UNIT 10 AN INITIAL BASIC FEASIBLE SOLUTION OF THE TRANSPORATION PROBLEM

Structure

10.1 INTRODUCTION

In Unit 9, you were introduced to the Transportation Problem TP, its formulation as a EP? and its representation in a tabular form. A few special characteristics of the TP were also described in section 9.5 of Unit 9 which give it a special structure. Being a special type of EPP, it could be solved by the Simplex Method. We already know that the initial basic feasible solution for this problem can be determined by adding an artificial variable to each constraint and then applying Phase I procedure. In this unit you will see that such an initial basic feasible solution can be obtained in a better way by exploiting the special structure of the problem. We shall discuss two methods to obtain an initial basic feasible solution of a TP namely North-West Corner Method and Matrix-Minima Method.

Suppose you solve the TP by simplex method. Observe that in any TP with 'm' sources and 'n' destinations there are m × n constraints in mn variables and each constraint is an equation. Thus, the first step is to add an artificial variable to each of the constraints resulting in mn+ (m + 11) variables. Next step is to apply Phase 1 Method to get an initial basic feasible solution to the problem. Thus, even for a moderate size problem where 'm' and 'n' are not very large, quite a large number of iterations are needed to obtain the initial basic feasible solution. In this unit, we shall see that structure of the problem allows us to write down the initial basic feasible solution directly and comfortably without adding artificial variable and applying Phase 1.

Objectives

In this unit you should be able to

- North West Corner Method or the Matrix-Millinla Method to obtain an initial basic feasible,solutions of a TP
- identify the degenerate solutions.

10.2 NORTH-WEST CORNER METHOD

Let us come back to our Sewing Machines TP discussed in Unit 9. In tabular form it is:

	D_1	D_2	D_3	a_i
S_1	5 (40)	15 (10)	10 x	50
S_2	10 x	8 (20)	20 (35)	55
$b_j \rightarrow$	40	30	35	

We may first point out that this method does not take into account the costs of transportations C_{ij} and the method is what the common sense dictates us.

In this method, we start with extreme North West comer cell (1, 1) connecting source **S,** to destination D_1. In this route, 50 units are available at S, and 40 are needed by D_1. The minimum of 50 and 40 cell written as **min** [50, 40] = 40 can be transported from source S, to the market D_1. Thus we decide to set x,, = 40, write it in the cell (1, 1) and encircle it as shown 40 resperesents the value of x_{11}.

Now, demand of D_1 is exactly met and therefore, it does not need any supply from S_2. Therefore, we set x_{21} = 0. We do not write it in the table as it is a non basic variable. We can put a 'x' in the cell (2, 1). S has still 50 – 40 = 10 units available with it. Demand of market D_2 is 30. Min [10, 30] = 10. Thus PO units can be supplied by S_1 to D_2. As before, we set x_{12} = 10 and encircle it. Now, availability at S, is exhausted. D_2 still needs 30 – PO = 20 units more. This can be supplied by S_2 only (as availability of S, is exhausted). Thus set x_{22} = Min [55, 20] = 20 and. encircle it. Now, S_2 has 55 – 20 = 35 units available and only 35 is the need of D_3 Why does it happen?

Clearly because the problem is balanced. So, set x_{23} = 35 and encircle it. You see that you have very conveniently obtained a basic feasible solution of the TP with x,, = 40, x_{12} = PO, x_{21} = 20, x_{23} = 35. These are the basic variables. All other variables are non basic and zero where you have put a 'x'. Count the number of basic variables. These are 4 viz x,,, x,,, x,,, x_{23}. You may like to verify what you have learnt in Unit 9. Number of basic variables is (m + n) –1 = (2 + 3) –1 = 4.

Having understood the technique of the method, let **us** take a 3 x 4 TP the tabular form of which is given as:

Table - I

	D_1	D_2	D_3	D_4	$a_i \downarrow$
S_1	5 (50)	7 (20)	6 x	4 x	70
S_2	2 x	8 (20)	3 (30)	1 x	50
S_3	1 x	7 x	4 (20)	5 (70)	90
$b_j \rightarrow$	50	40	50	70	$\Sigma a_i = \Sigma b_j$ = 210

We observe that $\Sigma a_i = \Sigma b_j = 210$, Thus, it is a balanced TP

Start with N.W. comer cell (1, 1)

Min $[a_1, b_1]$ = Min [70, 50] = 50

Set $x_{11} = 50$.

