UNIT 2 SUPERPOSITION OF SIMPLE HARMONIC OSCILLATIONS

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2.1 INTRODUCTION

In Unit 1, we studied simple harmonic motion and considered a number of examples from different areas of physics. We found that in each case the motion is governed by a homogeneous second order differential equation. The solution of this equation gives us information regarding displacement of the body as a function of time. In many situations, one has to deal with a combination of two or more simple harmonic oscillations. Do you know that our eardrums vibrate under a complex combination of harmonic vibrations? The resultant effect is given by the principle of superposition. You must have observed that oscillations of a swing gradually die out, when left to itself. This is due to factors like friction and air resistance. The system loses energy and its motion is said to be damped. We will discuss damped harmonic oscillations in the next unit.

In this unit we first discuss the principle of superposition. Then you will learn to apply this principle to situations where two (or more) harmonic oscillations are superposed, either along the same line or in perpendicular directions.

'Objectives
After studying this unit you should be able to
• state the principle of superposition
• apply the principle of superposition to two harmonic oscillations of (a) the same frequency and (b) different frequencies along the same line
• apply the methods of vector addition and complex numbers for superposition of many simple harmonic oscillations, and
• apply the principle of superposition to two mutually perpendicular harmonic oscillations of different frequencies/ phases and describe the formation of Lissajous figures.
2.2 PRINCIPLE OF SUPERPOSITION

We know that for small oscillations, a simple pendulum executes simple harmonic motion. Let us reconsider this motion and release the bob at the instant \( t = 0 \) when it has initial displacement \( a_0 \). Let the displacement at a subsequent time \( t \) be \( x_0 \). Let us repeat the experiment with an initial displacement \( a_2 \). Let the displacement after the same interval of time \( t \) be \( x_2 \). Now if we take the initial displacement to be the sum of the earlier displacements, viz. \( a_1 + a_2 \), then according to the superposition principle, the displacement \( x_3 \) after the same interval of time \( t \) will be

\[
x_3 = x_1 + x_2.
\]

You can perform this activity by taking three identical simple pendulums. Release all three bobs simultaneously such that their initial velocities are zero and initial displacements of the first, second and the third pendulum are \( a_1 \), \( a_2 \) and \( a_1 + a_2 \), respectively. You will find that at any time the displacement \( x_3 \) of the third pendulum will be the algebraic sum of the displacements of the other two. In general, the initial velocities may be non-zero. Thus, the principle of superposition can be stated as follows:

When we superpose the initial conditions corresponding to velocities and amplitudes, the resultant displacement of two (or more) harmonic displacements will be simply the algebraic sum of the individual displacements at all subsequent times.

You will note that the principle of superposition holds for any number of simple harmonic oscillations. These may be in the same or mutually perpendicular directions, i.e., in two dimensions.

In Unit 1, we observed that Eq. (1.3) describes SHM:

\[
d x^2 \over dt^2 = -\omega^2 \cdot x
\]

This is a linear homogeneous equation of second order.

Such an equation has an important property that the sum of its two linearly independent solutions is itself a solution. We have already used this property in Unit 1 while writing Eq. (1.4).

Let \( x_1(t) \) and \( x_2(t) \) respectively satisfy equations

\[
\frac{d^2 x_1}{dt^2} = -\omega^2 \cdot x_1
\]

and

\[
\frac{d^2 x_2}{dt^2} = -\omega^2 \cdot x_2
\]

Then by adding Eqs. (2.2) and (2.3), we get

\[
\frac{d^2 (x_1 + x_2)}{dt^2} = -\omega^2 \cdot (x_1 + x_2)
\]

According to the principle of superposition, the sum of two displacements given by

\[
x(t) = x_1(t) + x_2(t)
\]

also satisfies Eq. (2.1). In other words, the superposition of two displacements satisfies the same linear homogeneous differential equation which is satisfied individually by \( x_1 \) and \( x_2 \).

SAQ 1

For a simple pendulum we know that the equation of motion is

\[
\frac{d^2 \theta}{dt^2} = -\omega^2 \cdot \sin \theta
\]
If in this equation you use the expansion

\[ \sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots \]

will it remain linear in \( \theta \)? If you retain the first two terms and consider the equation for the two displacements \( \theta_1 \) and \( \theta_2 \), will the principle of superposition still hold? If not, why?

You will find that in the case of a simple pendulum you can apply the principle of superposition only for small oscillations, i.e., when \( \sin \theta \approx \theta \). Here we shall study only those oscillations for which the displacement satisfies linear homogeneous differential equations.

### 2.3 SUPERPOSITION OF TWO HARMONIC OSCILLATIONS OF THE SAME FREQUENCY ALONG THE SAME LINE

Let us superpose two collinear (along the same line) harmonic oscillations of amplitudes \( a_1 \) and \( a_2 \), having frequency \( \omega_0 \) and a phase difference of \( \pi \). The displacements of these oscillations are given by

\[
X_1(t) = a_1 \cos \omega_0 t \\
X_2(t) = a_2 \cos (\omega_0 t + \pi)
\]

According to the principle of superposition, the resultant displacement is given by

\[
X(t) = X_1(t) + X_2(t) = a_1 \cos \omega_0 t - a_2 \cos \omega_0 (t + \pi) = (a_1 - a_2) \cos \omega_0 t
\]

This represents a simple harmonic motion of amplitude \( a_1 - a_2 \). In particular, if two amplitudes are equal, i.e., \( a_1 = a_2 \), the resultant displacement will be zero at all times. Displacement-time graph depicting this situation is shown in Fig. 2.1.

![Fig. 2.1 Superposition of two collinear harmonic oscillations of equal amplitude but out of phase by \( \pi \).](image)

**SAQ 2**

Two harmonic oscillations of amplitudes \( a_1 \) and \( a_2 \) have the same frequency \( \omega_0 \) and are in phase. Show that their superposition gives a harmonic oscillation of amplitude \( a_1 + a_2 \).

