UNIT 7 WAVES AT THE BOUNDARY OF TWO MEDIA

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7.1 INTRODUCTION

In Unit 6 we discussed the basic characteristics of wave motion. The propagation of waves on strings and in fluids was discussed with particular reference to sound. You may now ask:

What happens to a wave when it encounters a rigid barrier, as for instance, in the case of a string whose one end is tied to a rigid wall. The wave energy will not flow into the wall. But the wave cannot stop there. Then where will its energy go? What happens is that the wave turns around and bounces back along the string. We say that the wave has been reflected.

You must have experienced sound reflection in the form of echoes in large halls or in the neighbourhood of hills. You must have also observed reflection of water (sea) waves from a fixed barrier (sea shore). In the case of light, reflection from silvered surfaces, say in a looking mirror, is the most common optical effect we know. The reflection of ultrasonic (sound) waves forms the operating principle of sonars in depth-ranging, navigation, prospecting for oil and mineral deposits. The reflection of electromagnetic waves governs the working of a radar for detection of aircrafts. Reflection of radio waves by the ionosphere makes signal transmission from one place to another possible and is so crucial in the area of communications.

You may now like to know what would happen to the incident wave. You would agree that the boundary is not very rigid and properties of the medium change suddenly. Now suppose that we connect two strings of different mass per unit lengths. We observe that in such a case energy is partly transmitted into the second string and the rest is reflected back along the first. The phenomenon of partial reflection and transmission at a junction of strings has its analog in the behaviour of all waves at interfaces between two different media. Shallow water waves are partially reflected if water depth changes suddenly. Light incident on our atmosphere undergoes partial reflection because of changes in the density of the medium. Partial reflection of ultrasonic waves at the interfaces of body tissues with different densities makes ultrasound a valuable diagnostic tool.

Does this mean that waves never undergo complete refraction? Were this true, we could not explain working of lenses, which is fundamental to seeing and our contact with the surroundings. You may have seen the sun before actual sunrise and after actual sunset. This is because of refraction of light in the atmosphere.

In Sections 7.2 and 7.3 you will learn, using the concepts of Huygens' construction and the concept of impedance, that when a wave is incident at a boundary separating two media, its wavelength changes but frequency remains constant. But there are many situations where frequency of a wave also undergoes a change. This effect is known as Doppler Effect. You will learn it in Section 7.5.
Objectives

After going through this unit, you will be able to
- define a wavefront
- construct the wave front for a given source
- explain reflection and refraction of waves using Huygens’ construction
- compute the reflection and transmission amplitude coefficients
- compute reflection and transmission energy coefficients
- compute the apparent frequency of sound when the source and/or the observer (listener) are in motion.

7.2 THE CONCEPT OF WAVEFRONT AND HUYGENS’ CONSTRUCTION

Let us consider propagation of a wave on the surface of water. If you dip your finger in water repeatedly, a series of crests and troughs travel out. That is, waves set out in all directions. At any instant, a trough or a crest is circular in shape. The locus of points in the same phase at a particular time is called a wavefront. The shape of the wavefront depends on the nature of source. In the case of waves from a point source in air, the wavefronts are spherical. (In two dimensions, as on the water surface, the wavefronts are circular.) If the source is a long slit, the wavefront will be cylindrical. At large distances from the source (whether point or slit), the wavefront appears to be a plane. To understand the formation of wavefronts, we use Huygens’ construction.

Following Huygens, we make the following assumptions:

i) Each point on a wavefront becomes a fresh source of secondary wavelets, which move out in all directions with the speed of the wave in that medium.

ii) The new wavefront, at any later time, is given by the forward envelope of the secondary wavelets at that time.

iii) In an isotropic medium, the energy carried by waves is transmitted equally in all directions.

If $S$ is the source of sound or light (Fig. 7.1a), then after an interval of time $t$, all particles of the medium lying on the surface $AB$ vibrate in the same phase. This is because all particles on the surface $AB$ are equi-distant from the source. Any disturbance emanating from $S$ is handed on to them at the same time.

According to Huygen's construction, surface $AB$ is called a primary wavefront. Each point on $AB$, like the $a$, $b$, $c$, etc. acts as secondary source (derived from the original source $S$). These secondary sources give out waves (or disturbances) in all directions as demonstrated by drawing circles around the points $a$, $b$, $c$, etc. The envelope of all these waves (which acts as a tangent to all of them at any given instant), like the one at $CD$, forms another wavefront, called the secondary wavefront. This, in short, means that the source $S$ gives out wavelets in all directions. The envelope of these wavelets acts as a primary wavefront. Each point on this primary wavefront acts as a source for secondary wavelets. An envelope of these secondary wavelets forms a secondary wavefront. Each point on this secondary wavefront gives out further wavelets to form further secondary wavefronts. This process goes on and the wave keeps on spreading in space.

The direction $SP$ (Fig. 7.1a) in which the disturbance (originating at $S$) propagates is called a ray. A ray is always normal to the expanding wavefront.

To visualise the Huygens' construction in space, you may imagine a point source to be at the centre of a hollow sphere. The outer surface of this sphere then acts as a primary wavefront. If this sphere is further enclosed by another hollow sphere of larger radius, the outer surface of the second hollow sphere will then act as a secondary wavefront. If this sphere is further enclosed by another sphere of still bigger radius, the surface of the outermost sphere becomes the secondary wavefront. For this, the surface of the inner sphere acts as the primary wavefront. In two dimensions, the primary and secondary wavefronts appear to be concentric circles, the parts of which are shown in Figs. 7.1a and 7.1b.
The formation of secondary sources as visualised by Huygens can also be understood pictorially through a simple diagram. If we place a screen XY with a tiny hole at S' in the path of waves emanating from the source S, S' acts as a secondary source (Fig. 7.1b). This gives out waves on the other side of the screen. These waves spread out from S' as if S' is an original source itself.

