UNIT 2 THE STANDARD CONICS

Structure

2.1 INTRODUCTION

In this unit you will be studying some curves which may be familiar to you. They were first studied systematically by the Greek astronomer Apollonius (approximately 225 B.C.). These curves are the parabola, ellipse and hyperbola. They are called conics (or conic sections) because, as you will see in this course, they can be formed by taking the intersection of a plane and a cone.

We start this unit by defining conics as curves that satisfy the 'focus-directrix' property. From this definition, we will come to the particular cases of standard forms of a parabola, ellipse and a hyperbola. The standard forms are so called because any conic can be reduced to one of these forms, and then the various properties of the conic under consideration can be studied easily. We will trace the standard forms and look at their tangents and normals. We will also discuss some other characteristics, along with some of their applications in astronomy, military science, physics, etc.

In the next unit we will discuss conics in general. And then, what you study in this unit will certainly be of help. If you achieve the following unit objectives, then you can be sure that you have grasped the contents of this unit.

Objectives

After studying this unit, you should be able to

- obtain the equation of a conic if you know its focus and directrix;
- obtain the standard forms of the Cartesian and polar equations of a parabola, an ellipse or a hyperbola;
- prove and apply the string property of an ellipse or a hyperbola;
- obtain the tangent or normal to a standard conic at a given point lying on it;
- check whether a given line is a tangent to a given standard conic or not;
- find the asymptotes of a hyperbola in standard form.

Let us now start our discussion on conics.

2.2 FOCUS-DIRECTRIX PROPERTY

Suppose you toss a ball to your friend. What path will the ball trace? It will be similar to the curve in Fig. 1, which is a parabola. With this section, we begin to take a close look at curves like a parabola, an ellipse or a hyperbola. Such curves are called conic sections, or conics. These curves satisfy a geometric property, which other curve satisfies. We treat this property as the definition of a conic section.

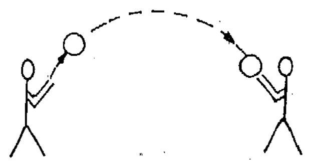


Fig. 1: The ball, when thrown, traces a parabola.

Definition: A conic section, or a conic, is the set of all those points in two-dimensional space for which the distance from a fixed point F is a constant (say, e) times the distance from a fixed straight line L (see Fig. 2).

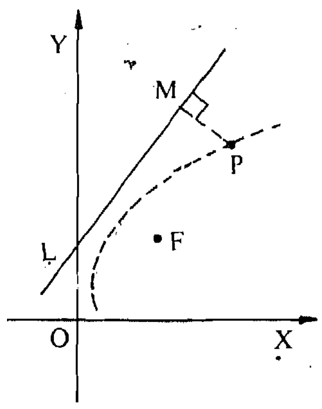


Fig. 2: The set of all points P, where PF = ePM is a conic.

The fixed point F is called a focus of the conic. The line L is known as a directrix of the conic. The number e is called the **eccentricity** of the conic.

Since there are infinitely many lines and points in a plane, you may think that there are infinitely many types of conics. This is not so. In the rest of this block we will list the types of conics that there are and discuss them in detail. As a first step in this direction, let us see what the definition means in algebraic terms.

We will obtain the equation of a conic section in the Cartesian coordinate system. Let F(a,b) be a focus of the conic, and $px+qy+r = 0$ be the directrix L (see Fig. 2). Let e be the eccentricity of the conic. Then a point P(x,y) lies on the conic iff

$$\sqrt{(x-a)^2+(y-b)^2} = e\left|\frac{px+qy+r}{\sqrt{p^2+q^2}}\right|,$$ by Formulae (1) and (10) of Unit 1.

$$\Leftrightarrow \{(x-a)^2+(y-b)^2\}(p^2+q^2) = e^2(px+qy+r)^2 \qquad \ldots(1)$$

Thus, (1) is the equation of the conic with a focus at (a, b), a directrix $px + qy + r = 0$ and eccentricity e.

For example, the equation of the conic with eccentricity 1/2, a focus at (1, 1) and a directrix $x + y = 1$ is

$$(x-1)^2+(y-1)^2 = \frac{1}{4}\cdot\frac{(x+y-1)^2}{2}$$

Why don't you try an exercise now?

E1) Find the equation of the conic section with

a) eccentridity 1, (2, 0) as its focus and $x = y$ as its directrix,

b) eccentricity 1/2, $2x+y = 1$ as its directrix and (0, 1) as its focus. (Note that in this case the focus lies on the directrix).

In E1 you have seen the two different possibilities that the focus may or may not lie on the directrix. Let us first consider the case when the focus does not lie on the directrix. In this case the conics we get are called non-degenerate conics. There are three types of such conics, depending on whether $e < 1$, $e = 1$ or $e > 1$.

