indicated by the displacement d is a polar vector. For example, the displacement d is a vector.

1.5.3 Quadruple Products of Vectors

Quadruple product means the product of four vectors. Some of the relevant quadruple products are (axb) x (cxd), (axb) x (cxd) and a x [b x (cxd)]. Here we will state, without proof, their simplified expressions

\[ (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \left( \mathbf{a} \cdot \mathbf{d} \right) \mathbf{b} - \left( \mathbf{a} \cdot \mathbf{b} \right) \mathbf{d} - \left( \mathbf{a} \cdot \mathbf{d} \right) \mathbf{b} + \left( \mathbf{a} \cdot \mathbf{b} \right) \mathbf{d} \]

\[ (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \left( \mathbf{a} \cdot \mathbf{d} \right) \mathbf{b} - \left( \mathbf{a} \cdot \mathbf{b} \right) \mathbf{d} - \left( \mathbf{a} \cdot \mathbf{d} \right) \mathbf{b} + \left( \mathbf{a} \cdot \mathbf{b} \right) \mathbf{d} \]

\[ \mathbf{a} \times [\mathbf{b} \times (\mathbf{c} \times \mathbf{d})] = \left( \mathbf{a} \cdot \mathbf{d} \right) \mathbf{b} - \left( \mathbf{a} \cdot \mathbf{b} \right) \mathbf{d} - \left( \mathbf{a} \cdot \mathbf{d} \right) \mathbf{b} + \left( \mathbf{a} \cdot \mathbf{b} \right) \mathbf{d} \]

You can prove the first one of these relations at the end of the unit.

Finally we would like to discuss the two categories of vectors into which all vector quantities of physics can be grouped. These are termed as proper vectors (or polar vectors) and pseudovectors (or axial vectors). The distinction between polar and axial vectors is important in certain areas of subatomic particle physics and in the discussion of symmetry of physical laws.

1.6 POLAR AND AXIAL VECTORS

So far you have studied examples of a variety of physical quantities that are vectors. These are essentially of two kinds. The direction of some of the vector quantities is clearly indicated by the direction of motion of a system. Examples of such vectors are displacement, velocity, acceleration etc. Such vectors are called polar vectors.

There is another category of vectors, namely, angular velocity, angular acceleration, angular momentum etc. associated with rotational motion. As you know, their direction does not indicate the direction of rotation of the body. Their direction is taken to be along the axis of rotation. But the sense or direction of rotation, i.e., clockwise or anti-clockwise, depends on the side from which you would look at the rotation. By convention, we determine the direction by the right-hand rule. We can as well use the left-hand rule which would then lead us to a left-handed coordinate system.

Let us now try to understand the difference between these two kinds of vector quantities. The difference is: Polar and axial vectors behave differently under the following coordinate transformation.

\[ x' = -x, \quad y' = -y, \quad z' = -z \]

Such a transformation is called the parity transformation (see Fig. 1.23). If a vector changes sign under the parity transformation, it is called a proper or a polar vector. For example, the displacement is a polar vector.
If you first rotate a vector around $z$ axis through $180^\circ$ and then reflect it through the $xy$ plane, it results in a parity transformation on the vector.

Fig. 1.23: The three-dimensional Cartesian coordinate system $(x'y'z')$ changes to the $(x'y''z'')$ system under parity transformation.

Let the displacement vector $\mathbf{d}$ have components

$$
d_1 = z_2 - x_1, \quad d_2 = y_2 - y_1, \quad d_3 = z_2 - x_1
$$

in a right-handed Cartesian coordinate system. Its components in the Coordinate system obtained under parity transformation are

$$
d'_1 = -x_2 + x_1 = -d_1, \quad d'_2 = -y_2 + y_1 = -d_2, \quad d'_3 = -z_2 + z_1 = -d_3,
$$

You can see that the components of $\mathbf{d}$ have changed their sign under parity transformation, i.e., $\mathbf{d}$ has changed sign. Other examples of polar vectors are velocity, acceleration, linear momentum, force, electric field etc. A more general definition of proper vectors is that these are vectors which transform according to Eqs. (1.21), (1.22) and (1.23).