In your school classes you have studied reflection and refraction of waves. We observe these whenever a wave travelling in one medium, say air, meets the boundary of another medium. Suppose we clamp one end of a string to a rigid wall and generate a pulse by moving the other end. You will observe that the pulse is reflected at the fixed end. Similarly, you can study reflection of ripples in a water basin. You will be surprised to know that same physical laws govern the reflection (refraction) of all waves, including light. We will now consider reflection and refraction of waves using Huygens' wave theory.

7.2.1 Reflection of Waves

Refer to Fig. 7.2. LM represents a part of a plane wavefront travelling towards a smooth reflecting surface SS$. It first strikes at A and then at successive points towards D. If v is the wave speed, the point $M on the wavefront reaches D at a time $ = DC/v later compared to the point $L. According to Huygens' Principle, each point on the reflecting surface will give rise to secondary wavelets. In this case we expect that they should constitute the reflected wavefront. Can you locate the reflected wavefront? To discover this, we note that at the instant D is just disturbed, the wavelet from A has grown for time DC/v and has travelled to E so that the distance AE is equal to DC. We can draw a circle of radius AE (= DC) to

Fig. 7.1: (a) Construction of Huygens' wavefront, (b) Depiction of a secondary source

Fig. 7.2: Huygens' construction for reflection of wave
represent this wavelet with $A$ as centre. Similarly, we can draw many circles from the intermediate points. The tangent or the envelope to these circles from $D$ defines the reflected wavefront.

From Fig. 7.2 it is clear that $ACD$ and $D EA$, are congruent. Hence

$$L \overline{CAD} = L \overline{EAD}$$

or

$$L i = L r \quad (7.1)$$

That is, the angle of incidence is equal to the angle of reflection. Moreover, you will note that the incident ray, the reflected ray and the normal at the point of incidence lie in the plane of the paper.

In this connection it is important to mention here that the reflected wavefront undergoes a phase change of $\pi$. In fact, it is true for any wave travelling in a rarer medium (air) and undergoing reflection at the interface with a denser medium (water). However, the reverse is not true.

### 7.2.2 Refraction of Waves

When a wave reaches the boundary of two different media, it may be partly reflected and partly transmitted. You can study this by joining two strings: one thick and another thin so that their mass per unit lengths are different. In Unit 6, you have learnt that velocity of a wave is inversely proportional to the density of the medium. This means that when a wave moves from a lighter to a denser medium, its velocity decreases. This results in a change (decrease) in wavelength. But the frequency remains the same. Fig. 7.3 depicts this situation when a wave is refracted (i.e., only transmitted).

\[ \lambda_1 > \lambda_2 \]

\[ v_1 > v_2 \]

\[ \lambda_1 = \frac{v_1}{f} \]

\[ \lambda_2 = \frac{v_2}{f} \]

Since $f$ is the same.

This relation holds for waves in water, air and string alike.

Using Huygens' principle, you can prove the laws of refraction as well (TQ1). But you will agree that Huygens' method is essentially geometrical and can be used when the wave is either reflected or refracted at the interface. You may now ask: Can we apply this method to study partial reflection and refraction, as in the case of two strings having different mass per unit length? In principle, we can do so but it is more convenient to study partial reflection and refraction in terms of impedance offered by a medium. To this end, we normally compute reflection and transmission amplitude coefficients. You will now learn to compute these in the following section.

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Waves at the Boundary of two Media
7.3 REFLECTION AND TRANSMISSION AMPLITUDE COEFFICIENTS

From Unit 6 you would recall that different media offer different impedances to waves travelling through them. These impedances depend on properties of the medium. You may like to know how waves respond to the abrupt change of impedance at the boundary of the media? We now answer this interesting question by considering transverse waves.

7.3.1 Transverse Waves

Let us reconsider the strings \(AO\) and \(OB\) joined together at \(O\) and kept under the same tension \(T\). Let us assume that they offer characteristic impedances of \(Z_1\) and \(Z_2\), respectively. A wave travelling in the positive \(x\)-direction (Fig. 7.4) gets partly reflected and partly transmitted at \(O\). The particle displacements due to incident, reflected and transmitted waves can be written as:

\[
y_i(x, t) = a \sin \left(\omega t - \beta_1 x\right) \quad (7.3)
\]

\[
y_r(x, t) = a \sin \left(\omega t + \beta_1 x\right) \quad (7.4)
\]

\[
y_t(x, t) = a \sin \left(\omega t - k_2 x\right) \quad (7.5)
\]

![Fig. 7.4: Transverse waves in strings having different mass per unit lengths](image)

where the subscripts \(i\), \(r\) and \(t\) on displacements and the amplitudes refer to the incident, reflected and transmitted waves, respectively. You will notice that the angular frequency of these waves remains the same. Moreover the propagation constant for the incident and the reflected waves is the same but differs for the transmitted wave. Do you know why? This is because the wave speed changes as density of the medium changes. You would also notice that for the reflected wave we have used a positive sign before \(k_2 x\). This is because it is travelling in the negative \(x\)-direction.

To give physical meaning to the reflection and transmission coefficients, we have to consider the boundary conditions. The boundary conditions are the conditions which must be satisfied at the interface where the two media meet. Here the total displacement and the total transverse component of tension on one side of the boundary are the result of the combination of incident and reflected waves. So the boundary conditions in this case are:

1. The particle displacements immediately to the left and the right of the boundary (i.e. at \(x = 0\)) must be the same. This implies that the particle velocities \(\frac{\partial y}{\partial t}\) should also be the same.

