UNIT 5 THE SPHERE

Structure

5.1 INTRODUCTION

With this unit we start our discussion of three-dimensional objects. As the unit title suggests, we shall consider various aspects of a sphere here. A sphere is not new to you. When you were a child you must have played with balls. You must also have eaten several fruits like limes, oranges and watermelons. All these objects are spherical in shape. But all of them are not spheres from the point of view of analytical geometry.

In this unit you will see what a geometer calls a sphere. We shall also obtain the general equation of a sphere. Then we shall discuss linear and planar sections of a sphere. In particular, we shall consider the equations of tangent lines and planes to a sphere. Finally, you will see what the intersection of two spheres is and how many spheres can pass through a given circle.

Spheres are an integral part of the study of the structure of crystals of chemical compounds. You find their properties used by architects and engineers also. Thus, an analytical study of spheres is not merely to satisfy our mathematical curiosity.

A sphere is a particular case of an ellipsoid as you will see when you study Block 3. So; if you have grasped the contents of this unit, it will be of help to you while studying the next block. In other words, if you achieve the following objectives, it will be easier for you to understand the contents of Block 3.

Objectives

After studying this unit you should be able to

- obtain the equation of a sphere if you know its centre and radius;
- check whether a given second degree equation in three variables represents a sphere;
- check whether a given line is tangent to a given sphere:
- obtain the tangent plane to a given point on a given sphere;
- obtain the angle of intersection of two intersecting spheres;
- find the family of spheres passing through a given circle.

Let us now see what a sphere is and how we can represent it algebraically.

5.2 EQUATIONS OF A SPHERE

In 2-space you know that the set of points that are at a fixed distance d from a fixed point is a circle. A sphere is a generalisation of this to 3-space (see Fig. 1).

Definition : The set of all those points in 3-space which are at a fixed distance d from a point C (a, b, c), is a sphere with **centre** C and **radius** d.

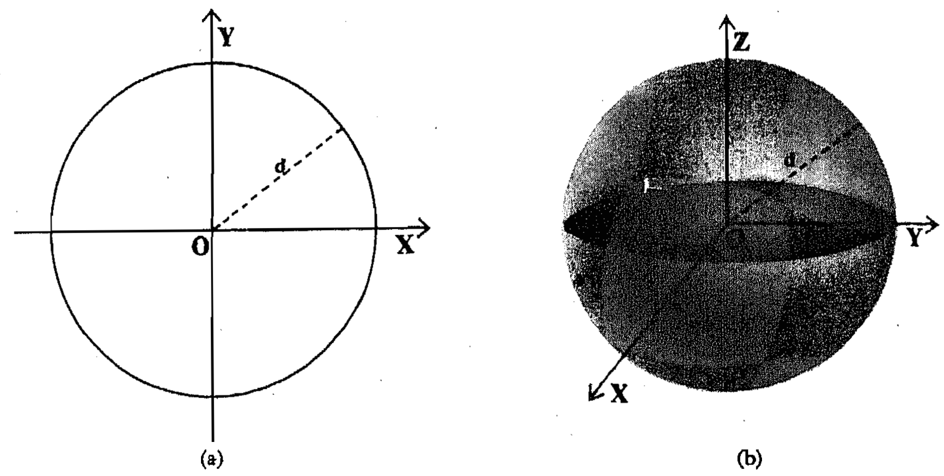

Fig 1 : (a) A circle, (b) a sphere, with origin as centre and radius d.

Spheres are, of course, not new to you. A ball and a plum are spherical in shape. However, whenever we talk of a sphere in analytical geometry, we mean the surface of a sphere. Thus, for us a hollow ball is a sphere, and a solid cricket ball is not a sphere.

Let us find the equation of a sphere with radius d and centre C (a, b, c) now. If P (x, y, z) is any point on the sphere, then, by the distance formula ((1) of Unit 4), we get

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = d^2, \qquad \text{......(1)}$$

which is the required equation,

For example, the sphere with centre (0, 0, 0) and radius 1 unit is, $x^2 + y^2 + z^2 = 1$.

Now, if we expand (1), we get the second degree equation
$x^2 + y^2 + z^2 - 2ax - 2by - 2cz + a^2 + b^2 + c^2 - d^2 = 0$.

Lqoking at this you could ask if every equation of the type

$$a(x^2 + y^2 + z^2) + 2ux + 2vy + 2wz + d = 0, \qquad \text{......(2)}$$

where a, u, v, w, d ∈ R, represents a sphere.

It so happens that if $a \neq 0$, then (2) represents a sphere. (What happens if a = 0? Unit 4 will give you the answer.)

Let us rewrite (2) as

$$x^2 + y^2 + z^2 + \frac{2u}{a}x + \frac{2v}{a}y + \frac{2w}{a}z = -\frac{d}{a}.$$

Adding $\frac{u^2}{a^2} + \frac{v^2}{a^2} + \frac{w^2}{a^2}$ on either side of this equation, we obtain

$$\left(x + \frac{u}{a}\right)^2 + \left(y + \frac{v}{a}\right)^2 + \left(z + \frac{w}{a}\right)^2 = \frac{u^2 + v^2 + w^2 - ad}{a^2}$$

Comparing this with (1), we see that this is a sphere with centre

$\left(-\frac{u}{a}, -\frac{v}{a}, -\frac{w}{a}\right)$ and radius $\frac{\sqrt{u^2+v^2+w^2-ad}}{|a|}$,

The following theorem summarises what we have said so far.

Theorem 1 : The general equation of a sphere is

$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$

Its centre is $(-u, -v, -w)$ and radius is $\sqrt{u^2+v^2+w^2-d}$.

Note that the general equation given above will be a real sphere iff $u^2+v^2+w^2-d \geq 0$. Otherwise it will be an imaginary sphere, that is, a sphere with no real points on it.

