UNIT 12 THE ASSIGNMENT PROBLEM

Structure

2.1 INTRODUCTION

In the first three units of this black, you have been introduced to the transportation problem and a computational method for solving it. In this unit we discuss the Assignment Problem, which is a special case of the transportation problem. You must have noticed that a transportation problem is a real life problem. In this unit, you will observe the same for an assignment problem. For example a factory manager may wish to assign three different jobs to three machines in such a way that the total cost is minimized. There are 3! = 6 ways of assigning 3 jobes to 3 machines. However, a problem of assigning 10 jobs to 10 machines requires 10! assignments to be examined, which clearly is not a simple task. Hence, the need to evolve efficient method to solve an assignment problem. In this case also, the simplex method is not very helpful although it can be used.

In 1931, the Hungarian mathematician D. Konig, published a theorem on graphs, whiehwas generalised in the same year by another Hungarian mathematician E. Eger-vary. This theorem was applied by I-I.W. Kuhn to solve the assignment problem. The method so evolved was suitably named the 'Hungarian-Method'. m this unit. we shall discuss the 'I-Iungarian Method' for solving an assignment problem.

Objectives

After completing this unit, you should be able to:

- identify an assignment problem,
- formulate an assignment problem,
- determine that an assignment problem is a special case of a transportation problem and so of a linear programming problem.
- solve an Assignment Problem by the Hungarian Method.

12.2 FORMULATION OF AN ASSIGNMENT PROBLEM

Let us consider the case of a factory which has three jobs to be done on the three available machines. Each machine is capable of doing any of the three jobs. For each job the machining-cost depends on the machine to which it is assigned. Costs incurred by doing various jobs on different machines are given below:

	Machine I	Machine II	Machine III
Job I	3	4	2
Job II	1	3	7
Job III	2	5	4

The problem of assigning jobs to machines, one to each, so as to minimize the total cost of doing all the jobs, is an assignment problem.

Each job machine combination which associates all jobs to machines on one-to-one basis is called an assignment. Every assignment corresponds to a 'total-cost'. The assignment problem, proposes to determine that assignment which corresponds to minimum total-cost. This will be called an optimal assignment.

In the example above, let us write all the possible assignments. Job I can be assigned to any of the three available machines. So there are three ways in which it can be done. Now, Job II cannot be assigned to the machine to which Job I has already been assigned. This is because no two jobs can be assigned to the same machine and conversely no two machines can be associated with the same job. So, job II can be associated to any of the two remaining machines, which can be done in two ways Ultimately job III has only one machine left for it and so has only one way. Combining the above, there are 3 x 2 × 1, ways in which all the jobs can be assigned to various machines, one to each. In other words, there are 3! number of possible assignments. The following table enumerates all these possible assignments and also mentions the 'total-cost' corresponding to each one of these.

Number	Assignment	Total-Cost
1.	Job I — Machine I Job II — Machine II Job III — Machine III	3 + 3 + 4 = 10
2.	Job I — Machine I Job II — Machine III Job III — Machine II	3 + 7 + 5 = 15
3.	Job I — Machine II Job II — Machine III Job III — Machine I	4 + 7 + 2 = 13
4.	Job I — Machine II Job II — Machine I Job III — Machine III	4 + 1 + 4 = 9
5.	Job I — Machine III Job II — Machine I Job III — Machine 11	2 + 1 + 5 = 8
6.	Job I — Machine III Job II — Machine II Job III — Machine I	2 + 3 + 2 = 7

Out of all these, the minimum total-cost i.e. 7 corresponds to the assignment

Job I — Machine III
Job II — Machine II
Job III— Machine I

which is called the **optimal assignment.**

Let us now generalise it and formulate it for n jobs. Let there be n jobs which are to be processed on n machines on one-job one-machine basis. Let the cost incurred by processing each jobs on each machine be known. Then the problem of processing all the jobs at minimum cost is known-as the assignment problem. We will develop a mathematical formulation for the assignment problem.