2. The transverse components of tension \(-T \frac{\partial y}{\partial x}\) must also be the same immediately on both sides of the boundary. These conditions require:

\[
y_i(x, 0) \bigg|_{x^+} + y_r(x, 0) \bigg|_{x^-} = y_t(x, 0) \bigg|_{x^-} \quad (7.6)
\]

and

\[
-T \frac{\partial y_i}{\partial x} \bigg|_{x^+} + T \frac{\partial y_r}{\partial x} \bigg|_{x^-} = -T \frac{\partial y_t}{\partial x} \bigg|_{x^-} \quad (7.7)
\]
Using Eqs. (7.3) to (7.5), the condition expressed by Eq. (7.6)

\[ a, \sin a_d + a, \sin a_d = a, \sin a_d \]

or

\[ a, + a, = a, \] (7.8)

The condition expressed by Eq. (7.7) gives:

\[ \alpha, k_1, T \cos a_d = a, k_1, T \cos a_d = a, k_2, T \cos a_d \]

or

\[ k_1, T a_d - a = k_2, T a_d . \] (7.9)

We know that

\[ k_1, T = \frac{2 \pi}{\lambda}, T = \frac{2 \pi v}{v}, T = 2 \pi \nu_{1,0} = 2 \pi v Z_1, \]

where \( Z_1 \) is impedance offered by the first medium.

In arriving at this result, we have used Eqs. (6.22) and (6.36b). Similarly, you can write

\[ k_1, T = 2 \pi v Z_1, \]

where \( Z_1 \) is the impedance offered by the second medium.

Using these results, we can rewrite Eq. (7.9) as

\[ 2 \pi v Z_1 (a, - a,) = 2 \pi v Z_1 a, \]

or

\[ Z_1, (a, - a,) = Z_1 a, \] (7.10)

Eqs. (7.8) and (7.10) enable us to calculate the ratios \( a/a, \) and \( a/a, \). These ratios give us the fractions of the incident amplitude reflected and transmitted at the boundary. These ratios are usually called the **reflection and transmission amplitude coefficients**. We will denote these by the symbols \( R_{12} \) and \( T_{12} \):

\[ R_{12} = \frac{a,}{a,}, \]

\[ T_{12} = \frac{a,}{a,}. \] (7.11)

\[ \frac{Z_1, - Z_2,}{Z_1, + Z_2,} \]

We note that the reflection and transmission amplitude coefficients depend only on the impedances of the two media.

Let us now consider the implications of results arrived at in Eqs. (7.11) and (7.12):

1. **Assume** that the string is rigidly fixed to a wall. This means that the second medium is extremely heavy, meaning thereby that \( Z_2 = \infty \). In such a case, \( R_{12} = -1 \) and \( T_{12} = 0 \). This result implies that \( a, = - a, \) and \( a, = 0 \). That is, **the amplitude of reflected wave is equal to the amplitude of incident wave with just a reversal of sign and there is no transmitted wave**. This means that the incident wave suffers a change of phase of **π** on reflection from a dense medium.

2. **When** \( Z_2 > Z_1 \), i.e., second string (medium) is denser, \( R_{12} \) is still negative implying a **phase change of π on reflection**. In this case, however, the incident wave is partly reflected and partly transmitted.

3. **When** \( Z_2 < Z_1, R_{12} \) is positive indicating no change of phase on reflection. Both transmitted and reflected waves exist in this case also.

4. **When** \( Z_2 = Z_1, R_{12} = 0 \) showing no reflected wave. In this case \( T_{12} = 1 \), which gives \( a, = a, . \) This means that the amplitude of a transmitted wave is equal to the amplitude of the incident wave.

The points (i), (ii), and (iii) above clearly show that if a wave travelling in a medium of lower impedance meets the boundary of a medium of higher impedance (air to water), the reflected wave undergoes a **phase change of π**. If, however, a wave travelling through a medium of higher impedance meets the boundary of a medium of lower impedance (water to air), no change of phase takes place for the reflected wave. You may also note that \( T_{12} \) is always **negative** indicating that there is no change of phase for the transmitted wave in any case. These results are depicted in Fig. 7.5.

From Eq. (6.36b, a), you will recall that for a given tension, the wave velocity will be lower in a medium of higher impedance. Using this observation, can you now connect the above discussion with the one given in Sec. 7.2.1? It is not a one to one correspondence between...
Waves

incident wave, boundary

\[ Z_i \]

transmitted wave

\[ Z_t \]

reflected wave

\[ Z_r \]

(a)

incident wave, boundary

\[ Z_i \]

transmitted wave

\[ Z_t \]

reflected wave

\[ Z_r \]

(b)

Fig. 7.5: Reflected and transmitted waves when the incident wave (a) travels from a medium of lower impedance to a medium of higher impedance, and (b) when reverse is the case.

the two cases? This explains why we expect all waves, whether sound waves, water waves, waves on string or light waves to follow the same laws.

Coming to the point (iv) above, we note that when \( Z_i = Z_t \), the two strings are made up of the same material and there effectively exists no boundary. That is why there is no reflection at all.

SAQ 1

Two strings of linear densities \( m_i \) and \( m_2 \) (= 4 \( m_i \)) are joined together and stretched with the same tension \( T \). For transverse waves, calculate the reflection and transmission amplitude coefficients.

7.3.2 Longitudinal Waves

To analyse the reflection and transmission of longitudinal waves, you can follow the same procedure as outlined for transverse waves. Let us consider a wave incident on a boundary at \( x = 0 \) separating media of acoustic impedances \( Z_i \) and \( Z_t \). As in the case of transverse waves, you can represent the particle displacements for the incident, reflected and transmitted waves by expressions similar to Eqs. (7.3), (7.4), and (7.5).

The boundary conditions in this case are:

i) The particle displacement \( w_i \) is continuous at the boundary. That is, it has the same value immediately to the left and right of the boundary at \( x = 0 \).

ii) The excess pressure is also the same immediately on two sides of the boundary.