Set $x_{21} = x_{31} = 0$ and cross 'x'. Why?

Because demand of D_1 is exactly met and it does not need any supply from S_2 or S_3.

availability at $S_1 = 70 - 50 = 20$

Demand for $D_2 = 40$

Min [20, 40] = 20

Set $x_{12} = 20$.

Set $x_{13} = x_{14} = 0$ and put 'x'. Why?

Because, supply at S_1 is exhausted. It cannot supply to markets D_3 and D_4.

D_2 still needs $40 - 20 = 20$ units which are to be supplied by next source S_2.

Min [20, 50] = 20

S_2 can supply all the 20 units needed 20.

Set $x_{22} = 20$.

Set $x_{32} = 0$ and put a 'x'. Why?

Because demand of D_2 is exactly met.

Next consider demand of D_3 which is 50 units. S_2 is left with $50 - 20 = 30$ units,?

Min [30, 50] = 30.

S_2 can supply 30 units needed by D_3.

Set $x_{23} = 30$.

Set $x_{24} = 0$ and put a 'x'. Why?

Demand of D_3 is $50 - 30 = 20$ units more.

S_3 has 90 units.

Min [90, 20] = 20

Set $x_{33} = 20$.

Demand of D_3 is exactly met.

Next consider the demand of D_4.

Demand of $D_4 = 70$ availability of $S_4 = 90 - \mathbf{20} = 70$.

Why it so happens?

Set $x_{34} = 70$ and get a basic feasible solution in which basic variables have values

$$x_{11} = 50,\ x_{12} = 20,\ x_{22} = 20,\ x_{23} = 30\ x_{33} = 20,\ x_{34} = 70.$$

These are 6 in number. What do you expect.

This should be $(m + n) - 1 = (3 + 4) - 1 = 6$ in number.

Now, non-basic variables are $x_{13} = x_{14} = x_{21} = x_{24} = x_{31} = x_{32} = 0$

and these are indicated by a cross in the table.

Yon may note in the above two examples that the circled basic variables move along horizontal and vertical lines or cells alternately. These could also move along vertical and horizontal lines if Min [a,, b_1] is equal to a,. In the above two cases it was b_1 in each case.

You may note that we have talked only about the number s f positive variables in a basic feasible solution. If you recall the definition of a basic feasible solution to the system of equations $\mathbf{A}\mathbf{x} = b$, $x \geq 0$ where **A** is a $m \times n$ matrix, and rank of $A = m$, then you require that:

1) number of positive variables is at the most 'm'.

2) Columns associated with the 'm' variables labeled as basic variables are linearly independent.

We are yet to establish the truth of the statement (2). In a TP, we shall say that basic variables are linearly independent or the basic cells (cells in which basic variables are circled) are in linearly independent position. We shall discuss this statement after outlining Matrix Minima Method to find the initial basic feasible solution of a TP.

Matrix Minima Method to find the initial basic feasible solution of a TP.

EXERCISE 1:

Using North-West Corner Method, find a basic feasible solution of the following TP.

(i)

	D_1	D_2	D_3	$a_i \downarrow$
S_1	6	3	4	10
S_2	2	1	7	50
S_3	1	4	2	20
$b_j \rightarrow$	40	10	30	

(ii)

	D_1	D_2	D_3	
S_1	8	9	5	25
S_2	4	5	3	35
S_3	1	6	7	15
S_4	9	2	12	30
$b_j \rightarrow$	30	40	35	

10.3 MATRIX-MINIMA METHOD

You have observed in the section 10.2, that we have given no consideration to the data c_{ij}. Now, we discuss method in which we transport starting with cheapest route and using the routes in ascending order of costs of transportation.

For demonstration, let us come back to two problems discussed in the section 10.2. The Sewing Machine TP is:

Table - II

	D_1	D_2	D_3	$a_i \downarrow$
S_1	5 (40)	15 x	10 (10)	50
S_2	10 x	8 (30)	20 (25)	55
$b_j \rightarrow$	40	30	35	

The collection of cells we can call a matrix. Thus it is a 2 X 3 matrix i.e. two rows and three columns.