Let J,, J, ..., J_n be the n jobs and let M,, M,, ..., M_n be the n machines. Also let c_{ij} be the cost of processing ith job J_i on jthe machine M_j. One job-one machine basis implies that not more than one job goes to the same machine and conversely that not more than one machine process the same job.

Let us define variable x_{ij} as follows,

$$x_{ij} = \begin{cases} 0 \text{ if ith job is not assigned to jth machine} \\ 1 \text{ if ith job is assigned to jth machine} \end{cases}$$

No job remains unprocessed and no machine remains idle. Note that the number of jobs is equal to tlie number of machines. The liypothesis of one job-one machine implies

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad (j = 1, 2, \ldots, n)$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad (i = 1, 2, \ldots, n)$$

In each of these summations, only one term on the left hand side has variable x_{ij} equal to one and the rest are zeros. Also, this particular term (for which $x_{ij} = 1$) pertains to that job and that machine which have been assigned together.

Then, the assignment problem is mathematically stated:

$$\left.\begin{array}{ll} \text{Minimise } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} & \\ \text{Subjectto } \sum_{i=1}^{n} x_{ij} = 1 & (j = 1, 2, \ldots, n) \\ \quad\quad\quad {}_{j=1} \;\; = 1 & (i = 1, 2, \ldots, n) \\ x_{ij} = 0 \text{ or } 1 \begin{pmatrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, n \end{pmatrix} & \end{array}\right] \text{(AP-1)}$$

It may be noted that assigning a non-negative integer value to x_{ij} is equivalent to assigning values 0 or 1 to x_{ij}. For this one has only to see the constraints of the problem (AP-1). This also justifies the way x_{ij}'s have been defined.

An assignment problem is known from its cost-matrix $[C_{ij}]$, which is given is:

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{ij} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2j} & \cdots & C_{2n} \\ \vdots & & & & & \\ C_{i1} & C_{i2} & \cdots & C_{ij} & \cdots & C_{in} \\ \vdots & & & & & \\ C_{n1} & C_{n2} & \cdots & C_{nj} & \cdots & C_{nn} \end{bmatrix}$$

If each row refers to a job and each column refers to a machine, then C_{ij} is the cost of processing ith job on jth machine. Clearly $[C_{ij}]$ is **a** square **matrix** of order **n.**

Assignment Problem as a special case of Transportation Problem.

Let us consider an m x n transportation problem:

$$\text{Minimise } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{S. t} \quad \sum_{i=1}^{n} x_{ij} = b_j \qquad (j = 1, 2, \ldots, n) \qquad \textbf{TP-1}$$

$$\sum_{j=1}^{n} x_{ij} = a, \qquad (i = 1, 2, \ldots, n)$$

$$x_{ij} \geq 0,$$

.where

a_i = availability at ith source

b_j = demand of jth destination

C_{ij} = per unit transportation cost from ith source to jth destination

and x_{ij} = number of units transported from ith source to jth destination.

we state a well known result here that wherever a, and b_j in TP-I are integers, then every basic feasible solution of TP-I has integret values.

Assignment problem can be viewed as a special'case of transportation problem. What we have to do is only to regard 'jobs' as 'sources' and 'machines' as 'destinations'. As each $a_i = 1$ and each $b_j = 1$, therefore

$\sum_{i=1}^{m} a_i = m$ and $\sum_{j=1}^{n} b_j = n$. In order to solve a transportation problem we 'balance' it, in case it is not already so. For this, we require m=n i.e. the number of jobs are taken equal to the number of machines. This is what we do in an assignment problem i.e. we take the number of jobs equal to the number of machines. Thus, we see that an assignment problem is a special case of transportation problem.

12.3 SOLVING AN ASSIGNMENT PROBLEM

Being a special case of transportation problem an assignment problem is a special type of linear programming problem. As a result, you can use simplex method to solve an assignment problem. In view of the special structure of the assignment problem, a very convenient method has been evolved for its solution. It is called the Hungarian Method. Before we discuss this method, let us take up the following results, which lead to its evolution.