We will now discuss the general case of superposition of two harmonic oscillations. Let one of these be characterised by amplitude \( a_1 \) and initial phase \( \phi_1 \), and the other with amplitude \( a_2 \) and phase \( \phi_2 \). Both oscillations have frequency \( \omega_0 \) and are collinear, i.e., they are along the same line. Then, we can write

\[
X_1(t) = a_1 \cos (\omega_0 t + \phi_1) \\
X_2(t) = a_2 \cos (\omega_0 t + \phi_2)
\]

According to the principle of superposition, the resultant displacement is the sum of \( X_1 \) and \( X_2 \) and we have

\[
X(t) = X_1(t) + X_2(t) = a_1 \cos (\omega_0 t + \phi_1) + a_2 \cos (\omega_0 t + \phi_2)
\]
Using the expression for the cosine of the sum of two angles, this can be written as

$$\cos(\omega t + \phi) = \cos \omega t \cos \phi - \sin \omega t \sin \phi$$

Collecting the coefficients of \( \cos \omega t \) and \( \sin \omega t \), we get

$$x(t) = (a_1 \cos \omega_1 t + a_2 \cos \omega_2 t) \cos \omega t$$

Since \( a_1, a_2, \phi_1, \) and \( \phi_2 \) are constant, we can set

$$a_1 \cos \phi_1 = a_1 \cos \phi_1 + a_2 \cos \phi_2$$

and

$$\phi = \frac{\phi_1 - \phi_2}{2}$$

where \( \alpha \) and \( \beta \) have to be determined. Then, we can rewrite Eq. (2.11) in the form

$$x(t) = a \cos (\omega t + \phi)$$

This equation has the same form as either of our original equations for separate harmonic oscillations. Hence, we have the important result that the sum of two collinear harmonic oscillations of the same frequency is also a harmonic oscillation of the same frequency and along the same line. But it has a new amplitude \( a \) and a new phase constant \( \phi \). The amplitude can easily be calculated by squaring Eqs. (2.12) and (2.13) and adding the resultant expressions. On simplification we have

$$\alpha^2 = a_1^2 + a_2^2 + \alpha_1 \alpha_2 \cos(\psi - \phi)$$

Similarly, the phase \( \phi \) is determined by dividing Eq. (2.13) by Eq. (2.12):

$$\phi = \tan^{-1} \left( \frac{a_1 \sin \phi_1 + a_2 \sin \phi_2}{a_1 \cos \phi_1 + a_2 \cos \phi_2} \right)$$

SAQ 3

Two harmonic oscillations of frequency \( \omega_0 \) have initial phases \( \phi_1 \) and \( \phi_2 \) and amplitudes \( a_1 \) and \( a_2 \). Their resultant has the phase

(a) \( \phi_1 - \phi_2 = 2\pi n \)

and

(b) \( \phi_1 - \phi_2 = (2n + 1) \pi \)

where \( n \) is an integer. Using Eq. (2.15), show that the amplitudes of the resultant oscillations are equal to \( (a_1 + a_2) \) and \( (a_1 - a_2) \), respectively.

SAQ 4

Two harmonic oscillations of frequency \( \omega_0 \) having an amplitude 1 cm and initial phases zero and \( \pi/2 \), respectively, are superposed. Calculate the amplitude and the phase of the resultant vibration.

### 2.4 SUPERPOSITION OF TWO COLLINEAR HARMONIC OSCILLATIONS OF DIFFERENT FREQUENCIES

In a number of cases, we have to deal with superposition of two or more harmonic oscillations having different angular frequencies. A microphone diaphragm and human eardrums are simultaneously subjected to various vibrations. For simplicity,
we shall first consider superposition of two harmonic oscillations having the same amplitude \( a \) but slightly different frequencies \( \omega_1 \) and \( \omega_2 \) such that \( \omega_1 > \omega_2 \):

\[
\begin{align*}
    x_1 &= a \cos(\omega_1 t + \phi_1) \\
    x_2 &= a \cos(\omega_2 t + \phi_2)
\end{align*}
\]

We note that the phase difference between these two harmonic vibrations is

\[
\phi = (\omega_1 - \omega_2) t + (\phi_1 - \phi_2)
\]

The first term \((\omega_1 - \omega_2) t\) changes continuously with time. But the second term \((\phi_1 - \phi_2)\) is constant in time and as such it does not play any significant role here. Therefore, we may assume that the initial phase of two oscillations are zero. Then two harmonic oscillations can be written as

\[
\begin{align*}
    x_1(t) &= a \cos \omega_1 t \\
    x_2(t) &= a \cos \omega_2 t
\end{align*}
\]

The superposition of two oscillations gives the resultant

\[
x(t) = x_1(t) + x_2(t) = a \cos \omega_1 t + a \cos \omega_2 t
\]

This equation can be rewritten in a particularly simple form using the formula

\[
\cos A + \cos B = 2 \cos \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right)
\]

\[
x(t) = 2a \cos \left( \frac{\omega_1 + \omega_2}{2} \right) t \cos \left( \frac{\omega_1 - \omega_2}{2} \right) t
\]

This is an oscillatory motion with angular frequency \( \left( \omega_1 + \omega_2 \right) / 2 \) and amplitude \( 2a \cos \left( \frac{\omega_1 - \omega_2}{2} \right) t \).

Let us define as average angular frequency

\[
\omega_{av} = \frac{\omega_1 + \omega_2}{2}
\]

and a modulated angular frequency

\[
\omega_{mod} = (\omega_1 - \omega_2) / 2
\]

Then we find that the modulated amplitude

\[
ad(t) = 2a \cos \omega_{mod} t
\]

varies with a frequency \( \omega_{mod} / 2\pi = (\omega_1 - \omega_2) / 4\pi \).

This also implies that in one complete cycle the modulated amplitude takes values of 
\(2a, 0, -2a, 0, 2a\) for \(\omega_{mod} t = 0, \pi/2, \pi, 3\pi/2, 2\pi\), respectively. The resultant oscillation can be written as

\[
x(t) = ad(t) \cos \omega_{av} t
\]

This equation resembles the equation of SHM. But this resemblance is misleading because its amplitude varies with time.
The displacement-time graph depicting the resultant of two \textbf{collinear} harmonic oscillations of different frequencies is shown in Fig. 2.2. You will \textit{note} that individual oscillations are harmonic but their superposition shows variation with time; it is periodic but not simple harmonic.

In the general case, we consider two harmonic oscillations having amplitudes $a_1$ and $a_2$ and angular frequencies $\omega_1$ and $\omega_2$. If their initial phases are zero, the resultant oscillation can be written as

$$x(t) = a_{\text{mod}}(t) \cos (\omega_{\text{mod}} t + \theta_{\text{mod}}).$$ \hspace{1cm} (2.22)

The modulated amplitude and phase constant are respectively given by

$$a_{\text{mod}}(t) = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos (2\omega_{\text{mod}} t)}$$ \hspace{1cm} (2.23)

and

$$\theta_{\text{mod}} = \frac{(a_1 - a_2) \sin \omega_{\text{mod}} t}{(a_1 + a_2) \cos \omega_{\text{mod}} t}.$$ \hspace{1cm} (2.24)

For $a_1 = a_2$ you will note that the expression for $a_{\text{mod}}(t)$ reduces Eq. (2.20c) and $\theta_{\text{mod}} = \theta$.

If $a_1$ and $a_2$ are nearly equal, $a_{\text{mod}}$ would be much less than $a_{\text{mod}}$ and modulated amplitude will vary very slowly with time. That is, for $a_{\text{mod}} < < a_{\text{mod}}$, one can regard $a_{\text{mod}}(t)$ as essentially constant over the period $2\pi/a_{\text{mod}}$. Then Eq. (2.22) will represent an almost harmonic oscillation of angular frequency $\omega_{\text{mod}}$.