The cheapest route is the one connecting source S_1 to market D_1 where cost per unit is 5. i.e. $C_{11} = 5$.

availability at S_1 is 50

Demand at D_1 is 40

Min [50, 40] = 40

Thus, S_1 can supply all the 40 units needed by D_1.

Set $x_{11} = 40$ and circle it.

Set $x_{21} = 0$ and cross 'x'. Why?

Next look for next cheapest cost among uncrossed cells.

(Crossed routes are no more needed).

This route is (2, 2) connecting source S_2 to market D_2 with $C_{22} = 8$

availability at $S_2 = 55$

Demand at $D_2 = 30$

Min [55, 30] = 30

Set $x_{22} = 30$ and circle 30.

Set $x_{12} = 0$ and put a 'x'. Why?

Now, look up for cheapest route in uncrossed or unused routes. It is (1, 3) with $C_{13} = 10$.

availability at $S_1 = 50 - 40 = 10$

Demand at $D_3 = 35$.

Min [10, 35] = 10.

set $x_{13} = 10$ and circle it.

Next cheapest and only unused route is (2, 3) with $C_{23} = 20$.

availability at S_2 = Demand at $S_2 = 25$. Why?

Set $x_{23} = 25$ and circle it.

This gives a basic feasible solution with $x_{11} = 40$, x,, = 10, $x_{22} = 30$, $x_{23} = 25$. Other variables are non basic.

Note that number of basic variables is (2 + 3) –1 = 4. Also, note that this basic feasible solution is different from corresponding solution that we obtained in section 10.2. It is because, now costs have been taken into consideration.

However, we have yet to establish Linear Independence of these basic variables,

We next find a basic feasible solution by Matrix Method for the second problem discussed in section 10.2. The problem is

Table - III

	D_1	D_2	D_3	D_4	$a_i \downarrow$
S_1	5 x	7 (40)	6 (10)	4 (20)	70
S_2	2 x	8 x	3 x	1 (50)	50
S_3	1 (50)	7 x	4 (40)	5 x	90
	50	40	50	70	

Minimum cost in the matrix is in the cell (3, 1) linking third source S_3 to Destination D_1 and in the cell (2, 4) linking source S_2 to destination D_4. We can choose any one of the two the cell. Let us choose (3, 1)

$a_3 = 90,\quad b_1 = 50$

Min [90, 50] = ***50***

Set $x_{31} = 50$ and circle it as a basic variable.

As demand of D_1 is exactly met, we put a cross in the cells (1, 1) and (2, 1) indicating that these are non basic i.e. $x_{,,} = x_{21} = 0$ and these are non basic variables.

Next equally cheap cost route is (2, 4) with $c_{24} = 1$.

$a_2 = 50,\quad b_4 = 70$

Min [50, 70] = 50

Set $x_{24} = 50$ and circle it as a basic variable

Set $x_{22} = x_{23} = 0$ as these are non basic variables and put a 'x'.

We now look for the cheapest route among open routes i.e. routes which have not been crossed. These are (3, 3) and (1, 4). We find that $c_{33} = 4$ and $c_{,,} = 4$. Choose any one of these say (3, 3).

Capacity available at S_3 is 90 – 50 = 40

Demand of D_3 = 50

Min [40, SO] = 40

Set $x_{33} = 40$ and circle it as a basic variable.

Set $x_{23} = x_{34} = 0$ and put a 'x'. Why?

Choose next equally cheaper cost route (1, 4)

availability at S_1 = 70

demand for D_4 is 70 – 50 = 20

Min [70, 20] = 20

Set x_{14} = 20 and circle it **as-a** basic variable.

Next higher cost in the matrix with routes unused is $c_{13} = \boldsymbol{6}$.

availability at S, = 70 – 20 = 50

demand at D_3 = 50 – 40 = 10

Min [50, 10] = 10

Set $x_{13} = 10$ and encircle it as a basic variable

Last unused route is (1, 2) with $c_{12} = 7$

Availability with $S_1 = 70 - (20 + 10) = 40$

demand at $D_2 = 40$.

Set $x_{12} = 40$ and circle it.

This gives you a basic feasible solution with $(4 + 3) - 1 = 6$ basic variables viz.

$x_{12} = 40, x_{13} = 10, x_{14} = 20$

$x_{24} = 50, x_{31} = 50, x_{33} = 40$

Other variables are non basic.