Using the boundary conditions stated above, you can show that the reflected and the transmitted longitudinal waves obey the same characteristics as transverse waves (TQ 5).
7.4 REFLECTION AND TRANSMISSION ENERGY COEFFICIENTS

We know that progressive waves are a useful means of transferring energy from one point to another in a medium. It is therefore interesting to consider as to what happens to the energy in a wave when it encounters the boundary between two media of differing impedances. As before, we will consider transverse as well as longitudinal waves.

You have seen in Unit 6 that when a string of mass per unit length $m$ vibrates with amplitude $a$ and angular frequency $\omega$, the total energy is given by

$$E = \frac{1}{2} ma^2 \omega$$  \hspace{1cm} (7.13)

Let us assume that the wave is travelling with a speed $v$. Then the rate at which the energy is carried along the string is obtained by multiplying the expression for energy with the speed of the wave and is equal to $\frac{1}{2} ma^2 \omega v$.

Now refer to the case of the transverse waves discussed in Section 7.3.1. The rate at which the energy reaches the boundary along with the incident wave is given by

$$P_i = \frac{1}{2} Z_1 a_1^2 \text{sin} \phi$$  \hspace{1cm} (7.14)

Similarly, the rates at which the energy leaves the boundary along with the reflected and the transmitted waves are

$$P_r = \frac{1}{2} Z_2 a_r^2 \cos \phi$$  \hspace{1cm} (7.15)

and

$$P_t = \frac{1}{2} Z_2 a_t^2 \cos \phi$$  \hspace{1cm} (7.16)

Using Eqs. (7.8) and (7.10), we can write $a_i$ and $a_o$ in terms of $a_r$ and $a_t$. Substituting the resulting expressions in Eqs. (7.15) and (7.16) we find that

$$P_r = \frac{1}{2} Z_2 a_r^2 \left( \frac{Z_1 - Z_2}{Z_1 + Z_2} \right)^2$$  \hspace{1cm} (7.17)

and

$$P_t = \frac{1}{2} Z_2 a_t^2 \left( \frac{Z_2 - Z_1}{Z_1 + Z_2} \right)^2$$  \hspace{1cm} (7.18)

These results can be used to obtain the reflection and transmission coefficients $R_i$ and $T_i$:

$$R_i = \frac{\text{Rate at which energy is reflected at the interface}}{\text{Rate at which energy is incident at the interface}} = \frac{P_r}{P_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$  \hspace{1cm} (7.19)

$$T_i = \frac{\text{Rate at which energy is transmitted at the interface}}{\text{Rate at which energy is incident at the interface}} = \frac{P_t}{P_i} = \frac{4Z_2 a_0^2}{(Z_1 + Z_2)^2}$$  \hspace{1cm} (7.20)

We note from Eq. (7.19) that if $Z_1 = Z_2$ (which is also possible if we have $\rho_1 v_1 = \rho_2 v_2$), $R_i = 0$. That is, no energy is reflected back when impedances match. Such an impedance matching plays a very important role in energy transmission. Long distance cables carrying energy need to be matched accurately at all joints; otherwise, a lot of energy will be wasted due to reflection. We need impedance matching when we wish to transfer sound energy from air in a loudspeaker to the air of the room. Similarly, when light waves travel from air into glass lens or a slab, we wish not to have reflections (as it will reduce intensity).

SAQ 2

Show that the energy is conserved when a transverse wave meets the boundary between two media of characteristic impedances $Z_1$ and $Z_2$.

For longitudinal waves, it is customary to calculate energy transfer in terms of their intensity. From Unit 6 we recall that intensity of sound waves in a gas is given by

$$P = \frac{1}{2} \rho v^2 a^2$$
Waves

\[ I = \frac{1}{2} \rho c \frac{d^2u}{dt^2} \vphantom{\frac{1}{2}} \]

\[ = 2\pi^2 \rho c \vphantom{\frac{1}{2}} \]

(7.21)

where \( Z \) is impedance offered by the medium to wave motion. Hence the incident, reflected and transmitted wave intensities can be written as

\[ I_r = 2\pi^2 \rho c \frac{d^2u}{dt^2} Z_i \]

\[ I_t = 2\pi^2 \rho c \frac{d^2u}{dt^2} Z_t \]

\[ I_i = 2\pi^2 \rho c \frac{d^2u}{dt^2} Z_i \]

(7.22)

(7.23)

(7.24)

Using these equations, you can easily show that the reflection and transmission energy coefficients are given by

\[ R = \frac{I_r}{I_i} = \left( \frac{Z_i - Z_t}{Z_i + Z_t} \right)^2 \]

\[ T = \frac{I_t}{I_i} = \frac{4Z_i Z_t}{(Z_i + Z_t)^2} \]

(7.25)

(7.26)

You will observe that these relations are the same as for transverse waves. This means that same conclusions hold even for longitudinal waves.

**SAQ 3**

Sound wave at an incident on the water-steel interface. Show that 86% of the energy is reflected back. Impedances of water and steel are respectively \( 1.43 \times 10^6 \) Nm\(^{-1}\)s and \( 3.9 \times 10^6 \) Nm\(^{-1}\).

### 7.5 The Doppler Effect

We have so far discussed the situations where the wavelength or the wave velocity undergoes a change but its frequency remains the same. Do you know of any situation where frequency of a wave changes, or at least appears to change? In this context we are reminded of an anecdote. The famous physicist W.L. Bragg jumped a red light while driving in London. We was booked for the offence. In the following lines we report the conversation Bragg had with the Magistrate when the latter asked him to appear in his court.

**Magistrate:** Why did you jump the red light?

**Bragg:** Sir, I saw it as green light.

**Magistrate:** At what speed of your vehicle do you see a red light as green?

**Bragg:** (on some calculation) He could do so if he was driving at about two hundred million kilometers per hour.

**Magistrate:** O.K. You are now fined for over-speeding.

This dialogue suggests that frequency can change with the speed of the observer or source. You all must have heard the whistle of a moving train. What do you feel when the train approaches you? The pitch of the whistle seems to rise. But when the engine passes by, the pitch appears to decrease. Similarly, while standing near a highway you may have also noticed that a loaded truck approaching you makes a relatively high-pitched sound “aaaaaaa” and stays low as the truck recedes. The apparent change of frequency due to the relative motion between the source and the observer (or the listener) is known as the Doppler Effect.