The amplitude of the resulting motion is maximum ($= a_1 + a_2$) when

$$\cos 2\omega_{\text{mod}} t = 1$$

This means that

$$\frac{2\omega_{\text{mod}} t = 2n\pi}{(\omega_1 - \omega_2) t = 2n\pi} \quad n = 0, 1, 2, \ldots$$

or

$$t = \frac{1}{(\nu_1 - \nu_2)}, \frac{1}{(\nu_1 - \nu_3)}, \ldots, \frac{1}{(\nu_1 - \nu_n)}$$ \hspace{1cm} (2.25)

where $(\nu_1 = \omega_1/2\pi)$ and $(\nu_2 = \omega_2/2\pi)$ are the frequencies of two harmonic oscillations.

Similarly, you will note that the amplitude of the resultant oscillation attains a minimum value $(a_1 - a_2)$ when

$$\cos 2\omega_{\text{mod}} t = -1$$

That is, when

$$t = \frac{1}{2 (\nu_1 - \nu_2)}, \frac{3}{2 (\nu_1 - \nu_3)}, \frac{5}{2 (\nu_1 - \nu_4)} \ldots$$ \hspace{1cm} (2.26)

### 2.5 SUPERPOSITION OF MANY HARMONIC OSCILLATIONS OF THE SAME FREQUENCY

In the preceding sections, we considered superposition of two collinear harmonic oscillations. How will you calculate the resultant of a number of harmonic oscillations of the same frequency? You may suggest that an obvious way is to extend the procedure outlined in Sec. 2.3. But we find that the mathematical analysis, though simple, becomes unwieldy. A convenient way out in such a case is to use either the method of vector analysis or complex numbers. We will now discuss these in turn.
2.5.1 Method of Vector Addition

This method is based on the fact that the displacement of a harmonic oscillation is the projection of a uniform circular motion on the diameter of the circle. Therefore, it is important to understand the connection between SHM and uniform circular motion.

Uniform Circular Motion and SHM

Let us suppose that a particle moves in a circle with constant speed $V$ (Fig. 2.3). The radius vector joining the centre of the circle and position of the particle on the circumference will rotate with a constant angular frequency. We take the $x$-axis to be along the direction of the radius vector at time $t=0$. Then the angle made by the radius vector with the $x$-axis at any time $t$ will be given by

![Fig. 2.3 Uniform circular motion and its connection with SHM.](image)

$$\theta = \frac{\text{length of the arc}}{\text{radius of the circle}} = \frac{Vt}{R}$$

The $x$- and $y$- components of the position of the particle at time $t$ are

$$x = R \cos \theta,$$

and

$$y = R \sin \theta,$$

Thus

$$\frac{dx}{dt} = -R \sin \theta \frac{d\theta}{dt} = -\omega R \sin \theta$$

since

$$\frac{d\theta}{dt} = \omega = \frac{V}{R}$$

Similarly, you can write

$$\frac{dy}{dt} = \omega R \cos \theta$$

Differentiating again with respect to time, we get

$$\frac{d^2x}{dt^2} = -\omega^2 x,$$

and

$$\frac{d^2y}{dt^2} = -\omega^2 y$$

These expressions show that when a particle moves uniformly in a circle, its projections along $x$- and $y$- axes execute SHM. In other words, a simple harmonic motion may be viewed as a projection of a uniformly rotating vector on a reference axis.

Suppose that the vector $\mathbf{OP}$ with $|\mathbf{OP}|=a_0$ is rotating with an angular frequency $\omega_0$ in an anticlockwise direction, as shown in Fig. 2.4. Let $P$ be the foot of the perpendicular drawn from $P'$ on $x$-axis. Then $OP$ is projection of $OP'$ on the $x$-axis. As vector $\mathbf{OP'}$ rotates at constant speed, the point $P$ executes simple harmonic motion along $x$- axis. Its period of oscillation is equal to the period of the rotating vector $\mathbf{OP'}$. Let $OP'$ be the initial position of a rotating vector. Its projection $OP$ on the $x$-axis is $a_0 \cos \phi$. If this rotating vector moves from $OP'$ to $OP$ in time $t$, then

$$\Delta P'OP = \omega_0 t$$

Then we can write
Thus, point \( P \) executes simple harmonic motion along the \( x \)-axis.

If you project \( OP' \) on the \( y \)-axis, you will find that the point corresponding to the foot of the normal satisfies the equation

\[ y = a_0 \sin (\omega t + \phi). \tag{2.28b} \]

This means that a rotating vector can, in general, be resolved into two orthogonal components, and we can write

\[ r = x + y \tag{2.29} \]

where \( x \) and \( y \) are unit vectors along the \( x \)-axis and the \( y \)-axis, respectively.

Vector Addition

Let us now consider superposition of \( n \) harmonic oscillations, all having the same amplitude \( a_0 \) and angular frequency \( \omega_0 \). The initial phases of successive oscillations differ by \( \phi_0 \). Let the first of these oscillations be described by the equation

\[ x(0) = a_0 \cos \omega_0 t \]

Then other oscillations are given by

\[
x_2(t) = a_0 \cos (\omega_0 t + \phi_0) \\
x_3(t) = a_0 \cos (\omega_0 t + 2\phi_0) \\
x_n(t) = a_0 \cos (\omega_0 t + (n-1)\phi_0) \tag{2.30}
\]

From the principle of superposition, the resultant oscillation is given by

\[ x(t) = a_0 \left[ \cos \omega_0 t + \cos (\omega_0 t + \phi_0) + \cos (\omega_0 t + 2\phi_0) + \ldots + \cos (\omega_0 t + (n-1)\phi_0) \right] \tag{2.31} \]

Let us denote the harmonic oscillations given in Eq. (2.31) as projections of rotating vectors \( OP_1, OP_2, OP_3, \ldots \) (Fig 2.4b).

To find the resultant of these vectors, we translate them parallel to themselves so that the head of the first coincides with the tail of the second and so on. You will observe that

i) combining vectors form successive sides of an incomplete \( n \)-sided polygon (Fig 2.4c)

ii) \( OP_1 \parallel OP_2 \parallel OP_3 \parallel OP_4 \parallel OP_5 \parallel \ldots \) and so on.
A circle is an infinite-sided regular polygon.