Observe that the basic feasible solution to the problem obtained by North-West Comer Method and one obtained by Matrix Minima Method arc distinct. You may also observe that if you were to solve this problem as a LPP, you would have added artificial va iables and apply Phase I Method to reach the initial basic feasible solution. Imagine the amount of labour saved. This is one of the reasons to discuss TP as a seperate unit.

EXERCISES 2:

Using Matrix Minima Method, find the initial basic feasible solution to the following TP.

(i)

	D_1	D_2	D_3	$a_i \downarrow$
S_1	3	2	3	30
S_2	1	5	7	40
S_3	8	6	2	30
$b_j \rightarrow$	50	10	40	

(ii)

	D_1	D_2	D_3	
S_1	1	7	6	40
S_2	4	2	3	30
S_3	3	5	4	20
S_4	2	1	8	30
	45	40	35	

So far, we have discussed two methods to find an initial basic feasible solution to a **given** TP. As already mentioned, we have yet to verify the fact that the solution so obtained is a basic feasible solution. We now state a rule or procedure for checking whether a given feasible solution is basic, i.e. cells are in linearly independent positions. This is called the **chain rule.**

Closed Chain

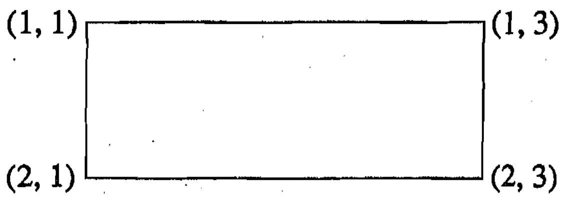
Consider the 2 × **3** TP discussed in the beginning of this unit.

The tabularrepresentation is given in table II viz.

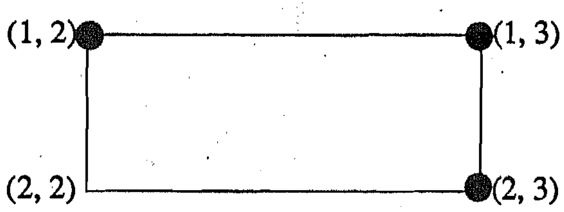
	D_1	D_2	D_3	$a_i \downarrow$
S_1	5	15	10	50
S_2	10	8	20	55
$b_j \rightarrow$	40	30	35	

Consider the set of cells (1, 1), **(1, 3)**, (2, 3), (2, 1). **Starting** with **any** one of the cells say (1, 1) **draw** alternately horizontal and vertical lines to reach the other cells. If doing so, we come back to the starting cell, we say that a closed chain is formed.

Thus in this case, the circuit is as follows:

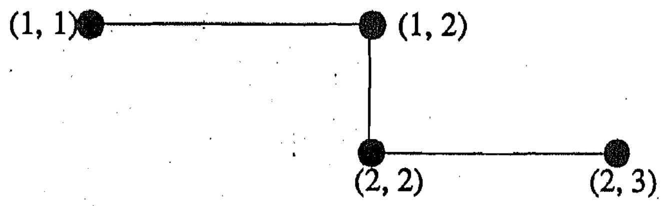


Similarly, we observe that (1, 2), (1, 3), (2, 3), (2, 2) forms a closed chain as given below:



Likewise, (1, 1), (1, 2), (2, 2), (2, 1) from a closed chain . Next let us consider the cells.

(1, 1), (1, 2) (2, 2) (2, 3)



In this case, drawing alternates horizontal and vertical line, we do not come back to the starting cell (1, 1). Thus, these cells do not form a closed chain and therefore, form simply a chain.

Similarly, (1, 1), (1, 3), (2, 3) form a chain,
(1, 1), (1, 3), (2, 2) form a chain.

We now give you a method or rule to test if given any number of cells, these are in linearly independent positions or in linear dependent positions.

Closed Chain Rule

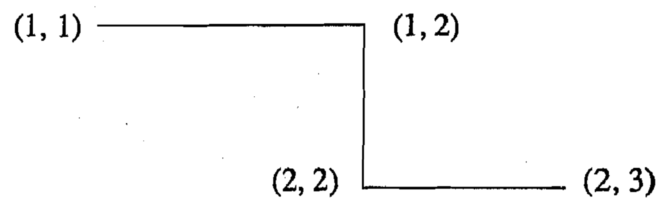
Given a feasible solution, if you can form a closed circuit involving all the basic cells or any subset or part of basic cells, then the cells are in linearly dependent position. If you can form no circuit involving the basic cells above, the cells are in linearly independent positions.