So, what we have seen is that

> a second degree equation in x, y and z represents a sphere iff
>
> i) the coefficients of x^2, y^2 and z^2 are equal, and
>
> ii) the equation has no terms containing xy, yz or xz.

Why don't you see if you've taken in what has been said so far?

E1) Find the centre and radius of the sphere given by $x^2+y^2+z^2-2x+4y-6z=11$.

E2) Does $2x^2+1+2y^2+3+2z^2+5=0$ represent a sphere?

E3) Determine the cenue and radius of the sphere $x^2+y^2+z^2=4z$.

Now, if you look at the general equation of a sphere, you will see that it has 4 arbitrary constants u, v, w, d. Thus, if we know 4 points lying on a sphere, then we can obtain its equation.

Let's consider an example.

Example 1 : Find the equation of the sphere through the points (0, 0, 0), (0, 1, –1), (–1, 2, 0) and (1, 2, 3).

Solution : Suppose the equation is

$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$

Since the 4 given points lie on it, their coordinates must satisfy this equation. So we get

$d=0$

$2+2v-2w+d=0$

$5-2u+4v+d=0$

$14+2u+4v+6w+d=0.$

Solving this system of simultaneous linear equations (see Block 2, MTE-04), we get

$u=-\frac{15}{14},\ v=-\frac{25}{14},\ w=-\frac{11}{14},\ d=0.$

Thus, the required sphere is

$7(x^2+y^2+z^2)-15x-25y-11z=0$

Note that it can happen that the system obtained by substituting the four points is inconsistent, that is, it does not have a solution. (Such a situation can occur if three of the points lie on one line.) In this case there will be no sphere passing through these points.

You can try some exercises now.

E4) Find the centre and radius of the sphere passing through (1, 0, 0), (0, 1, 0), (0, 0, 1) and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

E5) Is there a sphere passing through (4, 0, 1), (10, −4, 9), (−5, 6, −11) and (1, 2, 3)? If so, find its equation.

A diameter of a sphere is a line segment through its centre and with end points on the sphere.

Now if, instead of four points on the sphere, we only know the coordinates of the two ends of one of its diameters we can still determine its equation. Let us see how. Let A (x_1, y_1, z_1) and B (x_2, y_2, z_2) be the ends of a diameter of a sphere (see Fig. 2). Then, if P (x, y, z) is any point on the sphere, PA and PB will be perpendicular to each other. Thus, from (10) of Unit 4, we see that

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0. \quad \text{......(3)}$$

This is satisfied by any point on the sphere, and hence is the equation of the sphere.

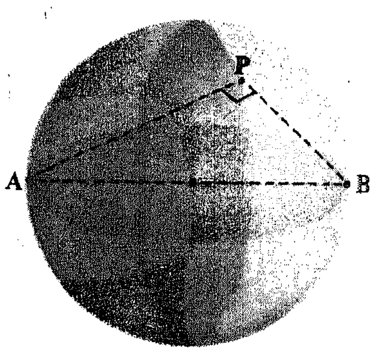

Fig. 2

For example, the equation of the sphere having the points (−3, 5, 1) and (3, 1, 7) as the ends of a diameter is $(x + 3)(x - 3) + (y - 5)(y - 1) + (z - 1)(z - 7) = 0$,

that is, $x^2+y^2+z^2 = 6y + 8z - 3$.

You can obtain the diameter form of a sphere's equation in the following exercise.

E6) Find the equation of the sphere described on the join of (3, 4, 5) and (1, 2, 3).

By now you must be familiar with spheres. Let us now see when a line or a plane intersects a sphere.

5.3 TANGENT LINES AND PLANES

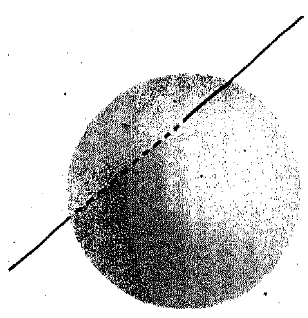

Fig. 3 : A line intersecting a sphere

In this section we shall first see how many common points a line and a sphere can have. Then we shall do the same for a plane and a sphcrc.

5.3.1 Tangent Lines

Suppose you take a hollow ball and pierce it right through with a knitting needle. Then he ball and the needle will have two points in common (see Fig. 3). Do you think this is true of any line that intersects a sphere? See what the following theorem has to say about this.

Theorem 2: A line and a sphere can intersect in at most two poinw,

Proof: Let $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ and $\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma} = t$ (say) be a given sphere and line, respectively. Then any point on the line is of the form $(\alpha t + a, \beta t + b, \gamma t + c)$, where $t \in \mathbf{R}$. If this lies on the sphere, then

$$(\alpha t + a)^2 + (\beta t + b)^2 + (\gamma t + c)^2 + 2u(\alpha t + a) + 2v(\beta t + b) + 2w(\gamma t + c) + d = 0$$

$$\Rightarrow (\alpha^2 + \beta^2 + \gamma^2)t^2 + 2t(a\alpha + b\beta + c\gamma + u\alpha + v\beta + w\gamma) + (a^2 + b^2 + c^2 + 2ua + 2vb + 2wc + d) = 0 \quad \text{......(4)}$$

This is a quadratic in t. Thus, it gives two values of t. For each value of t, we will get a point of intersection. Thus, the line and sphere can intersect in at most two points.

Note that (4) can have real distinct roots, real coincident roots or distinct imaginary roots. Accordingly, the line will intersect the sphere in two points, in one point, or not at all. This leads us to the following definitions.

Definitions: If a line intersects a sphere in two distinct points, it is called a secant line to the sphere.

If a line intersects a sphere in one point P, it is called a tangent to the sphere at the point P; and P is called the point **of contact** of the tangent.

For example, the line L in Fig. *3* is a secant line to the sphere; and the line L in Fig. 4 is a tangent to the sphere at the point P.