The vectors which do not change sign under a parity transformation are called axial vectors. The cross product of two polar vectors is the simplest example of axial vectors. Consider the cross product of two polar vectors $\mathbf{a}$ and $\mathbf{b}$: Let $(a_1, a_2, a_3)$ and $(b_1, b_2, b_3)$ be the components of $\mathbf{a}$ and $\mathbf{b}$ in a right-handed system. Then the components of $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ in that system are

$$
c_1 = a_2 b_3 - a_3 b_2, \quad c_2 = a_3 b_1 - a_1 b_3, \quad c_3 = a_1 b_2 - a_2 b_1.
$$

Now since $\mathbf{a}$ and $\mathbf{b}$ are polar vectors, $a_i$s and $b_i$s ($i = 1, 2, 3$) will change sign when they undergo a parity transformation. But since $c_i$ involves products of $a_i$s and $b_i$s, they will not change sign. Therefore, the cross product of two polar vectors is an axial vector. Some examples of the axial vector are angular momentum ($\mathbf{L} = \mathbf{r} \times \mathbf{p}$) and torque ($\mathbf{\tau} = \mathbf{r} \times \mathbf{F}$). In fact, all vectors associated with rotations, e.g., angular velocity, angular acceleration etc. are axial vectors.

Let us now summarise the contents of this unit.

1.7 SUMMARY

- Quantities which are completely specified by a number are called scalars. Appropriate units have to be attached to specify physical quantities which are scalars. Scalars are invariant under any coordinate transformation.
- Quantities which are specified by a positive real number (called magnitude or modulus) and a direction in space are called vectors. Vectors combine according to the following rules:

  $$
  \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}
  $$

  $$
  \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}
  $$

  $$
  m(\mathbf{a}) = (m\mathbf{a})
  $$

  $$
  (m + n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}
  $$

  $$
  m(\mathbf{a} + \mathbf{b}) = ma + mb
  $$

26
Any vector \( \vec{a} \) can be expressed as
\[
\vec{a} = a \hat{a}
\]
where \( \hat{a} \) is the unit vector in the direction of \( \vec{a} \).

Vectors \( \vec{a} \) and \( \vec{b} \), their magnitudes and directions can be expressed in terms of their components and the unit vectors in two and three-dimensional Cartesian coordinate systems, respectively, as
\[
\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}, \quad \vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}
\]

Here \( a_i \) and \( b_i \) are the components of \( \vec{a} \) and \( \vec{b} \), respectively. \( \hat{i}, \hat{j}, \hat{k} \) are unit vectors along the positive x, y, and z-axes. Here \( \theta \) is the angle \( \vec{a} \) makes with the x-axis. And \( \alpha, \beta, \gamma \) are the angles \( \vec{b} \) makes with the x, y, and z-axes, respectively.

A three-dimensional vector is defined as a set of three numbers (or components) referred to an orthogonal coordinate system. If \( a_1, a_2, a_3 \) are the components of \( \vec{a} \) in one system and \( a'_1, a'_2, a'_3 \) its components in another system, these two sets of components are related as follows
\[
\begin{align*}
\begin{bmatrix}
a'_1 \\
a'_2 \\
a'_3 \\
\end{bmatrix} &=
\begin{bmatrix}
c_{11} & c_{12} & c_{13} \\
c_{21} & c_{22} & c_{23} \\
c_{31} & c_{32} & c_{33} \\
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
\end{bmatrix}
\end{align*}
\]

Here the coefficients \( c_{ij} \) \((i = 1, 2, 3; j = 1, 2, 3)\) are defined by Eq. (1.21). The magnitude of a vector is invariant under any coordinate transformation.

Two vectors \( \vec{a} \) and \( \vec{b} \) can be added graphically using the triangle law or parallelogram law of vector addition. Their difference \( \vec{a} - \vec{b} \) is obtained by adding \((-\vec{b})\) to \( \vec{a} \). Given
\[
\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} \quad \text{and} \quad \vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}
\]

The scalar product of two vectors \( \vec{a} \) and \( \vec{b} \) is defined as
\[
\vec{a} \cdot \vec{b} = ab \cos \theta
\]
where \( \theta \) is the angle between \( \vec{a} \) and \( \vec{b} \), such that \( 0 \leq \theta \leq \pi \).