Applying this rule, you may observe that initial feasible solution obtained by North-West Corner Method or by Matrix Minima Method is indeed a basic feasible solution satisfying the criterion of cells being in independent positions.

As an illustration, the solution from the 2 × 3 TP is:

				a_i
	5 (40)	15 (10)	10	50
	10	8 (20)	20 (35)	55
$b_j \rightarrow$	40	30	35	

basic cells are (1, 1), (1, 2), (2, 2), (2, 3).



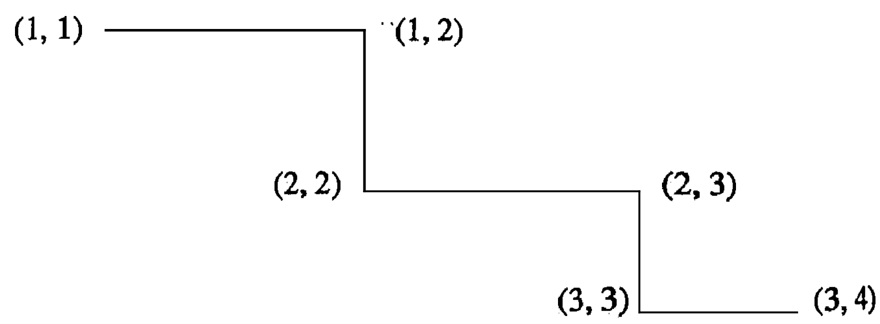
You observe that these cells do not from a closed chain. Thus these are in linearly Independent positions.

Next, consider the solution obtained for the second TP in section 10.2

(1, 1) with allocation 50, (1, 2) with allocation 20, (2, 2) with allocation 20, (2, 3) with allocation 30.

(3, 3) with allocation 20, (3, 4) with allocation 70.

These cells do not form a closed chain as shown below:



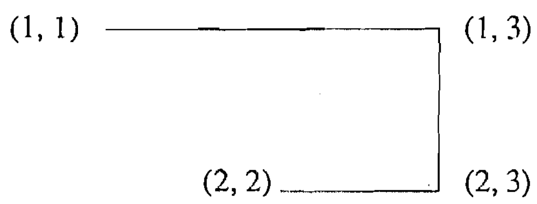
Thus, these cells are in linearly independent positions.

You may consider the solution obtained for the TP discussed in the beginning of section 10.3.

	D_1	D_2	D_3	a_i
S_1	5 (40)	15	10 (10)	50
S_2	10	8 (30)	20 (25)	55
$b_j \rightarrow$	40	30	35	

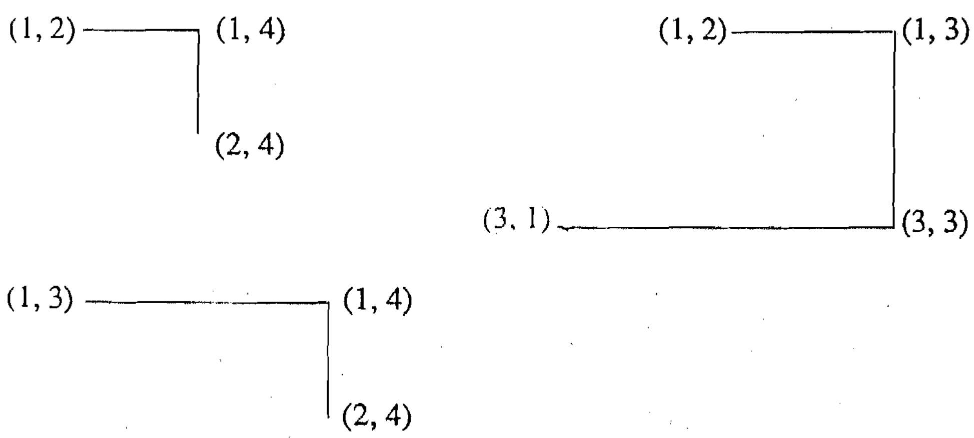
Basic cells are (1, 1), (1, 3), (2, 3), (2, 2)

They do not from a closed chain as shown below:



Similarly, for solution of the second problem discussed by Matrix Minima Method, basic cells are (1, 2), (1, 3), (1, 4), (2, 4), (3, 1), (3, 3).