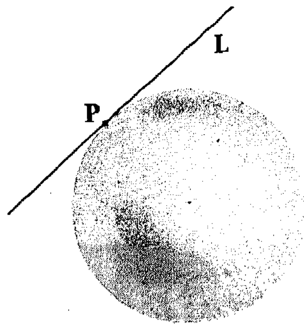

Fig. 4 : L intersects the sphere in only one point, P.

Now, (4) will have coincident roots iff

$(a\alpha + b\beta + c\gamma + u\alpha + v\beta + w\gamma)^2 = (\alpha^2 + \beta^2 + \gamma^2)(a^2 + b^2 + c^2 + 2ua + 2vb + 2wc + d)$ ......(5)

Thus, (5) is the condition for $\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma}$ to be a tangent to

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$

Let us consider an example.

Example 2: Find the intercept made by the sphere $x^2 + y^2 + z^2 = 9$ on the line $x - 3 = y = z$.

Solution: Any point on the line is of the form $(t + 3, t, t)$ where $\in$ R. This lies on the sphere if

$(t + 3)^2 + t^2 + t^2 = 9 \Rightarrow 3t^2 + 6t = 0 \Rightarrow t = 0, -2.$

Thus, the points of intersection are (3, 0, 0) and (1, –2, –2). Thus, the intercept is the distance between the two points, which is $\sqrt{4+4+4} = 2\sqrt{3}$.

You can try some exercises now.

E7) Check if $\frac{x-3}{4} = \frac{y+4}{3} = \frac{z}{5}$ is a tangent to the sphere
$x^2 + y^2 + z^2 + 4x + 6y + 10z = 0.$

E8) If we extend the rule of thumb to find the tangent to a conic (see Unit 3) to a sphere, will we get the equation of a tangent line to a sphere? Why?

Let us now discuss the intersection of a plane and a sphere.

5.3.2 Tangent Planes

Consider a sphere and a plane that intersects it. What do you expect the intersection to be? The following result will give you the answer.

Theorem 3: A planar section of a sphere is a circle.

Proof: Let *S* be a sphere with radius r and centre O (see Fig. 5), and let the plane Π intersect it.

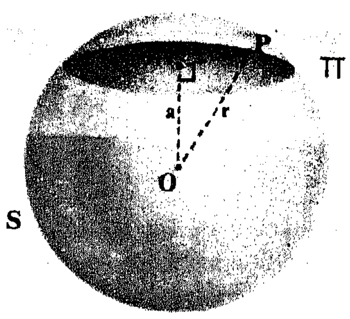

Fig. 5: A planar intersection of a sphere is a circle.

Drop a perpendicular ON from O onto Π, and let ON = a. Now let P be a point which belongs to Π as well as S. Then OP = r and $OP^2 = ON^2 + NP^2$.

Thus, $NP = \sqrt{r^2 - a^2}$, which is a constant.

Thus, the intersection of S and Π is the set of points in Π which are at a fixed distance from a fixed point N. Thus, it is a circle in the plane Π with centre N and radius $\sqrt{r^2 - a^2}$.

If a = 0 in the proof above, the plane passes through the centre of the sphere. In this case the circle of intersection is of radius r and is called a **great circle** (see Fig. **6)** of the sphere.

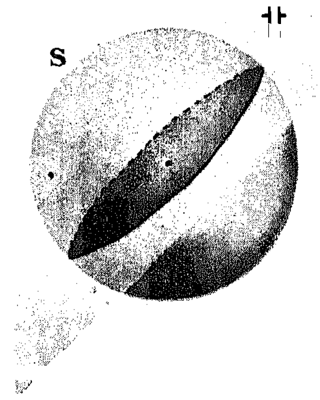

Fig. 6: The intersection of S and Π is a great circle of the sphere S.

Note that a sphere has infinitely many great circles, one for each plane that passes through the centre of the sphere.

We have seen that the planar section of a sphere is a circle. Now let us find its equation. Let the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, and that of the plane intersecting it be $Ax + By + Cz + D = 0$. Then the equation of the planar section can be written as

$$\left.\begin{array}{l} x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 = Ax + By + Cz + D, \text{ or} \\ x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,\ Ax + By + Cz + D = 0. \end{array}\right] \quad \ldots\ldots(6)$$

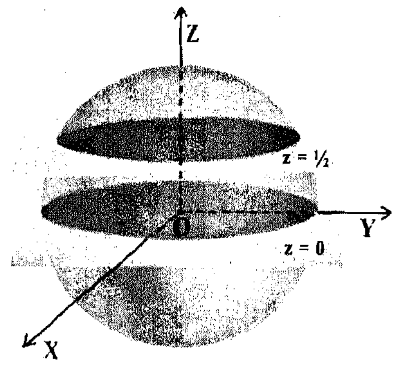

Fig. 7: Planar sections of $x^2 + y^2 + z^2 = 1$

For example, the equation of the planar section of the sphere $x^2 + y^2 + z^2 = 1$ by the plane $z = \frac{1}{2}$ (see Fig. 7) is $x^2 + y^2 + z^2 - 1 = z - \frac{1}{2} = 0$. This is the circle $x^2 + y^2 = \frac{3}{4}$, in the plane $z = \frac{1}{2}$.

Since the centre of the given sphere is (0, 0, 0), we can get a great circle of the sphere by intersecting it with z = 0. Thus, one great circle is $x^2 + y^2 = 1$ in the plane z = 0.

Let us consider an example of the use of Theorem 3.

Example 3: Find the centre and radius of the circle

$x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0,\ x - 2y + 2z = 3.$

Solution: The centre of the sphere is C(4, – 2, – *3)* and its radius is

$r = \sqrt{16 + 4 + 16 + 45} = 9.$

The distance of the plane from the centre of the sphere is

$$d = \frac{|4 + 4 - 8 - 3|}{\sqrt{1 + 4 + 4}} = 1.$$

Thus, the radius of the circle $= \sqrt{r^2 - d^2} = 4\sqrt{5}$.