In component form
\[
\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3
\]

The vector product of two vectors \( \vec{a} \) and \( \vec{b} \) is defined as
\[
\vec{c} = \vec{a} \times \vec{b} = a b \sin \theta \hat{c}
\]
where \( \theta \) is the angle between \( \vec{a} \) and \( \vec{b} \) such that \( 0 \leq \theta \leq \pi \). The direction of \( \hat{c} \) is obtained by the right-hand rule.

In component form
\[
\vec{a} \times \vec{b} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
\end{vmatrix}
\]

The scalar triple product of three vectors \( \vec{a}, \vec{b}, \) and \( \vec{c} \) is given by
\[
\vec{a} \cdot (\vec{b} \times \vec{c}) = a_1 (b_2 c_3 - b_3 c_2) + a_2 (b_3 c_1 - b_1 c_3) + a_3 (b_1 c_2 - b_2 c_1)
\]

where \( \beta \) is the angle between \( \vec{a} \) and \( (\vec{b} \times \vec{c}) \). In component form
\[
\vec{a} \cdot (\vec{b} \times \vec{c}) =
\begin{vmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3 \\
\end{vmatrix}
\]
The vector triple product of three vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) is given by
\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}
\]

Some quadruple products of vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) and \( \mathbf{d} \) are:
\[
\begin{align*}
\mathbf{a} \times \mathbf{b} \times (\mathbf{c} \times \mathbf{d}) &= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \mathbf{d} \\
(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \mathbf{d}
\end{align*}
\]

All vectors can be grouped into the two categories of polar vectors and axial vectors. The components of a polar vector change their sign under a parity transformation, while those of an axial vector do not.

### 1.8 TERMINAL QUESTIONS

1. a) Find a unit vector in the yz-plane such that it is perpendicular to the vector \( \mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k} \).

   b) A proton having a speed of \( 5.0 \times 10^6 \) m/s in a uniform magnetic field feels a force of \( 8.0 \times 10^{-14} \) N towards west when it moves vertically upward. When moving horizontally in a northerly direction it feels zero force. What is the magnitude and direction of the magnetic field in this region? Charge on the proton = \( 1.6 \times 10^{-19} \) C.

2. Prove that for any three vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \):
   
   \[
   (\mathbf{a} \times \mathbf{b}) \times (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = (\mathbf{a} \times \mathbf{c}) \times \mathbf{b} = 0.
   \]

3. Consider a current circuit in a prescribed magnetic field. The magnetic force on each circuit element \( dl \) is given by \( F = I dl \times B \). Let the circuit be allowed to move under the influence of magnetic forces, such that an element is displaced by \( dr \) and at the same time \( I \) is held constant. Show that the mechanical work done by the force is \( dW = I dr \Phi_B \), where \( \Phi_B \) is the additional flux through the circuit.

4. Prove Eq. (1.44a).

### 1.9 SOLUTIONS AND ANSWERS

Self-assessment Questions (SAQs)

3. a) The vectors \( \mathbf{d} \) and \( \mathbf{f} \).

   b) Two different sequences for obtaining the resultant \( \mathbf{F} \) are shown in Figs. 1.24a, b. The resultant is the same.
c) The graphical addition of the two vectors is shown in Figs. 1.25a, b and c.

\[ \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} \]

2.

a) (i) \( \frac{3}{2} \) V

(b) (i) \(-2\) V

2.) (i) \(-2300\) \(\text{j}\)

(ii) \(1700\) \(\text{j}\)

(c) See Fig. 1.26.

\[ \overrightarrow{a} - (\overrightarrow{b} + \overrightarrow{c}) \]

\[ \overrightarrow{a} - \overrightarrow{b} + \overrightarrow{c} \]

\[ (\overrightarrow{a} - \overrightarrow{b}) + \overrightarrow{c} \]

\[ \overrightarrow{a} - (\overrightarrow{b} + \overrightarrow{c}) \]

\[ \overrightarrow{a} - \overrightarrow{b} + \overrightarrow{c} \]

\[ (\overrightarrow{a} - \overrightarrow{b}) + \overrightarrow{c} \]

\[ \overrightarrow{a} - (\overrightarrow{b} + \overrightarrow{c}) \]

\[ a_1 = -1 \quad a_2 = 3 \quad b = \sqrt{17} \quad c = \sqrt{2} \tan \theta = -3 \]

Since both \( c_1 \) and \( c_2 \) are negative, \( \theta \) lies in the third quadrant.