THEOREM I

The optimal solution of an assignment problem remains the tame, if a constant is added or subtracted from any sow or column of the cost matrix.

PROOF

Let $[C_{ij}]$ and $[C^*_{ij}]$ with $C^*, = C_{ij} \pm u_i \pm v_j$ represent respectively the original assignment problem, obtained after adding or subtracting constants u_i's and v_j's from its rows and columns. Here, u_i is the constant added or subtracted from all the elements of ith row of matrix $[C_{ij}]$; and v_j is the constant added or subtracted from all the elements of the jth column of the matrix $[C_{ij}]$. Let

$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}\, x_{ij}$ be the objective function of the original problem with cost matrix $[C_{ij}]$. The objective function Z* for the resulting assignment problem with cost matrix $[C^*_{ij}]$ is given by,

$$Z^* = \sum_{i=1}^{n} \sum_{j=1}^{n} C^*_{ij}\, x_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(c_{ij} \pm u_i \pm v_j\right) x_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}\, x_{ij} \pm \sum_{i=1}^{n} \left(u_i \sum_{j=1}^{n} x_{ij} \right) \pm \sum_{j=1}^{n} \left(v_j \sum_{i=1}^{n} x_{ij} \right)$$

$$= Z \pm \sum_{i=1}^{n} u_i \pm \sum_{j=1}^{n} v_j\,, \quad \because \sum_{i=1}^{n} x_{ij} = 1 = \sum_{j=1}^{n} x_{ij}$$

$$= Z \pm K \text{ where } \pm \sum_{i=1}^{n} u_i \pm \sum_{j=1}^{n} v_j = \pm K$$

This shows that the minimization of the original objective function Z yields the same solution as the minimization of Z*. Only the optimal values differ.

In the light of the above theorem, let us take the assignment problem.

	M_1	M_2
J_1	5	3
J_2	2	6

In order to ensure that no element of the cost-matrix becomes negative, subtract the minimum element of each row from all the elements of that row. We get,

	M_1	M_2
J_1	2	0
J_2	0	4

In the above reduced cost-matrix, the optimal assignment yielding total-cost zero is J_1 M„ J_2 M_1. So for the original problem the optimal assignment is $J_1 M_2$, $J_2 M_1$, yielding optimal value;3 + 2 = **5.**

Let us consider a second problem.

	M_1	M_2
J_1	5	3
J_2	6	2

As before, substracting the minimum element of each row from all elements of that row, the reduced matrix obtained is,

	M_1	M_2
J_1	2	0
J_2	4	0

Now, subtracting the minimum element of each column from all elements of that column, the cost-matrix is further reduced to

	M_1	M_2
J_1	0	0
J_2	2	0

Keeping in mind the one job-one machine basis, the optimal assignment yielding total cost zero is J, M,, J_2 M_2. From the original cost-matrix of this problem the optimal assignment J, M_1, J_2 M_2 corresponds to total cost 5 + 2 = 7.

This is as far as the application of the theorem, given above, is concerned. Sometimes, even after this, optimal assignments in the reduced cost-matrix, yielding zero total-cost can not be located. For example, consider the problem,

	M_1	M_2	M_3
J_1	2	4	2
J_2	5	2	3
J_3	4	2	5

After subtracting the minimum of each row (column) from all elements of that row (column) the reduced matrix so obtained is,

	M_1	M_2	M_3
J_1	0	2	0
J_2	3	0	1
J3	2	0	3

It can be seen that on one job-one machine basis, an optimal assignment yielding total-cost zero cannot be obtained from this reduced matrix. Such a situation can be systematically identified by observing that all the zeros in the above reduced matrix can be covered by a minimum of 2 lines only (shown as dotted lines below)

$$\begin{array}{ccc} 0 & 2 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 3 \end{array}$$

This stalemate can be resolved by creating new zeros from amongst the elements uncovered by these *two* dotted lines. For this minimum of the uncovered elements i.e. 1 is subtracted from all uncovered elements; added to the elements at intersection of the dotted lines, leaving other covered elements unchanged. The cost-matrix further reduces to,

	M_1	M_2	M_3
J_1	0	3	0
J_2	2	0	0
J_3	1	0	2

Optimal assignment yielding zero total-cost can now be made as J, M,, J_2 M_3, J, M_2 which corresponds to the total cost $2 + 3 + 2 = 7$.