The centre of the circle is the foot of the perpendicular from C onto the plane. To find this, we first need to find the equations of the perpendicular. Its direction ratios are 1, –2, 2. Thus, its equations are

$$\frac{x-4}{1} = \frac{y+2}{-2} = \frac{z+4}{2}.$$

Therefore, any point on the perpendicular is given by (t + 4, –2t – 2, 2t – 4), where $t \in R$. This point will be the required centre of the circle if it lies on the plane, that is, if

$$(t + 4) - 2(-2t - 2) + 2(2t - 4) = 3 \Rightarrow t = \frac{1}{3}.$$

Hence, the centre of the circle is $\left(\frac{13}{3}, -\frac{8}{3}, -\frac{10}{3}\right)$.

You can do the following exercise on the same lines.

E9) Find the centre and radius of the circle
$x^2 + y^2 + z^2 + 12x - 12y - 16z + 111 = 0 = 2x + 2y + z - 17$.

Now, if we take $a = r$ in the proof of Theorem 3, then what happens to the circle of intersection? It reduces to a single point, that is, a point circle. And in this case the plane only touches the sphere (see Fig. 8).

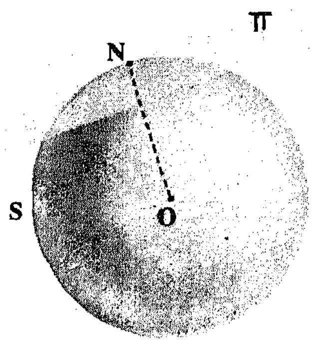

Fig. 8: The plane Π is tangent to the sphere S at the point N.

Definition: A plane is tangent to a sphere at a point P if it intersects the sphere in P only.

In this case we also say that the plane touches the sphere at P. P is called the point of tangency, or the point of contact, of the tangent plane.

Remark 1: If you go back to the proof of Theorem 3, you will see that the line joining the point of tangency to the centre of the sphere is perpendicular to the tangent plane. We will use this fact to obtain the equation of a tangent plane.

Let us find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ at the point P (a, b, c).

As P lies on the sphere,

$$a^2 + b^2 + c^2 + 2ua + 2vb + 2wc + d = 0. \quad \text{......(7)}$$

Also, the centre of the sphere is C (–u, –v, –w). Thus, the direction ratios of CP are $a + u$, $b + v$, $c + w$ (see Equation (8) of Unit 4).

Now, the tangent plane passes through P (a, b, c). Thus, its equation will be

$$f(x - a) + g(y - b) + h(z - c) = 0, \text{ for some } f, g, h \in \mathbb{R}. \quad \text{......(8)}$$

Now CP is perpendicular to (8), and hence, is parallel to the normal to (8). Further, f, g, h are direction ratios of the normal to the plane. Therefore, $a + u$, $b + v$, $c + w$ and f, g, h are proportional.

$$\therefore \frac{f}{a+u} = \frac{g}{b+v} = \frac{h}{c+w} = t, \text{ say.}$$

Then (8) gives us

$$(x - a)(a + u) + (y - b)(b + v) + (z - c)(c + w) = 0.$$

$$\Rightarrow xa + yb + zc + ux + vy + wz = a^2 + b^2 + c^2 + ua + vb + wc. \quad \text{......(9)}$$

Using (7) and (9), we get

$$xa + yb + zc + ux + vy + wz = -ua - vb - wc - d.$$

Thus,

> the equation of the tangent plane at the point (a, b, c) to the sphere*
> $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is
> $xa + yb + zc + u(x + a) + v(y + b) + w(z + c) + d = 0$.

Is this the equation you got while doing E8? From the equation you may have realised that there is a similar thumb rule for the tangent plane (and not tangent line!) to a sphere.

Rule of Thumb: To obtain the equation of the tangent plane to a sphere at the point (a, b, c), simply substitute ax for x^2, by for y^2, cz for z^2; and in the linear terms substitute $\frac{x+a}{2}$ for x, $\frac{y+b}{2}$ for y, $\frac{z+c}{2}$ for z in the equation of the sphere.

For example, the equation of the tangent plane to $x^2 + y^2 + z^2 = a^2$ at (α, β, γ) is $x\alpha + y\beta + z\gamma = a^2$.

Why don't you try an exercise now?

E10) Find the equations of the tangent planes
a) to $x^2 + y^2 + z^2 + 2z = 29$ at $(2, 3, -4)$.
b) to $x^2 + y^2 + z^2 - 4y - 6z + 4 = 0$ at $(2, 3, 1)$.

So, what we have seen so far is that if a plane is at a distance d from the centre of a sphere with radius r, then

i) if $r < d$, the plane and sphere do not intersect;

ii) if $r = d$, the plane is tangent to the sphere; and

iii) if $r > d$, the plane intersects the sphere in a circle of radius $\sqrt{r^2 - d^2}$.

Now, if you are given the equations of a sphere and a plane, can you tell if the plane is tangent to the sphere? An obvious way would be to check what the distance of the centre of the sphere from the plane is. Let us use this method to derive the condition for the plane $Ax + By + Cz + D = 0$ to be a tangent plane to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

Now, the radius of the sphere is $\sqrt{u^2 + v^2 + w^2 - d}$.

The length of the perpendicular to the plane from the centre $(-u, -v, -w)$ of the sphere is

$$\frac{|Au + Bv + Cw + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

This distance must equal the sphere's radius since the plane is tangent to the sphere.

$$\therefore (Au + Bv + Cw + D)^2 = (A^2 + B^2 + C^2)(u^2 + v^2 + w^2 - d), \qquad \ldots\ldots(10)$$

which is the required condition.