Let us now project each of these vectors along the x-axis. Then, we get

\[ x_1 = \text{Proj}(\mathbf{O}P_1) = a_0 \cos \omega t \]
\[ x_2 = \text{Proj}(\mathbf{P}_1\mathbf{P}_2) = a_0 \cos (\omega t + \phi_0) \]
\[ x_3 = \text{Proj}(\mathbf{P}_2\mathbf{P}_3) = a_0 \cos (\omega t + 2\phi_0) \]
\[ \vdots \]
\[ x_n = \text{Proj}(\mathbf{P}_{n-1}\mathbf{P}_n) = a_0 \cos \left[ \omega t + (n - 1)\phi_0 \right] \]

The law of vector addition implies that the resultant of \( \mathbf{OP}_1, \mathbf{P}_1\mathbf{P}_2, \mathbf{P}_2\mathbf{P}_3, \ldots \) is given by the vector \( \mathbf{OP}_n \), i.e.

\[ \mathbf{OP}_n = \mathbf{OP}_1 + \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_2\mathbf{P}_3 + \ldots + \mathbf{P}_{n-1}\mathbf{P}_n \]

Thus,

\[ \text{Proj}(\mathbf{OP}_n) = \text{Proj}(\mathbf{OP}_1) + \text{Proj}(\mathbf{P}_1\mathbf{P}_2) + \ldots \]

or

\[ x(t) = x_1(t) + x_2(t) + x_3(t) + \ldots \]

Let the length of \( \mathbf{OK} \) be \( a \) and its phase with respect to the first vector be \( \phi_0 \). Then the projection of \( \mathbf{OK} \) along x-axis is given by

\[ x(t) = \text{Proj}(\mathbf{OK}) = a_0 \cos (\omega t + \phi_0) \]

Hence the sum in Eq. (2.31) reduces to calculating a and \( \phi \) characterizing the resultant vector \( \mathbf{OP}_n \). To this end, we recall that any regular polygon will lie on a circle of radius \( r \), as shown in Fig. 2.4 b. The angle subtended at the centre \( C \) of the circle by individual vectors will be equal to \( \phi_0 \). Hence, the total angle subtended at \( C \) by the resultant vector \( \mathbf{OP}_n \) will be \( n\phi_0 \).

From the triangle \( \triangle OCP \), we note that

\[ a = \sqrt{2r^2 - 2r^2 \cos \phi_0} \]

Using the trigonometric relation \( \cos 2\theta = 1 - 2\sin^2\theta \) and simplifying the resultant expression, we get

\[ a = 2r \sin (\phi_0/2) \] \hspace{1cm} (2.33a)

Similarly, we can show that

\[ a_0 = 2r \sin (\phi_0/2) \] \hspace{1cm} (2.33b)

On combining Eqs. (2.33 a) and (2.33 b) we obtain the amplitude of resultant vector \( \mathbf{OP}_n \):\n
\[ a = a_0 \frac{\sin (n\phi_0/2)}{\sin (\phi_0/2)} \] \hspace{1cm} (2.34)

The phase difference \( \phi \) of the resultant oscillation relative to the first oscillation is given by

\[ \phi = \angle \mathbf{OP}_n - \angle \mathbf{OP}_1 \] \hspace{1cm} (2.35)
In the isosceles \( \triangle COA' \), \( \angle OCP' = \phi_0 \) and \( \angle COP' = \pi/2 \).

Since the sum of the angles of a triangle is \emph{equal} to \( \pi \), we can write

\[
\angle COA' = \pi - \angle OCP' - \angle COP' = \pi - \frac{\pi}{2} - \phi_0 = \frac{\pi}{2} - \phi_0
\]

(2.36 a)

Similarly, in the isosceles \( \triangle COP' \), \( \angle COP = \pi/2 \) and \( \angle COP' = \angle CP'O \).

Therefore,

\[
2 \angle COP' = \pi - n \phi_0
\]

or

\[
\angle COP' = \frac{\pi}{2} - \frac{n \phi_0}{2}
\]

Hence, by combining Eqs. (2.35) and (2.36), we get

\[
\phi = \left( \frac{\pi}{4} - \phi_0 \right) - \left( \frac{\pi}{2} - n \frac{\phi_0}{2} \right) = (n - 1) \frac{\phi_0}{2}
\]

(2.37)

That is, the initial phase of the resultant oscillation is equal to half the phase difference between the \( n \)th and the first oscillations. Hence,

\[
x(t) = a_0 \left[ \frac{\sin \left( \frac{n \phi_0}{2} \right)}{\sin \left( \frac{\phi_0}{2} \right)} \right] \cos \left[ \omega t + (n - 1) \frac{\phi_0}{2} \right]
\]

(2.38)

We shall obtain the same result in the next sub-section using the method of complex numbers. For the moment let us examine the behaviour of the amplitude of the resultant oscillation defined by Eq. (2.38):

\[
a = a_0 \left[ \frac{\sin \left( \frac{n \phi_0}{2} \right)}{\sin \left( \frac{\phi_0}{2} \right)} \right]
\]

You will note that the value of \( a \) depends on the value of \( \phi_0 \). When \( n \) is very large, \( \phi_0 \) becomes very small. Then using Eq. (2.37), we can write

\[
\phi = \left( \frac{n - 1}{2} \right) \phi_0 = \frac{n \phi_0}{2}
\]

so that

\[
\sin \frac{\phi_0}{2} \approx \frac{\phi_0}{2} = \frac{A}{n}
\]

Hence, for large \( n \), we have

\[
a = a_0 \frac{\sin \frac{\phi_0}{2}}{\phi} = \frac{n a_0 \sin \phi}{\phi}
\]

(2.39)
Oscillations

That is, the polygon becomes an arc of the circle with centre at \( O \) and length \( a_n \) = \( A \) with \( a \) as chord. The plot of \( A \) for different values of \( \phi \) is shown in Fig. 23.

The pattern is symmetric about \( \phi = 0 \) and is zero for \( \sin \phi = n \pi \) (\( n = 1, 2, \ldots \)) except at \( \phi = 0 \). When \( \phi = 0 \), \( \phi = 0 \) and the resultant of \( n \) oscillations (vectors) is a straight line of length \( A \). As \( \phi \) increases, \( A \) becomes the arc of the circle until \( \phi = \pi / 2 \) the last and first contributions are out of phase and the arc \( A \) becomes a semi-circle whose diameter is the resultant \( a \). A further increase in \( \phi \) curls the length \( A \) into the circumference of a circle (\( \phi = \pi \)) with a zero resultant and so on.

SAQ 5

Three collinear harmonic oscillations, represented by \( x_1 = 4 \cos 20\pi t \), \( x_2 = 4 \cos (20\pi t + \pi / 3) \), \( x_3 = 4 \cos (20\pi t + 2\pi / 3) \) are superposed. Determine the amplitude and, phase of the resultant vibration.