It can now be observed that we can make optimal assignments yielding zero total cost in the reduced cost matrix, only when, the minimum number of dotted horizontal and vertical lines needed to cover all the zeros is equal to the order of the given assignment problem. In the first two problems (each of order 2), we could make optimal assignments only when the minimum number of lines required to cover all the zeros was two. In the third problem of order **3** the optimal assignment could be made only when the minimum number of lines required to cover all zeros in the reduced matrix is equal to three.

All the above observations contribute to the following steps of the Hungarian-Method for solving an n **xn** assignment problem.

12.3.1 Hungarian Method :

Step 1

i) Subtract the minimum element of each row from all elements of that row.

ii) Subtract the minimum element of each column from all elements of that column.

The reduced matrix thus obtained, contains atleast one zero in each row and each column.

Step 2

Cover all the zeros in the reduced cost-matrix by minimum number of horizontal and vertical lines. Let the least number of such lines needed to cover all the zeros be r. If r = n, an optimal assignment can be made at this stage in this case go to Step 4. If r < n, an optimal assignment can not be made at this stage. In this case go to Step 3.

Step 3

Mere, the least number of lines needed to cover all the zeros is less than the order of the assignment problem.

Pick the minimum element not covered by these r covering-lines and,

i) Subtract it from all uncovered elements,

ii) add to all elements at intersection of two covering lines, and

iii) leave all other covered elements unchanged.

Thus we get a new reduced cost-matrix. Go to step 2.

Step 4

Here the minimum number of lines needed to cover all the zeros is exactly equal to the order of the assignment.

An optimal assignment shall be made now.

i) Examine the rows successively until a row with exnctly one zero is found. Encircle this zero and cross all other zeros in its column.

ii) Similarly, examine the columns successively until a column with exactly one zero is found. Encircle this zero and cross all other zeros in its row.

Repeating the above steps either of the following situations is encountered:

a) each row and each column has an encircled zero. In this case an optimal assignment has been made and the process terminates.

b) There lie more than one zero in some rows and columns which are not encircled. In such a case encircle any one of the zeros which is not encircled arbitrarily and cross all other zeros in its row and column, both.

Continuing in this way, we shall have exnctly one encircled zero in each row and each column.

Assignments are made corresponding to each encircled zero.

Step 5

For obtaining the minimum cost, refer to the original cost-matrix of the given problem. Optimum cost is obtained by adding costs c_{ij}'s at all the encircled-zero positions.

In order to illustrate the above method, let us consider the following examples.

EXAMPLE 1

Solve the cost-minimizing assignment problem.

Medium →	I	II	III	IV
Rows ↓				
A	10	12	9	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

Step 1

i) Subtracting the minimum element of each row from all elements of that row, **we** get

	I	II	III	IV
A	1	3	0	2
B	0	5	2	3
C	1	3	2	0
D	0	7	3	1

ii) Subtracting the **minimum** elements of each column from elements all of that column, **we** get

	I	II	III	IV
A	1	0	0	2
B	0	2	2	3
C	1	0	2	0
D	0	4	3	1

Step 2

Cover all the zeros by minimum number of horizontal and vertical lines. A **systematic** approach for this is to look for a **row** or column containing the maximum number of zeros. See that we can cover all the zeros by 3 lines onlys. So, r = 3 < 4 = n, so go to step 3.