In general, when the source approaches the listener or the listener approaches the source, or both approach each other, the apparent frequency is higher than the actual frequency of the sound produced by the source. Similarly when the source moves away from the listener, or when the listener moves away from the source, or when both move away from each other, the apparent frequency is lower than the actual frequency of the sound produced by the source.

Do you know that Doppler shift in ultra-sound reflected from moving body tissues allows measurement of blood flow? It is commonly used by obstetricians to detect foetal heart-beat. Do you know how it arises? As the heart muscle pulsatates, the reflected ultra-sound waves are Doppler shifted from the incident waves. Similarly, a sonar makes use of the Doppler effect in determining the velocity of a submarine relative to a ship.
The electromagnetic waves, including light, are also subject to the Doppler effect. In air navigation, radar works by measuring the Doppler shift of high frequency radio waves reflected from moving aeroplanes. The Doppler shift of star-light allows us to study stellar motion. When we examine light from stars in a spectrograph, we observe several spectral lines. These lines are slightly shifted as compared to the corresponding lines from the same elements on the earth. This shift is generally towards the red-end and is attributed to stellar motion. This is illustrated in Fig. 7.6 for hydrogen atoms in a double-star system. (The Doppler shift of light from distant galaxies is an evidence that our universe is expanding.)

**Figure 7.6**: The wavelength of light emitted by hydrogen atoms in a binary star reveals the stellar motion.

To study the Doppler effect for sound waves, we have to consider the following situations:

i) Whether the source is in motion, or the observer is in motion, or both are in motion.

ii) Whether the motion is along the line joining the source and the observer, or inclined (at an angle) to it.

iii) Whether the direction of motion of the medium is along or opposite to the direction of propagation of sound.

iv) Whether the speed of the source is greater or smaller than the speed of sound produced by it.

We will now consider some of these possibilities.

### 7.5.1 Source in Motion and Observer Stationary

Let us suppose that a source \( S \) is producing sound of frequency \( v \) and wavelength \( \lambda \). The waves emitted by the source spread out as spherical wavefronts of sound. When the velocity of source is less than the velocity of sound, wavefronts lie inside one another. The distance between successive wavefronts is minimum along the direction of motion and maximum in a direction opposite to it (Fig. 7.7).

**Figure 7.7**: Successive wavefronts emitted by a moving source.
Representing the same situation in terms of waves, as shown in Fig. 7.8a, we find that if y is the speed of sound produced, y waves occupy a length v in one second. If the source is stationary. After one second, when the source has moved a distance \( u_s \) towards the listener, the same number of waves per second crowded a length \( (v - u_s) \) as shown in Fig. 7.8(b).

The reduced wavelength \( \lambda' \) then becomes
\[
\lambda' = \frac{v - u_s}{v}
\]

The apparent frequency of sound (heard by the listener) is then
\[
y' = \frac{v}{\lambda'} = \frac{v}{v - u_s}
\]  

If, however, the source moves away from the observer (in a direction opposite to sound), \( u_s \) is negative and Eq. (7.27) becomes
\[
y' = \frac{v}{v + u_s}
\]  

To fix up the ideas discussed above, you may now like to solve a SAQ.

SAQ 4
A person is standing near a railway track. Rajdhani express approaches him/her with a speed of 72 km/h. The apparent frequency of the whistle heard by the person is 750 Hz. What is the actual frequency of the whistle? Use the speed of sound in air as 350 m/s.

7.5.2 Source Stationary and Observer in Motion
If the observer is at rest, the length of the block of waves passing him per second is \( v \), and contains \( y \) waves. However, when the observer moves with speed \( u_o \), he will be at \( O' \) after one second and find that only a block of waves with length \( (v - u_o) \) passes him in one second. For him the apparent frequency is then
\[
y' = \frac{v - u_o}{\lambda} = \frac{v - u_o}{v}
\]

If the listener moves towards the source, \( u_s \) is negative, and the apparent frequency is given by
\[
y' = \frac{v + u_s}{v}
\]
7.5.3 Source and Observer both in Motion

When both source and observer are in motion (and approach each other), we have to combine the results contained in Eqs. (7.27) and (7.29). The source in motion causes a change in wavelength. The listener in motion results in a change of number of waves received. In such a case, apparent frequency $v'$ is given by

\[ v' = \frac{\text{Length of block of waves received}}{\text{Reduced wavelength}} \]

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\[ v' = \frac{\text{Length of block of waves received}}{\text{Reduced wavelength}} \]

The apparent frequency $v'$ is given by

\[ v' = \frac{v}{\sqrt{1 - \frac{u_s}{v}}} \]

or

\[ v' = \sqrt{\frac{v}{1 - \frac{u_s}{v}}} \]

For electromagnetic waves, Eq. (7.31) has to be modified. For sound, $u_s$ and $v$ are measured relative to the medium. This is because the medium determines the wave speed. However, e.m. waves do not require a medium for propagation so that their speed relative to source or the observer is always the same. For these waves we have to consider only the relative motion of the source and the observer. If $u_s$, is the speed of sound relative to observer, and $u_s < v$, we can rewrite Eq. (7.31) as

\[ v' = \frac{v}{\sqrt{1 - \frac{u_s}{v}}} \]

In air navigation, we take $u_s$ to be twice the approach velocity of the aeroplane. This is because the radar detects e.m. waves sent by it and reflected back by the aeroplane.

\[ u_s = 2U_{\text{air}} \]

\[ v' = \sqrt{\frac{v}{1 - \frac{u_s}{v}}} \]

7.6 SHOCK WAVES

So far we have considered the cases where the velocity of sound is greater than the velocity of the source. As $v$ increases, Eq. (7.31) predicts that Doppler shifted frequency will increase gradually and diverge for $u_s = v$. What does this mean? When the source moves exactly at wave speed, wave crests emitted in the forward direction pile up into a very large amplitude at the front of the source, as shown in Fig. 7.10.