2.5.2 Method of Complex Numbers

In the preceding section we used a geometrical method of vector addition to calculate the resultant of \( n \) superposed harmonic oscillations. The same result can be obtained in a very convenient and compact form using the method of complex numbers. In fact, as you proceed you will observe that the use of complex numbers simplifies mathematical steps very much. We know that in complex number notation, a vector can be represented as \( z = a \exp (i \phi) \). The complex exponential \( \exp (i\theta) \) is given by

\[ \exp (i \theta) = \cos \theta + i \sin \theta \]

with \( \cos \theta = \text{Re} \{ \exp (i \theta) \} \) and \( \sin \theta = \text{Im} \{ \exp (i \theta) \} \).

Let us now see how this technique of complex numbers is used to obtain the resultant of \( n \) harmonic oscillations given by Eq. (2.30). In the complex exponential notation, we can write

\[ Z_1 = a_0 \exp (i\omega t + \phi_0) \]
\[ Z_2 = a_0 \exp [i(\omega t + \phi_0 + \phi)] \]
\[ Z_3 = a_0 \exp [i(\omega t + 2\phi_0)] \]

This series is in geometric progression with common ratio \( e^{i\phi} \). Its sum is given by

\[ Z = a_0 e^{i\omega t} \frac{1 - e^{i\phi}}{1 - e^{i\phi}} \]

Using the relation

\[ \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \]

we get

\[ Z = a_0 \exp (i\omega t) \frac{\sin \left( \frac{n \phi_0}{2} \right)}{\sin \left( \frac{\phi_0}{2} \right)} \exp \left( \frac{\omega \phi_0}{2} \right) \exp \left( -\frac{i\omega t}{2} \right) \exp \left( -\frac{i\phi_0}{2} \right) \exp \left( \frac{\omega (n-1)\phi_0}{2} \right) \]

Since \( Z = a \exp [i(\omega t + \phi)] \) we find that the amplitude and phase of the resultant vibration are the same as given by Eqs. (2.34) and (2.37), respectively.

The cosine form of the resultant oscillation is obtained by taking the real part of Eq. (2.42).
So far we have confined our discussion to harmonic oscillations in one dimension. But oscillatory motion in two dimensions is also possible. Most familiar example is the motion of a simple pendulum whose bob is free to swing in any direction in the \( x - y \) plane. (We call this arrangement a spherical pendulum.) We displace the pendulum in the \( x \)-direction and as we release it, we give it an impulse in the \( y \)-direction. What happens when such a pendulum oscillates? The result is a composite motion whose maximum \( x \)-displacement occurs when \( y \)-displacement is zero and \( y \)-velocity is maximum and vice versa. Remember that since the time period of the pendulum depends only on acceleration due to gravity and the length of the cord, the frequency of the superposed SHM's will be the same. The result is a curved path, in general, an ellipse.

We now apply the principle of superposition to the case where two harmonic oscillations are perpendicular to each other.

### 2.6.1 Superposition of Two Mutually Perpendicular Harmonic Oscillations of the Same Frequency

Consider two mutually perpendicular oscillations having amplitudes \( a_1 \) and \( a_2 \) such that \( a_1 > a_2 \) and angular frequency \( \omega_0 \). These are described by equations

\[
\begin{align*}
X &= a_1 \cos \omega_0 t \\
y &= a_2 \cos (\omega_0 t + \phi)
\end{align*}
\]

Here we have taken the initial phase of the vibrations along the \( x \) and the \( y \)-axes to be zero and \( \phi \) respectively. That is, \( \phi \) is the phase difference between the two vibrations.

We shall first find out the resultant oscillation for a few particular values of phase difference \( \phi \).

**Case I.** \( \phi = 0 \) or \( \pi \).

For \( \phi = 0 \)

\[
\begin{align*}
X &= a_1 \cos \omega_0 t \\
y &= a_2 \cos \omega_0 t
\end{align*}
\]

Therefore

\[
y/x = a_2/a_1
\]

or

\[
y = (a_2/a_1) X
\]

Similarly, for \( \phi = \pi \)

\[
X = a_2 \cos \omega_0 t
\]

and

\[
y = -a_2 \cos \omega_0 t
\]

so that

\[
y = -(a_2/a_1) x
\]

Eqs. (2.45) and (2.46) describe straight lines passing through the origin. This means that the resultant motion of the particle is along a straight line. However, for \( \phi = 0 \) the motion is along one diagonal (PR in Fig. 2.6a) but when \( \phi = \pi \) the motion is along the other diagonal (QS in Fig. 2.6b).

**Case II.** \( \phi = \pi/2 \)

In this case the two vibrations are given by

\[
\begin{align*}
X &= a_1 \cos \omega_0 t \\
y &= a_2 \cos (\omega_0 t + \pi/2) = -a_2 \sin \omega_0 t
\end{align*}
\]

On squaring these expressions and adding the resultant expressions, we get

\[
\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = \cos^2 \theta + \sin^2 \theta = 1
\]

This is the equation of an ellipse. Thus the resultant motion of the particle is along an ellipse whose principal axes are along the \( x \)- and the \( y \)-axes. The semi-major and semi-minor axes of the ellipse are \( a_1 \) and \( a_2 \). Note that as time increases \( x \) decreases from its
maximum positive value but $y$ becomes more and more negative. Thus the ellipse is described in the clockwise direction as shown in Fig. 2.6c. If you analyse the case when $\phi = 3\pi/2$ or $\phi = -\pi/2$, you will obtain the same ellipse. But the motion will be in anticlockwise direction (Fig. 2.6d).

When amplitudes $a_1$ and $a_2$ are equal, i.e., $a_1 = a_2 = a$, Eq. (2.47) reduces to

$$x^2 + y^2 = a^2$$

This equation represents a circle of radius $a$. This means that the ellipse reduces to a circle.

General case

We will now consider the general case for any arbitrary value of $\phi$. Let the two SHMs given by Eqs. (2.43) and (2.44) be superposed. To find the resultant oscillation, we write Eq. (2.44) as

$$\frac{Y}{a_2} = \cos (\omega t + \phi) = \cos \omega t \cos \phi - \sin \omega t \sin \phi$$

(2.48)

From Eq. (2.43)

$$\cos \omega t = \frac{x}{a_1}$$

so that $\sin \omega t = \sqrt{1 - (x^2/a_1^2)}$

Substituting these values of $\cos \omega t$ and $\sin \omega t$ in Eq. (2.48), we have

$$\frac{Y}{a_2} = \frac{x}{a_1} \cos \phi - \frac{y}{a_2} \sqrt{1 - (x^2/a_1^2)} \sin \phi$$

or

$$\frac{x}{a_1} \cos \phi - \frac{y}{a_2} \sqrt{1 - (x^2/a_1^2)} \sin \phi$$

Squaring both sides and rearranging terms, we get

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - 2 \frac{xy}{a_1a_2} \cos \phi = \sin^2 \phi$$

(2.49)

as the equation of the resultant path. This describes an ellipse whose axes are inclined to the coordinate axes.