	I	II	III	IV
A	1	0	0	2
B	0	2	2	3
C	1	0	2	0
D	0	4	3	1

Step 3

1 is the least uncovered element.

i) **Subtract** 1 from all uncovered elements

ii) add 1 to elements at intersection of the covering lines viz. 1 at position (1, 1) and 1 at position (3, 1).

iii) leave other covered elements unchanged.

The reduced cost-matrix so obtained is,

	I	II	III	IV
A	2	0	0	2
B	0	1	1	2
C	2	0	2	0
D	0	3	2	0

Again, cover the zeros by minimum number of horizontal and vertical lines. See that we require exactly 4 lines to cover all the zeros. As $r = 4 = n$: optimal assignment can be made at this stage, so go to step 4.

	I	II	III	IV
A	2	0	0	2
B	0	1	1	2
C	2	0	2	0
D	0	3	2	0

Step 4

For making assignments, proceed as follows:

	I	II	111	IV
A	2	~~0~~	⓪	2
B	⓪	1	1	2
C	2	⓪	1	~~0~~
D	~~0~~	3	2	⓪

i) 2nd row has only one zero in position **(2, 1),** so encircle this zero **and** cross all other zeros in its column **i.e.** the 1st column.

ii) **Now,** 4th row has only one zero in position (4, 4), so encircle this zero and cross all other zeros in its column **i.e.** the 4th column.

iii) **3rd** column contains only one zero in position **(3, 1),** so encircle it **and** cross all other zeros in its row **i.e.** the 1st row.

iv) There is **only one** zero in 3rd **row,** so encircle it.

As can be seen, each row and each column has a single encircled zero. The optimal assignment is given by : **A-111, B-I, C-II, D-IV**

Step 5

The minimum assignment cost is read from the original cost-matrix as,

$$C_{13} + C_{21} + C_{32} + C_{44} = 9 + 5 + 14 + 9 = 37$$

EXERCISE **1:** Solve the cost-minimizing assignment problem with **the** cost-matrix.

		Machines				
		I	II	III	IV	V
	A	11	10	18	5	9
	B	14	13	12	19	6
Jobs	C	5	3	4	2	4
	D	15	18	17	9	12
	E	10	11	19	6	14

EXAMPLE 2

Solve the cost-minimizing assignment problem whose cost matrix is given below,

	M_1	M_2	M_3	M_4
J_1	2	5	7	9
J_2	4	9	10	1
J_3	7	3	5	8
94	8	2	4	9

SOLUTION : Step 1

i) Subtracting the minimum element of each row from all elements of that row, the reduced cost-matrix is,

	M_1	M_2	M_3	M_4
J_1	0	3	5	7
J_2	3	8	9	0
J_3	4	0	2	5
J_4	6	0	2	7

ii) subtracting the minimum element of each column from all elements of that column, we get

	M_1	M_2	M_3	M_4
J_1	0	3	3	7
J_2	3	8	7	0
J_3	4	0	0	5
J_4	6	0	0	7

Step 2

Cover all the zeros by least number of horizontal and vertical lines. Exactly 4 lines are required to cover all the zeros. So, r = 4.

	M_1	M_2	M_3	M_4
J_1	0	3	3	7
J_2	3	8	7	0
J_3	4	0	0	5
J_4	6	0	0	7

As r = 4 = n, we can straightway go to step 4, and **make** the optimal assignment.

Step 4

i) There is only one zero in 1st row in position (1, **1**), so encircle this zero, and cross other zeros (if any) in its column **i.e.** 1st column.

ii) There is only one zero in 2nd row in position (2, 4), ***so*** encircle this zero and cross other zeros (if any) in is column **i.e.** 4th column.