Now you may ask: What happens when the speed of source exceeds the speed of sound waves as for supersonic planes? To discover the answer to this question, let us see if we can draw wave patterns similar to those shown in Fig. 7.7.

Fig. 7.10: Schematic representation of piling of waves when the source moves at the wave speed.
Let us suppose that the source is at point $A$ at $t = 0$. After time $t$, the waves emitted at $A$ are on a sphere of radius $vt$. Since $u > v$, the distance travelled by the source $AS = ut$ is more than the distance travelled by sound waves. The waves emitted at successive points, $B, C, D, E$, are on the line $AS$, where the circles are most crowded. We thus see that sound waves pile up on a cone whose half angle is given by $\sin \theta = \frac{v}{u}$ as shown in Fig. 7.1a. No sound waves are present outside the cone. The velocity of sound waves is normal to the surface of the cone. When this cone hits an observer, he detects the sudden arrival of a large amplitude wave, known as a shock wave. A supersonic aircraft generates shock waves, also called sonic booms, due to the formation of two principal shock fronts; one at its nose and the other at its tail (Fig. 7.1b). A strong boom can break window glasses or cause other damage to buildings.

![Shock wave cone](image)

Fig. 7.1: (a) Shock waves created by a sound source moving faster than the speed of sound. (b) Sonic booms produced by a supersonic aircraft.

Shock waves are also generated in a ripple tank by a moving source for Mach numbers greater than one. You can also observe that shock waves are formed by a boat moving faster than the speed of water waves.

### 7.7 SUMMARY

* The locus of points in a given phase is called a wavefront. The shape of a wavefront depends on the nature of the source.
* According to Huygens, each point on a wavefront becomes a fresh source of secondary wavelets, which move out in all directions with the speed of the wave in that medium.
* When waves travelling through one medium meet the boundary of another medium with a different impedance, they are partly reflected and partly transmitted. The reflection and transmission amplitude coefficients are respectively given by
  \[ R_{12} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \]
  and
  \[ T_{12} = \frac{2Z_1}{Z_1 + Z_2} \]
* When a wave travelling from a medium of lower impedance is reflected from a medium of higher impedance, a phase change of π takes place.

Doppler shift: The relative motion between the source of sound waves and the observer (listener), the apparent frequency of the sound is different from the actual one. This is known as the Doppler effect. The Doppler shifted frequency (when both approach each other) is given by

\[ v' = v \left( \frac{v - u_s}{v - u_o} \right) \]

7.8 TERMINAL QUESTION

1. Using Huygens' construction, verify that \( p_{12} = \frac{v_2}{v_1} \sin \theta \).

2. A sound wave travelling through air falls normally on the surface of water. Calculate the ratio of amplitude of sound wave that enters the second medium to the amplitude of incident wave. Use \( \rho = 1.29 \text{ kg m}^{-3} \). Speeds of sound in air and water are 350 m s\(^{-1}\) and 1500 m s\(^{-1}\) respectively.

3. A rope is made up of a number of identical strands twisted together. At one point, the rope becomes frayed so that only a single strand continues (Fig. 7.12). The rope is held under tension and a wave of amplitude 1.0 cm is sent from the single strand. The wave reflected back along the single strand has an amplitude of 0.45 cm. How many strands are 'in the rope'?

Fig. 7.12: A frayed rope

4. A car moving at a velocity 20 m s\(^{-1}\) passes by a stationary source of frequency 500 Hz. The closest distance between them is 20 m. Calculate the apparent frequency heard by the driver as a function of distance. Take \( v = 340 \text{ m s}^{-1} \).

5. Using boundary conditions for longitudinal waves, calculate the amplitude reflection and transmission coefficients.

7.9 SOLUTIONS

5AQs

1. We know that impedance is related to tension and mass per unit length through the relation:

\[ Z = \sqrt{\frac{m}{T}} \]

For the given strings

- \( Z_1 = \sqrt{m_1} \) and \( Z_2 = \sqrt{m_2} \)

\[ \frac{Z_1}{Z_2} = \frac{m_1}{m_2} = \frac{1}{2} \]
Waves

From Eqs. (7.11) and (7.12), we note that the reflection and transmission amplitude coefficients are

\[ R_i = \frac{a_i}{a_i} = \frac{Z_i - Z_\ast}{Z_i + Z_\ast} = \frac{Z_i/Z_\ast - 1}{Z_i/Z_\ast + 1} = \frac{1/2 - 1}{1/2 + 1} = -\frac{1}{3} \]

and

\[ T_i = \frac{a_i}{a_i} = \frac{2Z_i}{Z_i + Z_\ast} = \frac{2Z_i/Z_\ast}{Z_i/Z_\ast + 1} = \frac{2}{3} \]

The negative sign in \( R_i \) implies a phase change of \( \pi \) at the interface.