For some typical values of $\phi$ lying between 0 and 2 $\pi$, the resultant paths traced out by the resultant oscillation when two mutually perpendicular SHMs of equal frequency are superposed are shown in Fig. 2.7. These can be most easily demonstrated on a cathode ray oscilloscope.
We may thus conclude that an elliptical motion is a combination of two mutually perpendicular linear harmonic oscillations of unequal amplitudes and having a difference of phase. A circular motion is a combination of two harmonic oscillations of equal amplitudes.

SAQ 6
In a cathode ray oscilloscope, the deflection of electrons by two mutually perpendicular electric fields is given by

\[ x = 4 \cos 2\pi \frac{y}{t} \]

and

\[ y = 4 \cos (2\pi y + \pi/6) \]

What will be the resultant path of electrons?

2.6.2 Superposition of Two Rectangular Harmonic Oscillations of Nearly Equal Frequencies: Lissajous Figures

We now know that when two orthonormal vibrations have exactly the same frequency, the shape of the curve traced out by the resultant oscillation depends on the phase difference between component vibrations. For very small values of the phase difference \( \phi \) in the range 0 to \( 2\pi \) radian, these curves are shown in Fig. 2.6. When the two individual rectangular vibrations are of slightly different frequencies, the resulting motion is more complex. This is because the relative phase \( \phi = \omega_0 t + \phi_0 - \omega_1 t = (\omega_2 - \omega_0) t + \phi_0 \) of the two vibrations gradually changes with time. This makes the shape of the figure to undergo a slow change. If the amplitudes of vibrations are \( a_1 \) and \( a_2 \), respectively, then the resulting figure always lies in a rectangle of sides \( a_1 \) and \( 2a_0 \). The patterns which are traced out are called Lissajous figures. When the two vibrations are in the same phase, i.e., \( \phi = 0 \), the Lissajous figure reduces to a straight line and coincides with the diagonal \( y = (a_0/a_1)x \) of the rectangle. As \( 4 \) changes from \( 0 \) to \( 2\pi \), the Lissajous figure is an ellipse and passes through oblique positions in the rectangle. When \( 4 \) increases from \( \pi/2 \) to \( \pi \), the ellipse closes into a straight line which coincides with the other diagonal \( y = (a_0/a_1)x \) of the rectangle. Further, as \( \phi \) changes from \( \pi \) to \( 2\pi \), the series of changes mentioned above take place in the reverse order. In general, the shape of curve depends on the amplitudes, frequencies and the phase difference. All these changes are shown in Fig. 2.7. The phase \( \phi \) changes by \( 2\pi \) in the time interval \( 2\pi/(\omega_2 - \omega_0) \). Therefore, the period of the complete cycle of changes is \( 2\pi/(\omega_2 - \omega_0) \) and its frequency is \( \omega_2 - \omega_1 \), i.e., equal to the difference of the frequencies of individual vibrations.

Lissajous figures can be illustrated easily by means of a cathode ray oscilloscope (CRO). Different alternating sinusoidal voltages are applied at XX and YY deflection plates of the CRO. The electron beam traces the resultant effect on the fluorescent screen. When the applied voltages have the same frequency, we can obtain various curves of Fig. 2.7 by adjusting the phases and amplitudes.

If the frequencies of individual perpendicular vibrations are in the ratio 2:1, the Lissajous figures are relatively complex. It has the shape of parabola for \( \phi = 0 \) or \( \pi \) and for \( \phi = \pi/2 \) its shape is that of figure 8. To clarify this let us study the following example:

Two rectangular harmonic vibrations having frequencies in the ratio 2:1 are represented as follows:

\[ x = a_1 \cos (2\omega_0 t + \phi) \]

\[ y = a_2 \cos \omega_0 t \]

We will calculate the resultant motion for \( \phi = 0, \pi/2 \), and \( \pi \).

(i) When \( \phi = 0 \),

\[ x = a_1 \cos 2\omega_0 t = a_1 (2 \cos^2 \omega_0 t - 1) \]

\[ y = a_2 \cos \omega_0 t \]

We see that the motion along the y-axis is harmonic and the motion along the x-axis is a combination of two harmonic oscillations of unequal amplitudes. The difference of their amplitudes is equal to the difference of the amplitudes of the two individual vibrations. When \( \phi = 2\pi \), the resultant oscillation depends on the amplitudes, frequencies and the phase difference. All these changes are shown in Fig. 2.7. The phase \( \phi \) changes by \( 2\pi \) in the time interval \( 2\pi/(\omega_2 - \omega_0) \). Therefore, the period of the complete cycle of changes is \( 2\pi/(\omega_2 - \omega_0) \) and its frequency is \( \omega_2 - \omega_1 \), i.e., equal to the difference of the frequencies of individual vibrations.
Since \( \frac{y}{a^2} = \cos \omega t \), we can rewrite the above equation as

\[
\frac{x}{a_1} = \frac{2y}{a_2} - 1
\]

On rearranging terms, we get

\[
y' = -\frac{a_1}{2a_2} (x + a_1)
\]

This equation represents a parabola (Fig. 2.8 a)

(ii) When \( \phi = \frac{\pi}{2} \)

\[
x = -\frac{x}{a_1} \sin 2\omega t
\]

or \( \frac{x}{a_1} = 2 \sin \omega t \cos \omega t \)

and \( y = a_1 \cos \omega t \)

Since we can write

\[
\cos \omega t = \frac{y}{a_2}
\]

and \( \sin \omega t = \sqrt{1 - \frac{y^2}{a_2^2}} \)

the first of these equations reduces to

\[
-\frac{x}{a_1} = \frac{2y}{a_2} \sqrt{1 - \frac{y^2}{a_2^2}}
\]

On squaring and rearranging terms, we get

\[
\frac{dy}{a_1^2} \left( \frac{x^2}{a_1^2} - 1 \right) + \frac{x^2}{a_1^2} = 0
\]

which represents figure '8' in shape (Fig. 2.8b)

(iii) When \( \phi = \pi \)

\[
x = -a_1 \cos 2\omega t
\]

or \( \frac{x}{a_1} = 2 \cos^2 \omega t - 1 \)

and \( y = a_2 \cos \omega t \)

On combining these equations, we get

\[
\phi = 0 \quad \pi / 2 \quad \pi
\]

(i) (ii) (iii)

Fig. 2.21 Superposition of two harmonic oscillations having frequencies in the ratio 2:1 and phase difference (i) \( \phi = 0 \), (ii) \( \pi / 2 \), and (iii) \( \pi \) respectively.
This represents a parabola which is oppositely directed to the case when $\phi = 0$.