Let us consider an example.

Example 4: Show that $2x - y - 22 = 16$ touches the sphere $x^2 + y^2 + z^2 - 4x + 2y + 2z - 3 = 0$, and find the point of contact.

Solution: The centre of the sphere is $(2, -1, -1)$ and its radius is $\sqrt{2^2 + 1^2 + 1^2 + 3} = 3$.

The length of the perpendicular from the centre to the plane $2x - y - 22 - 16 = 0$ is

$$\frac{|2.2 + 1 + 2 - 16|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{9}{3} = 3,$$ which is the same as the radius of the sphere.

So the plane touches the sphere.

Let $(x,, y_1, z_1)$ be the point of contact. Then the equation of the tangent plane is

$$xx_1 + yy_1 + zz_1 - 2(x + x_1) + (y + y,) + (2 + z_1) - 3 = 0$$

$$\Leftrightarrow (x_1 - 2)x + (y_1 + 1)y + (z_1 + 1)z - 2x_1 + y_1 + z_1 - 3 = 0.$$

But this should be the same as the given plane $2x - y - 22 - 16 = 0$.

So the coefficients of x, y, z and the constant term in both these equations must be proportional.

$$\therefore \frac{x_1 - 2}{2} = \frac{y_1 + 1}{-1} = \frac{z_1 + 1}{-2} = \frac{2x_1 - y_1 - z_1 + 3}{16}$$

$\Rightarrow x_1 = -2y_1,\ z_1 = 1 + 2y_1$, and then

$$\frac{y_1 + 1}{-1} = \frac{2x_1 - y_1 - z_1 + 3}{16} = \frac{-7y_1 + 2}{16} \Rightarrow 9y_1 = -18 \Rightarrow y_1 = -2.$$

$\therefore x_1 = 4$ and $z_1 = -3$.

Thus, the point of contact is (4, –2, –3).

Using the same method, if we are given a point and a plane, we can find the sphere with the point as the centre and the plane as a tangent plane. In the example below we illustrate this.

Example 5: Find the sphere with centre $(-1, 2, 3)$, and which touches the plane $2x - y + 2z = 6$.

Solution: The distance of the plane from the point is

$$\frac{|-2 - 2 + 6 - 6|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{4}{3}.$$

This should be the radius of the sphere.

So, since the centre of the sphere is (–1, 2, 3), its equation will be

$$(x + 1)^2 + (y - 2)^2 + (z - 3)^2 = \frac{16}{9}.$$

Why don't you do some exercises now?

E11) Show that the plane $x + y + z = \sqrt{3}$ touches the sphere $x^2 + y^2 + z^2 = 1$. Find the point of contact.

E12) Show that the equation of the sphere which lies in the octant OXYZ and touches the coordinate planes is $x^2 + y^2 + z^2 - 2k(x + y + z) + 2k^2 = 0$, for some $k \in R$.

E13) Find the equation of the sphere with centre $(1, 0, 0)$, and which touches the plane $2x + y + z - 3 = 0$.

In this section we have seen what sets can be got by intersecting a line or a plane with a sphere. Now let us discuss what form the intersection of two or more spheres can take.

5.4 INTERSECTION OF SPHERES

In this section you will first see that the result of intersecting two spheres is the same as that obtained by intersecting a sphere and a plane, that is, a circle. And then you will see how to obtain infinitely many spheres whose intersection is a given circle.

5.4.1 Two Intersecting Spheres

Let us consider two spheres given by

$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$, and

$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0.$

Then each point that satisfies $S_1 = 0$ as well as $S_2 = 0$ will also satisfy the equation $S_1 - S_2 = 0$, that is,

$$2(u_1 - u_2)x + 2(v_1 - v_2)y + 2(w_1 - w_2)z + d_1 - d_2 = 0. \qquad \ldots\ldots(11)$$

Thus, thc spheres $S_1 = 0$ and $S_2 = 0$ intersect at 90° iff (13) is satisfied.

Lct us consider an example.

Example 6: Find the angle between $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 - 2x = 0$.

Solution: Here $u_1 = 0, v_1 = 0, w_1 = 0, d_1 = -4, u_2 = -1, v_2 = 0, w_2 = 0, d_2 = 0$.

Thus, the centres of the two spheres are (0, 0, 0) and (1, 0, 0), their radii are 2 and 1, rcspectivcly, and the distance between their centres is 1.

Therefore, by (12), the angle between the two spheres is

$$\cos^{-1}\left(\frac{2^2 + 1^2 - 1^2}{2(2)\ (1)}\right) = \cos^{-1}(1) = 0.$$

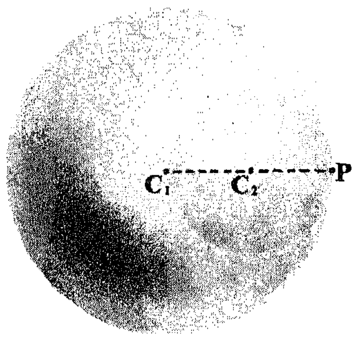

Fig. 11

You can see these spheres in Fig. 11. They intersect in only onc point P, and the x-axis is the normal from the centres of the spheres to both tangent planes.

You can try some exercises on intersecting spheres now.

E14) Find the angle of intersection of the spheres $x^2 + y^2 + z^2 - 2x + 2y - 42 + 2 = 0$ and $x^2 + y^2 + z^2 = 4$.

E15) Find the equation of the sphere touching the plane $3x + 2y - z + 2 = 0$ at P (1, –2, 1) and cutting the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$ orthogonally,

€16) a) Two spheres of radius r_1 and r_2 and with centres C_1 and C_2, respectively, will touch each other iff $r_1 + r_2 = C_1C_2$. True or false? Why?

b) Under what conditions on r_1, r_2 and C_1C_2 will the spheres not intersect?