2. From Eq. (7.14) we know that the rate at which energy reaches the boundary is given by

\[ P_i = \frac{1}{2} Z_i \omega \cdot a_i \]

Similarly, the rate at which energy leaves the boundary with reflected and transmitted waves is given by

\[ P_i = \frac{1}{2} Z_i \omega \cdot a_i + \frac{1}{2} Z_i \omega \cdot a_i \]

On substituting for \( a_i \) and \( a_i \), we get

\[ P_i = \frac{1}{2} Z_i \omega \left\{ \frac{Z_i - Z_\ast}{Z_i + Z_\ast} \right\} \cdot a_i + \frac{1}{2} Z_i \omega \left\{ \frac{2Z_i}{Z_i + Z_\ast} \right\} \cdot a_i \]

\[ = \frac{1}{2} Z_i \omega \left\{ \frac{Z_i - Z_\ast}{Z_i + Z_\ast} \right\} \cdot \frac{4Z_i}{Z_i + Z_\ast} a_i \int \frac{1}{2} Z_i \omega a_i \]

Since the rate at which energy arrives at the interface is equal to the rate at which energy leaves the interface (with reflected and transmitted waves), we can say that energy is conserved in this process.

3. Reflection energy coefficient

\[ R_i = \frac{Z_i - Z_\ast}{Z_i + Z_\ast} = \frac{(1.43 - 0.39) \times 10^9 \text{ N m}^{-2}}{(1.43 + 39) \times 10^8 \text{ N m}^{-2}} \]

\[ = \left( \frac{1}{4} \right) = 0.86 \]

This means that when sound waves are incident on water-steel interface, only 86% of the energy is reflected back.

4. From Eq. (7.27) we have

\[ \nu = \nu - \nu \]

Rearranging terms, we can write

\[ \nu = \nu - \nu \]

Here \( \nu = 350 \text{ m s}^{-1} \), \( \nu = 700 \text{ Hz} \)

and \( u_i = 72 \text{ km h}^{-1} = 20 \text{ m s}^{-1} \)

\[ \cdot \nu = \left( \frac{350 \text{ m s}^{-1}}{350 \text{ m s}^{-1}} \right) \times 700 \text{ Hz} = 660 \text{ Hz} \]

5. Since the wavelength increases, we can say that the star is moving away along the line of sight. This means that the frequency decreases. Using Eq. (7.28) for the case of light you can write

\[ \nu' = \nu \left( \frac{c}{c + u_i} \right) \]
Waves at the Boundary of Media

\[
\frac{1}{\lambda'} = \frac{1}{\lambda} \left( 1 - \frac{u_0}{c} \right)
\]

for \( u_0 \ll c \)

Since \( v = c/\lambda \), you can write

\[
\frac{1}{\lambda'} = \frac{1}{\lambda} \left( 1 - \frac{u_0}{c} \right)
\]

or

\[
u_0 = \frac{c}{\lambda'} (\lambda' - A)
\]

Here \( \lambda' = 4100 \lambda, A = 4000 \) and \( c = 3 \times 10^3 \) ms\(^{-1} \).

Hence

\[
u_0 = \frac{3 \times 10^3 \text{ m/s}}{4000 \lambda}
\]

\[
= 7.5 \times 10^2 \text{ m/s}
\]

\[
= 7.5 \times 10^2 \text{ km/h}
\]

**Q1.** Refer to Fig. 7.13. AB represents a part of a wavefront moving towards the interface \( S_1, S_2 \), which separates the two mediums say air and water. Let us assume that wave speeds in medium 1 and medium 2 be \( v_1 \) and \( v_2 \) respectively.

\[\text{Fig. 7.13: Huygen's construction to deduce the laws of refraction}\]

The wavefront will first strike at \( C \) and then at successive points towards \( D \). The point \( B \) on the wavefront reaches \( D \) at a time \( t = BD/v_1 \) later than the point \( A \) reaches \( C \). From each point on \( S_1, S_2 \), a secondary wavelet starts growing into the second medium at speed \( v_2 \) after the instant when \( D \) is just disturbed, the wavelet from \( C \) has grown for time

\[t = \frac{DC}{v_2}\]

You can represent this wavelet by drawing an arc of radius \( CC' \) with \( C \) as centre. Draw a tangent \( DC'' \) from \( D \) to the arc. If you repeat this process for other intermediate points between \( C \) and \( D \) you will observe that \( DC'' \) is a common tangent to all of them. Thus, \( DC'' \) represents the refracted wavefront.

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From As CB'D and CC"D, you can write
\[
\sin \theta = \frac{B'D}{CD} = \frac{BD}{CC''D}
\]
(ii)

Using the result contained in (i), you would get
\[
\frac{\sin \theta}{\sin \phi} = \frac{v_1}{v_2} = \text{a constant}
\]
(iii)

That is, the sine of the angle of incidence to the sine of the angle of refraction of the wavefront is equal to the ratio of the wave speeds and is a constant. This constant is known as the effective index of medium 2 with respect to medium 1. We denote it by the symbol \( \mu_{21} \).

For sound, with medium 1 as air and medium 2 as water
\[ \mu_{21} = 0.23 \]
and for light
\[ \mu_{21} = 1.33 \]

2. \( \rho_1 = 1.29 \text{ kg m}^{-3} \)
\( \rho_2 = 1000 \text{ kg m}^{-3} \)
\( \eta_1 = 350 \text{ m s}^{-1} \)
\( \eta_2 = 1500 \text{ m s}^{-1} \)

Since sound waves are longitudinal, from Eq. (7.12) we have
\[
a_1 = \frac{2\eta_1}{\rho_1 + \rho_1} = \frac{2\eta_1}{2(\rho_1 + \rho_1)}
\]
(i)

Since \( Z = \rho v \), we can write
\[
\frac{Z_1}{Z_2} = \frac{\rho_1 v_1}{\rho_2 v_2} = \frac{1.29 \text{ kg m}^{-3} \times 350 \text{ m s}^{-1}}{1000 \text{ kg m}^{-3} \times 1500 \text{ m s}^{-1}} = 0.023
\]

Using this result in (i), we get
\[
a_1 = \frac{2(0.023 \times 10^{-4})}{1 + (0.023 \times 10^{-4})} = 6.02 \times 10^{-4}
\]

3. From Eq. (7.11) you can write
\[
x = \frac{a_1}{a_2} = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{Z_1}{Z_2} - 1
\]
\( = \frac{(Z/Z_2) - 1}{Z_2/Z_2 + 1} \)