We may even try to form closed chains involving all the cells. We do not succeed as shown below:



Thus the cells are in linearly independent positions. Hence, the solution is a basic feasible solution.

We shall extend this rule in the next section to identify the cells at zero level in case, the solution obtained by any of the methods or even an any stage (after going through the next unit) is a degenerate solution.

10.4 DEGENERATE BASIC FEASIBLE SOLUTION

You may recall that in a LPP, a basic feasible solution is said to be degenerate if certain basic variables (one or more) are at zero level, **i.e.** number of positive variables is less then m for the system of equation $\mathbf{A}\mathbf{x} = b, \mathbf{x} \geq 0$, $\mathbf{A}$ being $\mathbf{m} \times \mathbf{n}$ and rank of $\mathbf{A}$ is $\mathbf{m}$. However, in a LPP you were not faced with the problem of identifying the basic variable at zero level. You got these variables at zero level in the process of solving the LPP.

Let us see what happens in a Transporation Problem **(TP)** while **determining** basic feasible solution by North West Corner Method or by Matrix Minima Method. In certain cases, you may discover that the number of basic variables circled are less than required $(m + n) - 1$ was number. **This** is illustrated by the following example:

Table - II

	D_1	D_2	D_3	$a_i \downarrow$
S_1	5 (50)	15	10	50
S_2	10	8 (20)	20 (35)	55
$b_j \rightarrow$	50	20	35	

The initial basic feasible solution has been obtained using North-West Corner Method. Observe that the data of the Sewing Machine TP has been slightly altered to get such a degenerate solution. What do you observe? The number of basic variables is **3** which is less than $(3 + 2) - 1 = 4$. Such a solution in LPP you call a degenerate basic feasible solution. We must now identify the additional basic variable at zero level to make it a complete basic feasible solution having $(m + n - 1)$ variables. To do it, we give the following rule:

Rule

Select a variable as a candidate for basic variable at zero level. Suppose it is x_{12} in the above example. **i.e.** cell (1, 2). It is called a candidate cell. Start joining it by horizontal lines and vertical lines alternately to basic variable , cells with circled allocation,

In this process, if we can come back to the candidate cell, this is disqualified. Why? Because they form a linearly dependent set. Otherwise, we can put a zero in the cell and circle it as (0) .

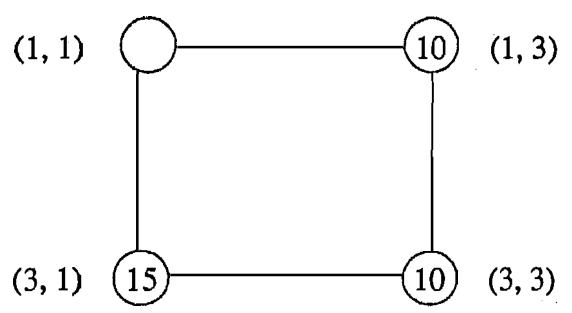
To illustrate the rule, (1, 2) can be joined only to (1, 1) by a horizontal line and then (1, 1) cannot be joined to any basic cell by a vertical line. So, we cannot come back to (1, 2). Thus (1, 2) qualifies and we can put a 0 in the cell (1, 2) and circle it. You may observe that (1, 3) and (2, 1) also qualify to become basic variables at zero level. Remember only one cell is to be marked to make $(3 + 2) - 1 = 4$ basic variables.

To make it more clear, let us take another TP as follows:

	D_1	D_2	D_3	D_4	D_5	$a_i \downarrow$
S_1	4	5	3 (10)	1 (10)	7	20
S_2	10	8	8	6	2 (20)	20
S_3	3 (15)	6 (25)	4 (10)	5	4	50
$b_j \rightarrow$	15	25	20	10	20	

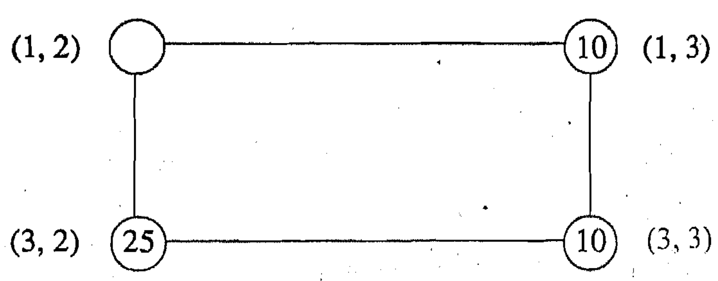
Note that the basic feasible solution has been written using Matrix Minima Method. Number of basic variables needed is $(5 + 3) - 1 = 7$. We find only 6 of these, in the process. Thus an addition basic variable at zero level is to be identified.