E17) Show that the spheres $x^2 + y^2 + z^2 - 2x - 4y - 4z = 0$ and $x^2 + y^2 + z^2 + 10x + 2z + 10 = 0$ touch each other. What is the point of contact?

So far you have seen that two sphercs intersect in a circle. Now let us see whether, given a circle, wc can find two or more sphercs passing through it.

5.4.2 Spheres Through a Given Circle

Suppose we are given a circle. Can we find two distinct sphercs whose intersection the circle is'? In fact, we can construct many spheres passing through a given circle (see Fig. 12).

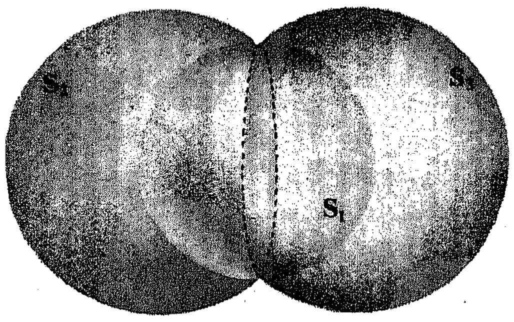

Fig. 12: Part nf a family of sphercs through a circle.

In Fig. 12, the circle is a great circle of the sphere S_1, but not of S_2, S_3, etc. Let us see what the method of construction of this kind of family is.

You know that a circle is the intersection of a sphere and a plane. So its equation is of the form $S = 0, \Pi = 0$, where

$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d$, and $\Pi \equiv Ax + By + Cz + D$.

Any sphere through this circle will be given by

$S + k\Pi = 0,$ ......(14)

where k is an arbitrary constant. Do you agree? Now, if you apply Theorem 1, you can see that (14) represents a sphere.

Further, every point that lies on the circle must satisfy (14). Thus, (14) represents a sphere through the given circle.

So, for each value of $k \in R$ in (14) we get a distinct sphere passing through the given circle. Thus, we have infinitely many spheres that intersect in the given circle.

Now, a circle can also be represented as the intersection of two spheres $S_1 = 0$ and $S_2 = 0$. In this case what will the equation of any sphere containing it be? It will be $S_1 + kS_2 = 0$, where $k \in R$. Thus, the infinite set $\{S_1 + kS_2 = 0 \mid k \in R\}$ gives us the family of spheres passing through the given circle.

Let us consider some examples of the use of (14).

Example 7: Show that the circles $x^2 + y^2 + z^2 - y + 2z = 0$, $x - y + z - 2 = 0$ and $x^2 + y^2 + z^2 + x - 3y + z - 5 = 0$, $2x - y + 4z - 1 = 0$ lie on a sphere, and find its equation.

Solution: Tho equation of any sphere through the first circle is

$x^2 + y^2 + z^2 - y + 2z + k(x - y + z - 2) = 0$, that is,

$x^2 + y^2 + z^2 + kx - (k + 1)y + (k + 2)z - 2k = 0,$ ......(15)

for some $k \in R$.

Similarly, the equation of any sphere through the second circle is

$x^2 + y^2 + z^2 + (2k_1 + 1)x - (k_1 + 3)y + (4k_1 + 1)z - (k_1 + 5) = 0,$ ......(16)

for some $k_1 \in R$.

To get a common sphere containing both circles, we must see if (15) and (16) coincide for some k and k_1 in R. Comparing the coefficients of x, y and z, and the constant terms in (15) and (16), we get

$k = 2k_1 + 1, k + 1 = k_1 + 3, k + 2 = 4k_1 + 1, 2k = k_1 + 5.$

These equations are satisfied for $k = 3$ and $k_1 = 1$.

Thus, there is a sphere passing through both the circles and its equation is

$x^2 + y^2 + z^2 + 3x - 4y + 5z - 6 = 0.$

Example 8: Find the equation of the sphere through the circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and the origin.

Solution: Let the equation of the sphere be $x^2 + y^2 + z^2 - 9 + k(2x + 3y + 4z - 5) = 0$, where $k \in R$.

Since, it passes through (0, 0, 0), we get $-9 - 5k = 0$, that is, $k = -\frac{9}{5}$.

Thus, the required equation is

$5(x^2 + y^2 + z^2) = 9(2x + 3y + 4z).$

Example 9: Find the path traced by the centre of a sphere which touches the lines $y = x$, $z = 1$ and $y = -x$, $z = -1$.

Solution: Let $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ be the equation of a sphere that touches the two lines. Since $y = x, z = 1$ touches it, the intersection of the line and the sphere must be only one point. Any point on the line is (t, t, 1), where $t \in R$. It lies on the sphere if

$t^2 + t^2 + 1 + 2ut + 2vt + 2w + d = 0.$

This equation has equal roots if

$(u + v)^2 = 2(1 + 2w + d)$.

Similarly, since $y = -x$, $z = -1$ touches the sphere, we get

$(u - v)^2 = 2(1 - 2w + d)$.

Subtracting these two conditions, we get

$4uv = 4w(1 + 1)$, that is, $uv = 2w$.

Thus, the centre of the sphere, $(-u, -v, -w)$, satisfies the equation $xy + 22 = 0$.

This is true for any sphere satisfying the given conditions. Thus, the required path is $xy + 2z = 0$.

Now, why don't you check if you've understood what we have done in this section so far?

E18) Prove that the circles
$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0, 5y + 6z + 1 = 0$ and
$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0, x + 2y - 7z = 0$
lie on the same sphere. Find its equation also.

E19) Find the equations of the spheres that pass through $x^2 + y^2 + z^2 = 5, 2x + y + 3z = 3$ and touch the plane $3x + 4y = 15$.

E20) Find the equation of the sphere for which the circle $2x - 3y + 4z = 8$, $x^2 + y^2 + z^2 + 7y - 22 + 2 = 0$ is a great circle.