\( \theta \) (b) and (c) See Figs. 1.27a and b.

4. Proof of Eqs. (1.16c and d): In analogy with Eqs. (1.14c to e) we can write

\[ a = \cos \theta \cos \phi + \sin \phi \sin \theta \]

Thus \( a = c_1 a_1 + c_2 a_2 \)

where \( c_1 = \cos a \) and \( c_2 = \sin a \), \( c_3 = -\sin a \cos \phi \), \( c_4 = \cos a \cos \phi \).
Proof of Eq. (1.17): From Eqs. (1.16c, d) and e we get
\[ \mathbf{a} = \sqrt{a_1^2 + a_2^2} = \sqrt{(c_{11} a_1 + c_{21} a_2)^2 + (c_{12} a_1 + c_{22} a_2)^2} \]
\[ = \sqrt{c_{11} a_1^2 + c_{12} a_1^2 + 2c_{11} c_{21} a_1 a_2 + c_{12} c_{22} a_2^2} \]
\[ = \sqrt{\cos^2 \alpha a_1^2 + \sin^2 \alpha a_2^2 - 2 \cos \alpha \sin \alpha a_1 a_2 + \sin^2 \alpha a_2^2 \cos^2 \alpha a_1^2 + \cos^2 \alpha a_2^2 + 2 \sin \alpha \cos \alpha a_1 a_2} \]
\[ = \sqrt{a_1^2 + a_2^2} \]

5. From Eq. (1.18b)
\[ \mathbf{f} \cdot \mathbf{a} = f_1 (a_1 a_1 + a_2 a_2 + a_3 a_3) \]
\[ = a_1 f_1 + a_2 f_2 + a_3 f_3 + \mathbf{a} \cdot \mathbf{k}' \]
\[ \text{Since} \]
\[ \mathbf{f} \cdot \mathbf{f}' = 1.1 \cos 90^\circ = 0 \]
\[ \mathbf{f} \cdot \mathbf{f}' = 1.1 \cos 0 = 1, \quad \mathbf{f} \cdot \mathbf{k}' = 1.1 \cos 90^\circ = 0 \]
\[ \therefore \mathbf{f} \cdot \mathbf{a} = a_1 f_1 + a_2 f_2 + a_3 f_3 + \mathbf{a} \cdot \mathbf{k}' \]
\[ \text{Thus} \]
\[ a_1 f_1 + a_2 f_2 + a_3 f_3 + \mathbf{a} \cdot \mathbf{k}' \]
\[ \text{Similarly, you can show that} \]
\[ a_2 = a_1, \quad a_3 = a_1 \]
\[ \mathbf{f} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{f} = a_1 \mathbf{f}_1 + a_2 \mathbf{f}_2 + a_3 \mathbf{f}_3 + \mathbf{a} \cdot \mathbf{k}' \]
\[ \text{Since scalar product is commutative we have, using Eq. (1.21) that} \]
\[ a_1 = a_1 c_{11} + a_2 c_{12} + a_3 c_{13} \]

6. Let \( \mathbf{d}_1 \) be its displacement vector at initial sighting be
\[ \mathbf{d}_1 = 500 \text{ m} \quad \text{and} \quad \theta_1 = 30^\circ. \]
\[ \therefore \mathbf{d}_{1y} = d_1 \cos \theta_1 = 500 \cos 30^\circ \approx 250 \text{ m} \]
\[ \mathbf{d}_{1x} = d_1 \sin \theta_1 = 500 \sin 30^\circ = 250 \text{ m} \]