	M_1	M_2	M_3	M_4
J_1	⓪	3	3	7
J_2	3	8	7	⓪
J_3	4	⓪	$\cancel{0}$	5
J_4	6	$\cancel{0}$	⓪	7

iii) Now, observe that 3rd and 4th rows as well as 2nd and 3rd columns contain two zeros each. To break this, and make an assignment, we pick any zero arbitrarily. Say, **we** pick zero in position **(3, 2)** and encircle it. Now, cross all zeros in its row **i.e.** 3rd row as well as its column i.e. 2nd column.

iv) There is only one zero left in position (4, **3),** Encircle it to get the optimal assignment as J, M_1, J_2 M_4, J_3 M_2, J_4 M_3.

v) J_1 M_1, J_2 M_2, J_3 M_3, J_4 M_2 is an alternative optional assignment.

Step 5

For **determining** minimum total cost, refer to the original cost-matrix **of** this problem and add the costs corresponding to J, M_1, J_2 M_4, J_3 M_3, J_4 M_3. **This gives** the minimum assignment cost as.

$$2 + 1 + 3 + 4 = 10.$$

Note that any arbitrary choice in step 4 (iii), of the zero to be encircled, would yield the same minimum total-cost.

EXERCISE 2: Solve the cost-minimizing assignment problem.

	I	II	III	IV	V	VI
A	7	8	3	7	6	2
B	3	7	9	3	1	6
C	5	3	7	5	6	3
D	8	4	8	7	2	2
E	6	7	8	6	9	4
F	5	7	7	5	5	7

EXAMPLE 3

The owner of a small machine shop has 4 mechanists available to do 4 jobs. Jobs are offered with expected profits for each mechanist as follows:

Machanists

		I	II	III	IV
	A	6	7	5	2
	B	4	3	2	8
Jobs	C	2	4	9	4
	D	5	3	1	7

Find by using the assignment method, the assignment of mechanists to jobs that will result in a maximum profit.

SOLUTION

From linear programming we know that a maximization problem can be converted into a minimization problem by replacing the costs with their negatives. It is also known that an assignment problem'is a linear programming problem. So, we can convert the above maximizing assignment problem into the usual minimizing assignment problem, by replacing costs with their negatives and proceed with the Hungarian Method.
The corresponding minimizing assignment problem has thk cost-matrix given below:

	I	II	III	IV
A	–6	–7	–5	–2
B	–4	–3	–2	–8
C	–2	–4	–9	–4
D	–5	–3	–1	–7

Step I

i) Substract the minimum element –7 from all elements of 1st row. Similarly, subtract –8, –9'and –7 respectively from all elements of 2nd, 3rd and 4th rows. The reduced matrix is,

	I	II	III	IV
A	1	0	2	5
B	4	5	6	0
C	7	5	0	5
D	2	4	6	0

(Note that this step amounts to subtracting each element of the original matrix (of the profit maximizing assignment problem) from the corresponding maximum element of their rows respectively. In other words, substract all elements of 1st row from the maximum element i.e. 7 of the 1st row. Similarly, for the other rows).

ii) Subtract the minimum element of each column from all elements of that column. The reduced matrix so obtained is,

	I	II	III	IV
A	0	0	2	5
B	3	5	6	0
C	6	5	0	5
D	1	4	6	0

Step 2

Cover all the zeros by minimum number of horizontal and vertical lines. Observe that only 3 lines can cover all the zeros. So, $r = 3$. As $3 = r < n = 4$, so we go to step 3.

	I	II	III	IV
A	0	0	2	5
B	3	5	6	0
C	6	5	0	5
D	1	4	6	0

Step 3

The minimum uncovered element is 1, so

i) subtracting 1 from all uncovered elements

ii) adding 1 to elements at intersection of horizontal and vertical lines viz. elements at positions (1, 4) and (3, 4).

iii) leaving all other covered elements unchanged, we get,

	I	II	III	IV
A	0	0	2	6
B	2	4	5	0
C	6	5	0	6
D	0	3	5	0

Observe that now we require exactly 4 lines to cover all the zeros i.e. now r = n. So, we can go to step 4, and make optimal assignment.