We will stop our discussion on spheres for now, though we shall refer to them off and on in the next block. Let us now do a quick review of what we have covered in this unit.

5.5 SUMMARY

In this unit we have covered the following points.

1) The second degree equation
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$
represents a sphere with centre $(-u, -v, -w)$ and radius $\sqrt{u^2 + v^2 + w^2 - d}$.
Conversely, the equation of any sphere is of this form.

2) A line intersects a sphere in at most two points. It is a tangent to the sphere it' it intersects the sphere in only one point.

3) A plane intersects a sphere in a circle. When this circle reduces to a point circle P, then the plane is tangent to the sphere at P.

4) The equation of the tangent plane to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ at the point (x_1, y_1, z_1) is $xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$. This is perpendicular to the line joining (x_1, y_1, z_1) to the centre of the sphere.

5) Two spheres intersect in a circle,

6) The angle of intersection of the two intersecting spheres $x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$ and $x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$ is $\cos^{-1}\left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$,
where r_1 and r_2 are their radii and d is the distance between their centres.
In particular, the two spheres are orthogonal if $2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$.

7) There are infinitely many spheres that pass through a given circle.

You may now like to go back to Sec. 5.1 and go through the list of **unit objectives** to see if you have achieved them. If you want to see what our solutions to the exercises in the unit are, we have given them in the following section.

5.6 SOLUTIONS/ANSWERS

E1) Its centre is $\left(-\left(\frac{-2}{2}\right), -\left(\frac{4}{2}\right), -\left(\frac{-6}{2}\right)\right) = (1, -2, 3).$

Its radius is $\sqrt{(-1)^2 + 2^2 + (-3)^2 - (-11)} = 5.$

E2) We can rewrite the equation as $x^2 + y^2 + z^2 + \frac{9}{2} = 0.$

This represents an imaginary sphere with centre at the origin.

E3) Its centre is (0, 0, 2) and radius is 4.

E4) Let the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$. Then, since the given points lie on it, we get

$1 + 2u + d = 0$
$1 + 2v + d = 0$
$1 + 2w + d = 0$

$1 + \frac{2}{\sqrt{3}}(u + v + w) + d = 0$

On solving these equations, we find that $u = v = w = 0$, $d = -1$.
Thus, the centre of the sphere is (0, 0, 0) and radius is 1.

E5) Suppose such a sphere exists. Let its equation be
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$
Since the given points lie on it, we get the linear system
$17 + 8u + 2w + d = 0$
$197 + 20u - 8v + 18w + d = 0$
$192 - 10u + 12v - 22w + d = 0$
$14 + 2u + 4v + 6w + d = 0.$
You can check that this system is inconsistent. Thus, the points do not lie on a sphere.

E) The required equation is
$(x - 3)(x - 1) + (y - 4)(y - 2) + (z - 5)(z - 3) = 0.$
$\Leftrightarrow x^2 + y^2 + z^2 - 4x - 6y - 8z + 26 = 0.$

E7) Any point on the line is $(4t - 3, 3t - 4, 5t)$, where $t \in R$. This will lie on the sphere if
$(4t - 3)^2 + (3t - 4)^2 + 25t^2 + 4(4t - 3) + 6(3t - 4) + 10(5t) = 0.$
$\Leftrightarrow 50t^2 + 36t - 11 = 0$

$$\Leftrightarrow t = \frac{-36 \pm \sqrt{(36)^2 + 2200}}{100}.$$

Since these are real distinct roots, the line will intersect the sphere in two distinct poinb. Hence, it will not be a tangent to the sphere.

E8) If we extend the rule of thumb to obtain the tangent at a point $P(x_1, y_1, z_1)$ on the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, we get
$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0.$
This is a linear equation, and hence represents a plane, and not a line. Thus, it cannot represent the tangent line,

E9) The centre of the sphere is C $(-6, 6, 8)$.
Its radius is $r = 5$.
The distance of C from the plane is $d = 3$.
Thus, the radius of the circle $= \sqrt{r^2 - d^2} = 4$.

The equations of the perpendicular from C onto the plane are $\frac{x+6}{2} = \frac{y-6}{2} = z - 8$.

Thus, the centre of the circle is $(-4, 8, 9)$.

E10) a) $2x + 3y - 4z + (z - 4) = 29 \Leftrightarrow 2x + 3y - 3z - 33 = 0$
b) $2x + 3y + z - 2(y + 3) - 3(z + 1) + 4 = 0$
$\Leftrightarrow 2x + y - 2z - 5 = 0$.

E11) The radius of the sphere = 1:
The distance of the plane from the centre (0, 0, 0) of the sphere = 1.
Thus, the plane is tangent to the sphere.
If the point of contact is (a, b, c), then the equation of the plane is $ax + by + cz - 1 = 0$, as well as $x + y + z - \sqrt{3} = 0$.

$\therefore \frac{a}{1} = \frac{b}{1} = \frac{c}{1} = \frac{1}{\sqrt{3}}$.

Thus, the point of contact is $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

E12) Let the equation be
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.
Since the plane $x = 0$ is a tangent to it, the distance of $(-u, -v, -w)$ from $x = 0$ must be $\sqrt{u^2 + v^2 + w^2 - d} = r$, say.
$\therefore -u = r$.
(Note that $|-u| = -u$, since the centre lies in the octant in which the x, y and z coordinates are all positive.)
Similarly, $-v = -w = r$,
Then $u^2 + v^2 + w^2 - d = r^2 \Rightarrow d = 2r^2$.
Thus, the equation of the sphere is
$x^2 + y^2 + z^2 - 2r(x + y + z) + 2r^2 = 0$.

E13) Its radius should be $\frac{|2-3|}{\sqrt{6}} = \frac{1}{\sqrt{6}}$.
Thus, its equation is
$(x - 1)^2 + y^2 + z^2 = \frac{1}{6}$.
$\Leftrightarrow 6(x^2 + y^2 + z^2) - 12x + 5 = 0$.