Let \( \mathbf{d}_2 \) be its displacement vector at final sighting. Then
\[ \mathbf{d}_2 = 1000 \text{ m}, \quad \theta_2 = 120^\circ \quad \text{and} \quad \theta_2 = 30^\circ \pm 150^\circ. \]
\[ \therefore \mathbf{d}_{2y} = d_2 \cos \theta_2 = 1000 \cos 150^\circ = -500 \sqrt{3} \text{ m}, \]
\[ \mathbf{d}_{2x} = d_2 \sin \theta_2 = 1000 \sin 150^\circ = 500 \text{ m} \]
The resultant displacement is then \( d = d_2 - d_1 \)
\[
= (d_2 - d_1) \hat{i} + (d_2 - d_1) \hat{j}
\]
\[
= (-500 \sqrt{2} - 250 \sqrt{3}) \hat{i} + (500 - 250 \sqrt{3}) \hat{j}
\]
or
\[
d = (-750 \sqrt{3}) \hat{i} + (250) \hat{j}
\]

8. \( \mathbf{a} \cdot \mathbf{b} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) \)

Using Eqs. (1.7b and c)
\[
\mathbf{a} \cdot \mathbf{b} = a_1 b_1 \hat{i} \cdot \hat{i} + a_2 b_2 (\hat{j} \cdot \hat{j}) + a_3 b_3 (\hat{k} \cdot \hat{k})
\]
\[
+ a_1 b_2 (\hat{j} \cdot \hat{k}) + a_2 b_1 (\hat{k} \cdot \hat{i}) + a_3 b_3 (\hat{i} \cdot \hat{j})
\]

Using Eqs. (1.7a and b) we get \( \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \).

9. a) The work done will be
i) zero when \( \mathbf{F} \) is perpendicular to the displacement \( \mathbf{d} \)
ii) maximum when \( \mathbf{F} \) is along \( \mathbf{d} \).

b) i) \( |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{(3)^2 + (-2)^2 + (1)^2} = \sqrt{14} \)
\( |\mathbf{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{(-3)^2 + (4)^2 + (2)^2} = \sqrt{29} \)
\( \theta = \cos^{-1} \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \right) = \cos^{-1} \left( \frac{3 \cdot 2 + 4 \cdot 4 + 2 \cdot 1}{\sqrt{14} \cdot \sqrt{29}} \right) = \cos^{-1} \left( \frac{3}{2 \sqrt{21}} \right) \)

ii) Consider \( \mathbf{a} \cdot \mathbf{c} = -3 \hat{i} + 2 \hat{j} = 7 \)
\( \mathbf{a} \cdot \mathbf{d} = -9 \hat{i} + 1 \hat{j} = -8 \)
\( \mathbf{a} \cdot \mathbf{e} = 6 \hat{i} - 2 \hat{j} = 0 \)
\( \therefore \mathbf{a} \cdot \mathbf{c} = 0, \mathbf{e} \) is perpendicular to \( \mathbf{a} \).

ii) The projection \( p \) is
\[
p = \left( \frac{a + b}{2} \right) \cdot \mathbf{a} = \frac{1}{|\mathbf{a}|} \left( \frac{1}{2} a \cdot \mathbf{a} \right) = \frac{14 + (-6)}{\sqrt{14}} = \frac{11}{\sqrt{14}}
\]

10. a) Eq. (1.35c) follows from Eq. (1.9b),
\( |\mathbf{i} \times \mathbf{k}| = 1\ |\mathbf{a} \times \mathbf{b}| = \sin 90^\circ = 1 \)
and \( \mathbf{i} \times \mathbf{k} = -\mathbf{j} \) from the right-hand rule.

Similarly
\( \mathbf{k} \times \mathbf{i} = -\mathbf{j} \), \( \mathbf{j} \times \mathbf{i} = \mathbf{k} \), \( \mathbf{i} \times \mathbf{j} = -\mathbf{k} \)