(Fig. 2.8~)

### 2.7 SUMMARY

The principle of superposition states that if we superpose the initial amplitudes and velocities corresponding to two (or more) harmonic oscillations, then the resultant displacement is the algebraic sum of individual displacements at all subsequent times:

$$ x(t) = x_1(t) + x_2(t) $$

- When two collinear harmonic oscillations of the same frequency, given by
  
  \begin{align*}
  x_1 &= a_1 \cos(\omega t + \phi_1) \\
  x_2 &= a_2 \cos(\omega t + \phi_2)
  \end{align*}

  are superposed, the resultant is given by

  \[ x = a \cos(\omega t + \phi) \]

  where

  \[ a = [a_1 + a_2 + 2a_1a_2 \cos(\phi_2 - \phi_1)]^{1/2} \]

  and

  \[ \phi = \tan^{-1} \left( \frac{a_2 \sin \phi_2 + a_1 \sin \phi_1}{a_1 \cos \phi_1 - a_2 \cos \phi_2} \right) \]

- When two collinear harmonic oscillations of different frequencies are superposed, the modulated oscillation is represented as

  \[ x = a_{\text{mod}} \cos(\omega t + \phi_{\text{mod}}) \]

  where

  \[ a_{\text{mod}} = \frac{a_1 + a_2}{2} \]

  \[ \phi_{\text{mod}} = \frac{\phi_1 + \phi_2}{2} \]

- Superposition of $n$ harmonic collinear oscillations of the same amplitude ($a_0$) and frequency ($\omega_0$) but having a constant phase difference ($\phi_0$) between successive oscillations yields a harmonic oscillation. It is given by

  \[ x(t) = a \cos(\omega_0 t + \phi) \]

  where

  \[ a = \frac{a_0 \sin \left( \frac{n\phi_0}{2} \right)}{\sin \left( \frac{\phi_0}{2} \right)} \]

  and

  \[ \phi = \left( n - 1 \right) \frac{\phi_0}{2} \]

- When two mutually perpendicular harmonic oscillations are superposed, the resultant form traces out different curves. If the oscillations have equal frequencies, the shape of the curve depends on the phase difference. In general, the curve is elliptical but for certain phases, it closes into a straight line. When the frequencies are nearly equal, the curves are termed Lissajous figures.
2.8 TERMINAL QUESTIONS

1. The motion of a simple pendulum is described by the differential equation

\[ \frac{d^2x}{dt^2} + 4x = 0 \]

Solve it for the following initial conditions: (1) at \( t = 0 \), \( x = 3 \) cm and \( \frac{dx}{dt} = 0 \);
(2) at \( t = 0 \), \( x = 2 \) cm and \( \frac{dx}{dt} = 4 \) cm s\(^{-1}\). Denote these two solutions by \( x_1 \) and \( x_2 \).

Show that for a new displacement \( x_3 = x_1 + x_2 \), the initial conditions of the bob are the superposition of the initial conditions of \( x_1 \) and \( x_2 \).

2. Two simple harmonic vibrations are represented by

\[ x_1 = 3 \sin (20\pi t + \pi/6) \]
\[ x_2 = 4 \sin (20\pi t + \pi/3). \]

Find the amplitude, phase constant and the period of resultant vibration.

3. Consider the following two simple harmonic oscillations

\[ x_1 = a_1 \cos \omega_1 t \]
\[ x_2 = a_2 \cos \omega_2 t \]

Use complex number analysis to obtain the following expressions of the amplitude for the resultant motion:

\[ a = \left[ |a_1|^2 + |a_2|^2 + 2|a_1|a_2 \cos (\omega_1 - \omega_2)t \right]^{1/2} \]

Show that the resultant amplitude oscillates between the values \( a_1 + a_2 \) and \( a_1 - a_2 \).

4. Two tuning forks A and B of frequencies close to each other are used to obtain Lissajous figures and it is observed that the figure goes through a cycle of changes in 20 s. Now if A is loaded slightly with wax, the figure goes through a cycle of changes in 10 s. If the frequency of B is 300 Hz, what is the frequency of A before and after loading.

2.9 SOLUTIONS

SAQ
1. On using the given expansion, we get

\[ \frac{d^2\theta}{dt^2} + \omega_0^2 \left[ \theta - \frac{\theta_0}{3} + \frac{\theta_0}{5} - \cdots \right] = 0 \]

Since this equation contains terms of power higher than \( \theta_i \), it is not linear.

Even if we retain the first two terms in the expansion, the resulting equation will not be linear and hence the principle of superposition will not hold.

2. \( x_1 = a_1 \cos \omega_1 t \)
\( x_2 = a_2 \cos \omega_2 t \)

According to the principle of superposition

\[ x = x_1 + x_2 = (a_1 + a_2) \cos \omega_1 t \]

Since the cosine function varies between +1 and −1, the amplitude of the resultant oscillation is \( |a_1 + a_2| \).

3. The resultant of two harmonic oscillations having amplitudes \( a_1 \) and \( a_2 \) and initial phases \( \phi_1 \) and \( \phi_2 \) is given by

\[ a = \sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos (\phi_1 - \phi_2)} \quad (i) \]

(a) When \( \phi_1 - \phi_2 = 2n\pi \), \( \cos (\phi_1 - \phi_2) = 1 \) and Eq. (i) reduces to

\[ a = \sqrt{a_1^2 + a_2^2} = (a_1 + a_2) \]

so that \( a = (a_1 + a_2) \). The negative sign is dropped as it will be physically absurd.
(b) When $\phi_1 - \phi_2 = (2n + 1)\pi$, $\cos (\phi_1 - \phi_2) = -1$. Then Eq. (i) reduces to

$$a^2 = a_1^2 + a_0^2 - 2a_0a_1 \cos (\phi_1 - \phi_2)$$

so that

$$a = (a_1 - a_0)$$

as before, the negative sign is dropped.

4. From Eq. (2.15), we get for $a_1 = a_2$:

$$a = \sqrt{2a_0 (1 + \cos (\phi_1 - \phi_2))}$$

Since $\phi_1 - \phi_2 = \pi/2$, this expression reduces to $a = \sqrt{2a_0} = \sqrt{2}$ cm since $a_0 = 1$ cm

Similarly, from Eq. (2.16), we get

$$\tan \phi = 1 \text{ or } \phi = \pi/4.$$

5. Here $n = 3$, $a_0 = 4$ units and $\phi_0 = \pi/3$ rad. From Eq. (2.34) we note that the amplitude of the resultant oscillation is given by

$$a = a_0 \left( \frac{\sin \left( \frac{\pi}{2} \right)}{\sin \left( \frac{3 \times \pi}{2 \times 3} \right)} \right)$$

$$= a_0 \left( \frac{\sin \left( \frac{\pi}{2} \right)}{\sin \left( \frac{\pi}{6} \right)} \right)$$

Since $\sin \frac{\pi}{2} = 1$, and $\sin \frac{\pi}{6} = \frac{1}{2}$, we get

$$a = 2a_0 = 8 \text{ units}$$

The phase of the resultant oscillation is given by Eq. (2.37):

$$\phi = (n - 1) \frac{\phi_0}{2}$$

$$= 2 \times \frac{\pi}{6}$$

$$= \frac{\pi}{3} \text{ rad.}$$

6. Using Eq. (2.49), we have

$$x^2 + y^2 - \frac{2xy}{4} \cos \frac{\pi}{6} = \sin^2 \frac{\pi}{6}$$

or

$$x^2 + y^2 - \frac{2xy}{4} \sqrt{3} = \frac{1}{4}$$

or

$$x^2 + y^2 - \sqrt{3} xy - 4 = 0$$

The resultant path is an ellipse.