Step 4

i) There is a single zero in 2nd row in the position (2, 4). Encircle this zero and cross all other zeros in its column i.e. 4th column.

ii) There is a single zero in 3rd row in the position (3, 3). Encircle this zero and cross all other zeros (if any) in its columns i.e. 3rd column.

iii) Now, there is only one unmarked zero in 4th row in the position (4, 1). Encircle this zero and cross all other zeros in its column i.e. 1st column.

	I	II	III	IV
A	$\cancel{0}$	⓪	2	6
B	2	4	5	⓪
C	6	5	⓪	6
D	⓪	3	5	$\cancel{0}$

iv) There is now a single zero in 1st row in the position (1, 2). Encircle it to get the optimal assignment as **AII,** B N , CIII, **DI.**

Step 5

Adding costs cox-responding to these assignments from the original profit maximizing matrix we get the maximum profit as,

$$7 + 8 + 9 + 5 = 29.$$

EXERCISE **3:** There are 5 jobs to be done on 5 available machines. The following matrix shows the return in rupees on assigning various jobs to different machines. Determine an assignment which maximizes the total return.

	M_1	M_2	M_3	M_4	M_5
J_1	5	11	10	12	4
J_2	2	4	6	3	5
J_3	3	12	5	14	6
J_4	6	14	4	11	7
J_5	7	9	8	12	5

12.4 SUMMARY

Assignment problem has been studied in rhis unit. Given an equal number of jobs and machines the problem consists of determining an optimal assignment. If the objective is to minimize the total cost incurred, the problem is known as cost minimizing assignment problem or simply assignment problem. In section 12.1, an assignment problem has been introduced systematically taking real life situations. Mathematical formulation of the problem has been developed in section 12.2, It has been shown that an assignment problem is represented completely by its cost-matrix.

An assignment problem is a special type of transpoi-tation problem and so of a linear programming problem. Consequently, it can be solved by the transportation technique or by the simplex method. Because of special structure of an assignment problem, a simple method called the Hungarian Method for solving it has been developed in Section 12.3.

Some examples illustrating the Hungarian Method have been discussed. A number of exercises have also been given which involve assignment problem in which the objective function is to be maximized.

12.5 ANSWERS/HINTS/SOLUTIONS

E1 Optimal assignment : AII, BV, CIII, DIV, EI

Minimum assignment cost = 39

E2 Optimal assignment : AIII, BV, CII DVI, EI, FIV or AIII, BIV, CII, DIV, EVI, FI

Minimum assignment cost = 20.

E3 Optimal assignment : $J_1 - M_3, J_2 - M_5, J_3 - M_4, J_4 - M_2, J_5 - M_1$

Maximum total return = 50.

REVIEW

This block has again four units namely units 9, 10, 11 and 12. In block I ,the Linear Programming was introduced and its solution only in two variables was discussed by the graphical method. In block 2, the algebraic method known as Simplex algorithm was used to solve the problem having any number of variables.

In this block, you have studied some typical special linear programming problems namely the transportation problem and the assignment problem. You must check for yourself to know whether you have successfully achieved the desired objectives in this block. For this, you will do well, if you try the following self-check problems and verify your answers given at the end of the block:

P1 Give the Mathematical Model of the TP represented in tabular form as given below:

	D_1	D_2	$a_i \downarrow$
S_1	4	3	50
S_1	7	2	30
S_1	5	1	40
$b_j \rightarrow$	40	80	

P2 Compute a basic feasible solution, using North West Corner Method, for the following TP

	D_1	D_2	D_3	D_4	D_5	$a_i \downarrow$
S_1	1	6	3	1	7	15
S_2	5	2	1	4	2	70
S_3	6	1	3	9	5	20
$b_j \rightarrow$	25	20	15	25	30	

B3 Use Matrix Minima Method to determine a basic feasible solution to the following TP.