11. a) If \( \mathbf{B} \) and \( \mathbf{C} \) are to be parallel to \( \mathbf{A} \) then \( \mathbf{B} \times \mathbf{A} = 0 \), \( \mathbf{C} \times \mathbf{A} = 0 \)
\( \mathbf{B} \times \mathbf{A} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times (5 \hat{i} + 3 \hat{j}) = 6a_1 \hat{k} - 15a_3 \hat{i} = 0 \Rightarrow 6a_1 - 15 = 0 \)
\( \therefore \hat{k} = 0 \)
\[ x = \frac{15}{6} = 2.5 \]
\( C \times \mathbf{A} = (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) \times (5 \hat{i} + 3 \hat{j}) = 12b_1 \hat{k} - 5b_3 \hat{i} = 0 \Rightarrow 5b_3 - 12 = 0 \)
\( \therefore \hat{k} = 0 \)
\( \therefore y = \frac{12}{5} = 2.4 \).

b) The displacement of \( \mathbf{P} \) with respect to \( \mathbf{O} \) is \( \mathbf{r} \) where \( \mathbf{r} = 7\hat{i} + 3\hat{j} + 4\hat{k} \)
\( \mathbf{F} \times \mathbf{r} = (7\hat{i} + 3\hat{j} + 4\hat{k}) \times (-3\hat{i} + 3\hat{j} + 5\hat{k}) \)
From Eq. (1.35b), \( \mathbf{r} \times (\hat{k} + 1) + (\hat{j} - 3 - 35) + (4\hat{i} + 7\hat{j} + 9\hat{k}) \)
\( = (14\hat{i} - 38\hat{j} + 16\hat{k}) \text{ Nm} \).
12. a) \[ \mathbf{b} \times \mathbf{c} = \hat{k}(b_2c_1 - b_1c_2) + \hat{i}(b_1c_3 - b_3c_1) + \hat{j}(b_3c_2 - b_2c_3) \]

b) \[ a \times (b \times c) = \left( a_3 \hat{k} - a_2 \hat{j} + a_1 \hat{i} \right) \times \left[ \left( b_2 \hat{k} - b_3 \hat{j} \right) + \left( b_3 \hat{i} - b_1 \hat{k} \right) + \left( b_1 \hat{j} - b_2 \hat{i} \right) \right] \]

\[ = a_3(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_1(b_1c_2 - b_2c_1) \]

(1.40)

13. \[ V = \frac{1}{3} a \cdot (b \times c) \]

So from Eq.

\[ V = \frac{1}{6} \left[ \begin{array}{ccc}
2 & 3 & -4 \\
1 & 2 & -1 \\
2 & 3 & 4
\end{array} \right]
\]

or \[ V = \frac{1}{6} \left[ (2 \times 2 \times 4 + 1 \times 3) + 3 \times (-1 \times 2 \times 1 \times 4) - 4 \times (1 \times 3 - 2 \times 2) \right] \]

14. The component \( y \) of \( a \times d \) is \[ a_1d_1 - a_2d_2 = a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1) \]

\[ = b_2(a_3c_3 + a_1c_1 - a_2c_2) - c_2(a_1b_3 + a_2b_1 + a_3b_2), \]

where we have added and subtracted \( a_2b_2c_2 \).

or \[ (a \times d)_y = b_2(a \cdot c) - c_2(a \cdot b) \]

Similarly, \( z \) component of \( a \times d = b_3(a \cdot c) - c_3(a \cdot b) \)

Terminal Questions

1. a) Since the unit vector is in the \( yz \) plane, its components will be along the positive \( y \) and \( z \) directions. Let it be the vector \( \mathbf{b} = m \hat{j} + n \hat{k} \), where \( m \) and \( n \) are numbers.

Since \( \mathbf{b} \) is a unit vector, its magnitude \[ |\mathbf{b}| = \sqrt{m^2 + n^2} = 1 \]

or \[ m^2 + n^2 = 1 \]

Since \( \mathbf{b} \) is perpendicular to \( \mathbf{a} = \hat{i} + \hat{j} + \hat{k} \)

\[ \mathbf{a} \cdot \mathbf{b} = (\hat{i} + \hat{j} + \hat{k}) \cdot (m \hat{j} + n \hat{k}) = 0 \]

or \[ m + n = 0 \]

Substituting \( m = -n \) in \( m^2 + n^2 = 1 \) we have \[ m^2 + n^2 = 2n^2 = 1 \]

or \[ n = \frac{1}{\sqrt{2}} \]

\[ m = -\frac{1}{\sqrt{2}} \]