Following the rule, we see that (1, 1) does not qualify as we have the following closed chain.

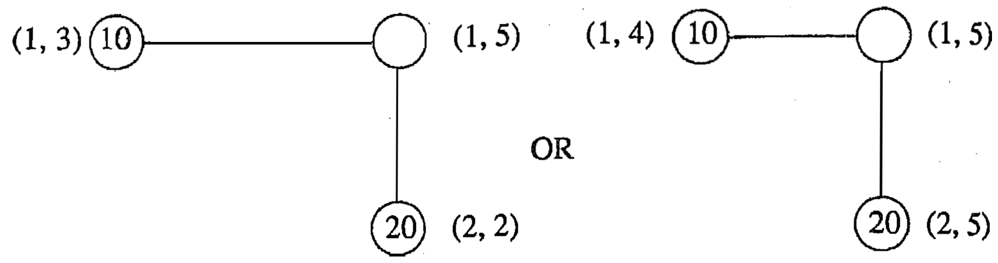


Thus (1, 1) forms a closed chain with basic cells or variables at (1, 3), (3, 3) and (3, 1).

(1, 2) also does not qualify as it forms a closed chain as shown below.



You may note that (1, 5) qualifies as it does not form a closed chain.



You may similarly discover that **(2,** 1), (2, 2), (2, 3), **(2, 4)** and (3, 5) qualify and (3, 4) does not. Thus you can put a zero or (0) in **any** of these cells (1, 5), **(2,** 1), (2, 2), **(2,** 3), (2, 4), but not in (1, 1), (1, 2) and (3, 4).

Remember you have to choose only one to make a total of **(5 + 3) − 1 = 7** basic variables or cells.

EXERCISE 3: Using North West comer Method obtain a basic feasible solution of the following TP. Indicate also the basic variables at zero level.

2	3	5	50
1	4	7	30
3	2	6	20
40	40	20	

EXERCISE 4: Determine a basic feasible solution by Matrix Minima Method Indicate thebasic variable at zero level.

2	4	8	14	80
7	6	1	10	50
9	5	7	3	40
30	60	50	30	

10.5 SUMMARY

Having gone through this unit, you are able to appreciate that although a TP is a LPP, you can find its initial basic feasible solution very conveniently and comfortably without adding artificial variables and applying Phase I Method. Also, you are at ease finding a basic feasible solution either using North-West Corner Method or Matrix Minima Method. In case you encounter degeneracy at any stage, you can identify the basic variable to be set equal to zero and thus write the complete degenerate basic feasible solution.

10.6 ANSWERS/HINTS/SOLUTIONS

E1 (i) $x_{11} = 10$, $x_{21} = 30$, $x_{22} = 10$.

$x_{23} = 10$, $x_{33} = 20$.

(ii) $x_{11} = 25$, $x_{21} = 5$, $x_{22} = 30$,

$x_{32} = 10$, $x_{33} = 5$, $x_{43} = 30$.

E2 (i) $x_{11} = 10$, $x_{12} = 10$, $x_{13} = 10$,

$x_{21} = 40$, $x_{33} = 30$.

(ii) $x_{11} = 40$, $x_{22} = 10$, $x_{23} = 20$,

$x_{31} = 5$, $x_{33} = 15$, $x_{42} = 30$.

E3 (i) $x_{11} = 40$, $x_{12} = 10$, $x_{22} = 30$,

$x_{23} = 0$, $x_{33} = 20$.

(Other basic variables at zero level also possible, e.g. $x_{23} = 0$ or $x_{32} = 0$.)

E4 $x_{11} = 30$, $x_{12} = 50$, $x_{23} = 50$,

$x_{24} = 0$, $x_{32} = 10$, $x_{34} = 30$.

(Other basic variables at zero level are also possible, e.g. $x_{13} = 0$, $x_{33} = 0$.)