**Terminal Questions.**

1. $\frac{d^2 x}{dt^2} + 4x = 0$

Comparing it with the standard equation for SHM $\frac{d^2 x}{dt^2} + \omega^2 x = 0$, (i)
we find that the solution of this equation is
\[ x = a \cos(2t + \phi) \]  

Differentiating Eq. (ii) with respect to \( t \), we get
\[ \frac{dx}{dt} = -2a \sin(2t + \phi) \]  

(1) Since \( x(0) = 0, x = 3 \text{ cm} \) and \( \frac{dx}{dt} = 0 \), from Eqs. (ii) and (iii), we obtain
\[ 3 = a \cos \phi \]  
and
\[ 0 = -2a \sin \phi \]

The latter of these two relations implies that \( \phi = 0 \). Using this in the former, we get
\[ a = 3 \text{ cm} \]

Therefore, the complete solution is
\[ x(t) = 3 \cos(2t + \phi) \]  

(iv)  

(2) Again if at \( t = 0, x = 2 \text{ cm} \) and \( \frac{dx}{dt} = -4 \text{ cm s}^{-1} \), we find
\[ 2 = a \cos \phi \]  
and
\[ -4 = -2a \sin \phi \]

or
\[ 2 = a \sin \phi \]

On dividing one by the other, we get
\[ \tan \phi = -1 \text{ or } \phi = -\frac{\pi}{4} \]. Hence \( a = 2\sqrt{2} \text{ cm} \).

Therefore, the solution in the second case is
\[ x_2 = 2\sqrt{2} \cos \left(2t - \frac{\pi}{4}\right) \text{ cm} \]  

(v)

Since superposition of \( x_1 \) and \( x_2 \) yields \( x_3 \), from Eqs. (iv) and (v) we get
\[ x_3 = x_1 + x_2 = 3 \cos 2t \text{ cm} + 2\sqrt{2} \left[ \cos \left(2t - \frac{\pi}{4}\right) \text{ cm} \right] \]
\[ = 3 \cos 2t \text{ cm} + 2\sqrt{2} \left( \frac{1}{\sqrt{2}} \cos 2t + \frac{1}{\sqrt{2}} \sin 2t \right) \text{ cm} \]
\[ = 5 \cos 2t \text{ cm} + 2 \sin 2t \text{ cm} \]  

(vi)

Now if we superpose the initial conditions of \( x_1 \) and \( x_2 \), we have
at \( t = 0, x = 5 \text{ cm} \) and \( \frac{dx}{dt} = 4 \text{ cm s}^{-1} \),
\[ 5 = a \cos \phi \]  
and
\[ 4 = -2a \sin \phi \]

Hence \( \tan \phi = -\frac{2}{5} \text{ or } \phi = -\frac{2}{\sqrt{29}} \text{ rad} \),
\[ \cos \phi = \frac{5}{\sqrt{29}} \]  
and
\[ a = \sqrt{29} \text{ cm} \].

Therefore the solution obtained on superposing initial conditions is
\[ x_3 = \sqrt{29} \cos (2t + \phi) \text{ cm} = \sqrt{29} \left[ \cos 2t \cos \phi - \sin 2t \sin \phi \right] \text{ cm} \]

On substituting for \( \cos \phi \) and \( \sin \phi \), we get
\[ x_3 = 5 \cos 2t \text{ cm} + 2 \sin 2t \text{ cm} \]  

(vii)

This is the same as given by Eq. (vi) and obtained by the superposition of \( x_1 \) and \( x_2 \).
2. \( x_1 = 3 \cos(20\pi t + \pi/6 - \pi/2) \text{cm} \)
and \( x_2 = 4 \cos(20\pi t + \pi/3 - \pi/2) \text{cm} \)
Or \( x_1 = 3 \cos(20\pi t - \pi/3) \text{cm} \)
and \( x_2 = 4 \cos(20\pi t - \pi/6) \text{cm} \)

Hence, resulting vibration is defined by \( x = a \cos(20\pi t + \phi) \text{cm} \)
where \( a = (3^2 + 4^2 + 2 \times 3 \times 4 \times \cos \pi/6)^{1/2} \text{cm} \)
\( = (9 + 16 + 12\sqrt{3}/3)^{1/2} \text{cm} \)
\( = 6.77 \text{cm} \)
and \( \phi = \tan^{-1} \left( \frac{3 \sin \pi/3 + 4 \sin \pi/6}{3 \cos \pi/3 + 4 \cos \pi/6} \right) \)
\( = -0.24\pi \)

3. \( Z = a_1 \exp(i\omega_1 t) + a_2 \exp(i\omega_2 t) \)
\( a^2 = (a_1 e^{-i\omega t} + a_2 e^{-i\omega t})^2 \times (a_1 e^{i\omega t} + a_2 e^{i\omega t}) \)
\( = a_1^2 + a_2^2 + 2a_1a_2 \cos(\omega_1 - \omega_2)t \) \[ + a_1a_2 \exp \left( -i(\omega_1 + \omega_2)t \right) \]

On taking the real part, we get \( a = [a_1^2 + a_2^2 + 2a_1a_2 \cos(\omega_1 - \omega_2)t]/2 \)
When \( (\omega_1 - \omega_2)t = \pi \text{ or } (2\pi + 1) \pi, a_{\text{real}} = a_1 - a_2 \)
When \( (\omega_1 - \omega_2)t \neq \pi \text{ or } (2\pi + 1) \pi, a_{\text{real}} = a_1 + a_2 \)

Hence the resultant amplitude oscillates between the values \( a_1 + a_2 \) and \( a_1 - a_2 \).

4. \( V_A - V_C = \pm 0.05 \text{ Hz} \)
Now on loading the prong of the tuning fork A with wax, the frequency of A will decrease. However now the cycle of changes of figures is completed in 10 s and hence the frequency difference increases to 0.1 Hz. This means that the frequencies of A before and after loading are respectively \((300 - 0.05) \text{ Hz} = 299.95 \text{ Hz} \) and \((300 - 0.1) \text{ Hz} = 299.9 \text{ Hz} \).