	D_1	D_2	D_3	D_4	D_5	$a_i\downarrow$
S_1	40	50	30	10	70	20
S_2	100	80	75	65	20	20
S_3	30	60	45	50	40	50
$b_j\rightarrow$	15	25	20	10	20	

P4 **A** company has three plants at locations A, B and C; which supply to warehouses located at D, E, F, G and H. Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400,400, 200, 400 and 800 units respectively. Unit transportation costs (in rupees) are given below:

	D	E	F	G	H
A	5	8	6	6	3
B	4	7	7	6	6
C	8	4	6	6	3
	400	400	200	400	800

Determine an optimum distribution for the company in order to minimize the total transportation cost.

P5 **A** company has 4 warehouses and 5 stores. The surplus in the warehouses, the requirements of the stores and cost (in rupees) of transporting one unit of commodity from warehouse i to store j are given below:

		Store					
		1	2	3	4	5	Surplus
Warehouse	1	10	15	10	12	20	100
	2	5	10	8	15	10	150
	3	15	10	12	12	10	120
	4	20	15	15	10	12	90
Requirements		80	100	100	90	90	

P6 A team of 5 horses and 5 riders has entered a jumping-show contest. The number of penalty points to be expected when each rider rides any horse is given below:

		Riders				
		R_1	R_2	R_3	R_4	R_5
Horses	H_1	5	3	4	7	1
	H_2	2	3	7	6	5
	H_3	4	1	5	2	4
	H_4	6	8	1	2	3
	H_5	4	2	5	7	1

How should the horses be allotted to the riders so as to minimize the total expected loss of the team?

P7 Five operators have to be assigned to five machines. The assignment costs are given as:

		Machines				
		I	II	III	IV	V
Operator	A	5	5	–	2	6
	B	7	4	2	3	4
	C	9	3	5	–	3
	D	7	2	6	7	2
	E	6	5	7	9	1

Operator A cannot operate machine III and operator C cannot operate machine IV. Find an assignment schedule that minimizes the total cost.

REVIEW/ANSWERS/HUNTS/SOLUTIONS

P1 Minimize

$$Z = 4x_{11} + 3x_{12} + 7x_{21} + 2x_{22} + 5x_{31} + x_{32}$$

Subject to

$$\begin{aligned} &x_{11} + x_{12} = 50, \qquad &&x_{11} + x_{21} + x_{31} = 40 \\ &x_{21} + x_{22} = 30, &&x_{12} + x_{22} + x_{32} = 80 \\ &x_{31} + x_{32} = 40, && \\ &\quad x_{ij} \geq 0. && \end{aligned}$$

P2 $x_{11} = 25$, $x_{12} = 15$, $x_{22} = 20$
$x_{23} = 15$, $x_{24} = 25$, $x_{25} = 10$
$x_{35} = 20$.

P3 $x_{14} = 10$, $x_{13} = 10$, $x_{25} = 20$
$x_{31} = 15$, $x_{32} = 25$, $x_{33} = 10$
$x_{35} = 0$ (or any other basic variable that does not form a closed chain).

P4 Optimum distribution for the company is:

A to H	800
B to D	400
B to G	100
C to E	400
C to F	200
C to G	300

The minimum transportation cost is Rs. 9,200.

P5

From warehouse 1 to store 3	100 units
From warehouse 2 to store 1	80 units
From warehouse 2 to store 5	70 units
From warehouse 3 to store 2	100 units
From warehouse 3 to store 5	20 units
From warehouse 4 to store 4	90 units.

Minimum cost of transportation = Rs. 4,200

P6 Optimal assignment : $H_1 - R_5$, $H_2 - R_1$, $H_3 - R_4$, $H_4 -$ R„ $H_5 - R_2$

Minimum penalty points = 8

P7 Hint : Take as infinite, all cost entries where an assignment is not to be made. In the problem, take $C_{13} = C_{34} = \infty$. Then solve the problem by the usual Hungarian Method.

Optimal assignment : AIV, BIII, CV, DII, EI

Minimum assignment cost = 15
(Alternative optimal assignment also exists)