Thus \( \mathbf{b} = \hat{j} + \frac{1}{\sqrt{2}} \hat{k} \)

You can see that it can also be the vector \[ \mathbf{b} = \frac{1}{\sqrt{2}} \hat{j} - \frac{1}{\sqrt{2}} \hat{k} \]

b) The force on the proton is given by \( \mathbf{F} = q \mathbf{v} \times \mathbf{B} \)

When the proton's velocity is in the north direction in the horizontal plane, the force on it is zero. This implies that \( \mathbf{v} \) is parallel to \( \mathbf{B} \). The direction of \( \mathbf{B} \) is then in the north direction in the horizontal plane. Therefore, when the proton moves vertically upward \( \mathbf{v} \) is perpendicular to \( \mathbf{B} \) and
2. From Eqs. (1.9a) and (1.42b) we have that
\[(a \times b) \times c + (b \times c) \times a + (c \times a) \times b\]
\[= (a \times b) \times c - (b \times c) \times a - (c \times a) \times b\]
\[= - (a \times b) + (b \times c) + (c \times a) = 0,\]
since scalar product is commutative, all the terms cancel out in pairs.

3. Work done \(dW = F \cdot dr\)
\[= I (d \mathbf{A} \times B) \cdot dr = dW = I (d \mathbf{A} \times B)\]
Using Eq. (1.41c) we get
\[dW = I (d \mathbf{A} \times d \mathbf{l}) \cdot B\]
Now let \(d \mathbf{A}\) define the area formed by \(dr\) and \(d \mathbf{l}\), then
\[d \mathbf{A} = (dr \times d \mathbf{l}) \cdot B\]
\[= I \cdot d \mathbf{A} \cdot B\]
\[= I \cdot d \Phi_B\]
where \(d \Phi_B\) is the flux of the magnetic field through \(d \mathbf{A}\), i.e., it is the additional flux through the circuit.

4. \((a \times b) \cdot (c \times d)\)
\[= [(a_3 b_2 - a_2 b_3) + b_3 a_1 b_2 - a_1 b_3 + a_2 b_1 a_3 b_2 - a_3 b_1 a_2 b_3 + a_1 b_2 a_3 b_1] \cdot [(c_2 d_3 - c_3 d_2) + b_3 (c_1 d_2 - c_2 d_1)]\]
\[= (a_2 b_3 c_1 d_2 - a_3 b_1 c_2 d_3) + (a_1 b_2 c_3 d_1 - a_3 b_1 c_2 d_3) + (a_1 b_2 c_3 d_1 - a_3 b_1 c_2 d_3)\]
\[= a_2 b_3 c_1 d_2 + a_1 b_2 c_3 d_1 - a_2 b_3 c_1 d_2 - a_3 b_1 c_2 d_3 + a_3 b_1 c_2 d_3 + a_1 b_2 c_3 d_1 - a_2 b_3 c_1 d_2\]
\[= a_2 b_3 c_1 d_2 + a_1 b_2 c_3 d_1 - a_2 b_3 c_1 d_2 - a_3 b_1 c_2 d_3\]
\[= a_2 b_3 c_1 d_2 + a_1 b_2 c_3 d_1 - a_2 b_3 c_1 d_2 - a_3 b_1 c_2 d_3\]
\[= a_2 b_3 c_1 d_2 + a_1 b_2 c_3 d_1 - a_2 b_3 c_1 d_2 - a_3 b_1 c_2 d_3\]
Adding and subtracting the terms \(a_1 b_2 c_3 d_1, a_2 b_3 c_1 d_2, a_3 b_1 c_2 d_3\) we get
\[(a \times b) \cdot (c \times d) = a_1 b_2 c_3 d_1 - a_3 b_1 c_2 d_3 - a_2 b_3 c_1 d_2 - a_3 b_1 c_2 d_3\]
\[= a_1 b_2 c_3 d_1 + a_2 b_3 c_1 d_2 - a_3 b_1 c_2 d_3 - a_2 b_3 c_1 d_2 - a_3 b_1 c_2 d_3\]
\[= (a \cdot c) (b \cdot d) - (a \cdot d) (b \cdot c)\]