For a string under tension, \( Z \propto \sqrt{m} \). So we can write
\[
\frac{Z_1}{Z_2} = \sqrt{\frac{m_1}{m_2}}
\]

Hence
\[
x = \frac{\sqrt{m_1/m_2} - 1}{\sqrt{m_1/m_2} + 1}
\]

Assume that first portion has \( n \) strands. Then
\[
x = \frac{\sqrt{n} - 1}{\sqrt{n} + 1}
\]
Solving this for \( n \), we find that
\[
\sqrt{n} = \frac{1 + x}{1 - x} = \frac{1 + 0.45}{1 - 0.45} = \frac{1.45}{0.55} = 2.65
\]
Hence
\[
\mu = \left( \frac{1.45}{0.55} \right)^2 = 7.6
\]

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4. In this case, the velocity of the car is not directed towards the sound source (Fig. 7.14a), and we have to find the component of the velocity vector directed towards the source. Referring to Fig. 7.14b it is given by

![Diagram](image)

Fig. 7.14: (a) Observer moving along a line not intersecting the line of motion of source

(b) The component of velocity towards the source is responsible for Doppler shift.
Waves

Then the space dependence Doppler-shifted frequency is given by

\[
\nu'(x) = \frac{\nu + v_0 \cos \theta}{v}
\]

You can plot this as a function of \( x \) for \(-100 \leq x \leq 100\) m. At \( x = 0 \), the car is moving perpendicular to the wave and at the instant when the car passes this point, the driver hears the true frequency, 500 Hz.

5. The particle displacement for the incident, reflected and transmitted waves are

\[
\psi(x, t) = a_0 \sin \left( \frac{\omega_0}{c} - kx \right) \quad (i)
\]

\[
\phi(x, t) = a_1 \sin \left( \omega_0 t + kx \right) \quad (ii)
\]

\[
\lambda(x, t) = a_0 \sin \left( \omega_0 t - kx \right) \quad (iii)
\]

The boundary conditions in this case are:

1. The particle displacement \( \psi(x, t) \) is continuous at the boundary. That is, it has the same value immediately to the left and the right of the boundary at \( x = 0 \).

2. The excess pressure is same on the two sides of the boundary.

The first condition implies that

\[ a_0 + a_1 = a_0 \quad (iv) \]

For a longitudinal wave, \( \Delta p = -\frac{\partial E}{\partial x} \) where \( E \) is elasticity. Since \( E = V p \), where

\[ V = \frac{C_0}{C} \]

and \( p_0 \) is equilibrium pressure, we find that \( p_0 \) cancels out on both sides and the second condition implies that

\[
\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial x} = \frac{\partial \lambda}{\partial x} \quad \text{Eq. (v)}
\]

Eq. (v) gives:

\[-a_0 k \cos \omega_0 t + a_0 k \cos \omega_0 t = -a_1 k_1 \cos \omega_0 t\]

giving

\[ k_1 (a_0 - a_1) = k_0 a_0 \]

We know that

\[ k_1 = \frac{\omega_0}{V_1} \]

Multiplying by \( p_0 \nu_1 \), we get

\[ k_1 = \frac{\omega_0 p_0 \nu_1}{p_0} = \frac{\omega_0 Z_1}{\gamma p_0} \]

as \( Z_1 = p_0 \nu_1 \) and \( V_1 = \sqrt{\gamma p_0} \). Similarly, you can show that

\[ k_2 = \frac{\omega_0 Z_2}{\gamma p_0} \]

Using these results in (vi), we find that

\[
\frac{\omega_0 Z_2}{\gamma p_0} (a_0 - a_1) = \frac{\omega_0 Z_2}{\gamma p_0} a_0
\]

Since relations (vi) and (vii) connecting the incident, reflected and transmitted amplitudes are exactly the same as in the transverse case, the reflection and transmission amplitudes coefficients are also given by the same relations.
8.1 INTRODUCTION

You have studied in Unit 2 of Block 1, how a particle acted upon simultaneously by two simple harmonic oscillations gives rise to the formation of Lissajous figures.

You have also read about the general characteristics of waves in Unit 6 of this block; and of their behaviour at the interface of two media in Unit 7. In this unit you will study about the principle of superposition of waves. Under certain conditions, the superposition of waves leads to some interesting phenomena like the formation of stationary waves, beats, wave groups, interference, diffraction etc. In the present unit you will study the phenomena of stationary waves, wave group and beats. The other two topics, viz. Interference and Diffraction, will be discussed in Unit 9 of this Block.

In the present unit you will study the basic features, especially the sound producing part of the woodwind instruments. There are two basic types of pipes, viz. flute pipe and reed pipe, which you will study in this unit. Stationary waves are formed when two waves of the same angular frequency (i.e., same ω), same wavelength (i.e., of the same wave vector or propagation constant k) and of same amplitude, travelling on opposite directions superpose on each other. On the other hand, if two sound waves of slightly different frequencies are superposed, they produce beats.

Wave groups, sometimes also called the wave packets, are the result of superposition of waves of slightly different frequencies. The concept of wave packet is of great importance in the study of quantum mechanics, which we consider later.

In the next Unit you will study the superposition of two waves, which leads to the phenomena of interference. There you will also study about the necessary conditions for the interference of two waves. Towards the end, you will learn about diffraction of waves and some typical cases of diffraction phenomena.

Objectives

After going through this unit, you will be able to:

- Describe the principle of superposition of waves
- Explain the ideas underlying the formation of stationary waves
- Identify the positions of nodes and antinodes on a stationary wave
- List the characteristics of stationary waves
- Describe the formation of wave groups
- Compute the value of group velocity knowing the dependence of wave velocity on wavelength
- Calculate the number of beats produced if the frequencies of two superposing notes are known.