E14) Their centres are C_1 $(1, -1, 2)$ and C_2 $(0, 0, 0)$, respectively.
Both their radii are 2, and $C_1C_2^2 = 6$.
Thus, the angle of intersection is

$$\cos^{-1}\left(\frac{4 + 4 - 6}{2(2)(2)}\right) = \cos^{-1}\left(\frac{1}{4}\right).$$

E15) Let the sphere be given by
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.
Then the plane $3x + 2y + z + 2 = 0$ is the same as
$x - 2y + z + u(x + 1) + v(y - 2) + w(z + 1) + d = 0$
$\Leftrightarrow x(1 + u) + y(v - 2) + z(w + 1) + u - 2v + w + d = 0$

$$\frac{1+u}{3} = \frac{v-2}{2} = \frac{w+1}{-1} = \frac{u - 2v + w + d}{2}$$

$\therefore\ v = \frac{2u+8}{3},\ w = \frac{-u-4}{3},\ d = \frac{4u+22}{3}.$

Further, this sphere cuts
$x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$ orthogonally.
Thus, using (13) we see that
$-4u + 6v = d + 4.$

Substituting the values of v and d, we get $u = \frac{7}{2}$. And then $v = 5$, $w = -\frac{5}{2}$, $d = 12$.

Thus, the required sphere is $x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0$.

E16) a) This is true only if one sphere doesn't lie inside the other. Otherwise, as in Fig. 11, $C_1C_2 \neq r_1 + r_2$.

b) If one lies outside the other and $r_1 + r_2 > C_1C_2$, then they won't intersect.
If one lies inside the other and $|r_1 - r_2| > C_1C_2$, they won't intersect.

E17) Their centres are C_1 (1, 2, 2) and C_2 (–5, 0, –1).
$\therefore C_1C_2 = 7$ = sum of their radii.
Thus, they touch each other.
The plane $S_1 - S_2 = 0$ is the common tangent plane, where $S_1 = 0$ and $S_2 = 0$ are the two spheres.
This will be $6x + 2y + 3z + 5 = 0$.
The point of contact will be the intersection of the line C_1C_2 with this plane. Now, C_1C_2 is given by $\frac{x+5}{6} = \frac{y}{2} = \frac{z+1}{3}$. Any point on this is $(6t - 5, 2t, 3t - 1)$. This lies on the tangent plane if $6(6t-5) + 2(2t) + 3(3t-1) + 5 = 0 \Rightarrow t = \frac{4}{7}$.

Thus, the point of contact is $\left(\frac{-11}{7}, \frac{8}{7}, \frac{5}{7}\right)$.

E18) Solving this on the lines of Example 7, you can check that they lie on the sphere
$x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0.$

E19) Any such sphere is given by $x^2 + y^2 + z^2 - 5 + k(2x + y + 3z - 3) = 0$, where $k \in \mathbf{R}$.

Its centre is $\left(-k, -\frac{k}{2}, -\frac{3k}{2}\right)$. Its distance from $3x + 4y = 15$ is the same as the radius of the sphere.

$\therefore k^2 + \frac{k^2}{4} + \frac{9k^2}{4} + (3k + 5) = (k + 3)^2$

$\Rightarrow k = 2, -\frac{4}{5}.$

Thus, the two spheres that satisfy the given conditions are
$x^2 + y^2 + z^2 + 4x + 2y + 6z - 11 = 0$ and $5(x^2 + y^2 + z^2) - 8x - 4y - 12z - 13 = 0$.

E20) Any such sphere will be given by
$x^2 + y^2 + z^2 + 7y - 2z + 2 + k(2x - 3y + 4z - 8) = 0$, where $k \in \mathbf{R}$.
Since the given circle is a great circle of the sphere, the centre of the sphere must lie on the plane $2x - 3y + 4z = 8$.

$\therefore 2(-k) - 3\left(\frac{3k-7}{3}\right) + 4(1 - 2k) = 8.$

$\Rightarrow k = \frac{13}{29}.$

Thus, the equation of the sphere is

$x^2 + y^2 + z^2 + \frac{26}{29}x + \frac{164}{29}y - \frac{6}{29}z - \frac{46}{29} = 0$

$\Rightarrow 29(x^2 + y^2 + z^2) + 26x + 164y - 6z - 46 = 0.$

UNIT 6 CONES AND CYLINDERS

Structure

6.1 INTRODUCTION

In the previous unit we discussed a very commonly found three-dimensional object. In this unit we look at two more commonly found three-dimensional objects, namely, a cone and a cylinder. But, what you will see in this unit may surprise you—what people usually call a cone or a cylinder are only portions of very particular cases of what mathematicians refer to as a cone or a cylinder.

We shall start our discussion on cones by defining them, and deriving their equations. Then we shall concentrate on cones whose vertices are the origin. In particular, we will obtain the tangent planes of such cones.

The other surface that we will discuss in this unit is a cylinder. We shall define a general cylinder, and then focus on a right circular cylinder.

The contents in this unit are of mathematical interest, of course. But, they are also of interest to astronomers, physicists, engineers and architects, among others. This is because of the many applications that cones and cylinders have in various fields of science and engineering.

The surfaces that you will study in this unit are particular cases of conicoids, which you will study in the next block. So if you go through this unit carefully and ensure that you achieve the following objectives, you will find the next block easier to understand.

Objectives

After studying this unit you should be able to

- obtain the equation of a cone if you know its vertex and base curve;
- prove and use the fact that a second degree equation in 3 variables represents a cone with vertex at the origin if it is homogeneous;
- obtain the tangent planes to a cone;
- obtain the equation of a right circular cylinder if you know its axis and base curve.

Let us now see what a cone is.