When $e < 1$, the conic is an ellipse; when $e = 1$, we get a parabola; and when $e > 1$, we get a hyperbola. We shall discuss each of these conics in detail in the following sections.

Let us start with the non-degenerate conics with eccentricity 1.

2.3 PARABOLA

In this section we will discuss the equation and properties of a parabola. Let us first define a parabola.

Definition: A **parabola** is the set of all those points in two-dimensional space that are equidistant from a line L, and a point F not on L. L is its directrix and F is its focus.

Let us use (1) to obtain the equation of a parabola. To start with let us assume F is (0, 0) and L is the straight line $x + c = 0$, where $c > 0$. (Thus, L is parallel to the y-axis and lies to the left of F.) Then, using (1), we see that the equation of the parabola is

$x^2 + y^2 = (x + c)^2$, that is,

$$y^2 = c(2x + c). \quad ...(2)$$

Now, to simplify the equation let us shift the origin to $\left(-\frac{c}{2}, 0\right)$. If we put $\frac{c}{2} = a$, then we are shifting the origin to (–a, 0). From Sec. 1.4.1 you know that the new coordinates x' and y' are given by

$x = x' - a$ and $y = y'$

Thus, (2) becomes

$y'^2 = 4ax'$.

This parabola has a focus at (a, 0) (in the $X'Y'$-system) and the equation of the directrix is

$x' + a = 0$.

So, what we have found is that the equation

$$y^2 = 4ax \quad ...(3)$$

represents a parabola with $x + a = 0$ as its directrix and (a, 0) as its focus. This is one of the standard forms of the equation of a parabola. There are three other standard forms of the equation of a parabola. They are

$$x^2 = 4ay, \quad ...(4)$$

$$y^2 = -4ax, \text{ and} \quad ...(5)$$

$$x^2 = -4ay, \quad ...(6)$$

where $a > 0$.

These equations are called standard forms because, as you will see in Unit 3, we can transform the equation of any parabola into one of these forms. The transformations that we use are the rigid body motions given in Sec. 1.4. So they do not affect the geometric properties of the curve that is being transformed. And, as you will see in the following sub-sections, the geometry of the standard forms are very easy to study. So, once we have the equation of a parabola, we transform it to a standard form and study its properties. And these properties will be the same as the properties of the parabola we started with.

Now let us see what the standard forms look like.

2.3.1 Description of Standard Forms

Let us now see what a parabola looks like. We start with tracing (3). For this, let us see what information we can get from the equation. Firstly, the curve intersects each of the axes in (0, 0) only.

Next, we find that for the points (x, y) of the parabola, $x \geq 0$, since $y^2 \geq 0$. Thus, the curve lies in the first and fourth quadrants.

Further, as x increases, y also increases in magnitude.

And finally, the parabola (3) is symmetric about the x–axis, but not about **the** y-axis or the origin (see Sec. 1.3). Thus, the portions of the curve in the first quadrant and the fourth quadrant are mirror images of each other.

Using all this information about the curve $y^2 = 4ax$, we trace it in Fig. 3.

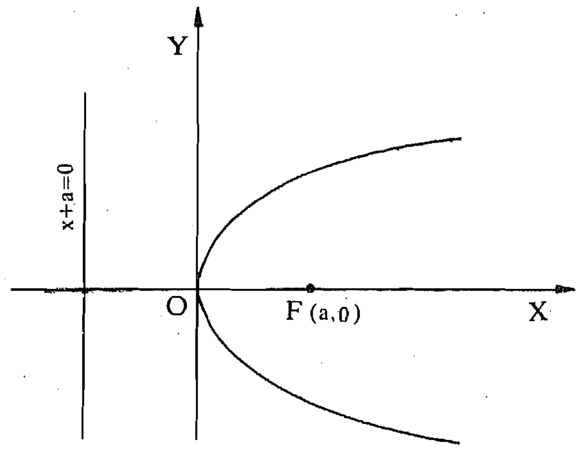


Fig. 3: $y^2 = 4ax$, $a > 0$.

The line through the focus and perpendicular to the directrix is called the ***axis*** of the parabola. Thus, in this case the x-axis is the axis of the parabola.

The point at which the parabola cuts its axis is called its ***vertex.*** Thus, (0, 0) is the ***vertex*** (plural 'vertices') of the parabola in Fig. 3.

Now, what happens if we interchange x and y in (3)? We will get (4). This is also a parabola. Its focus is at (0, a), and directrix is $y + a = 0$. If we study the symmetry and other geometrical aspects of the curve, we find that its geometrical representation is as in Fig. 4. Its vertex is also at (0, 0) but its axis is not the same as that of (3). Its axis is $x = 0$, that is, the y-axis.

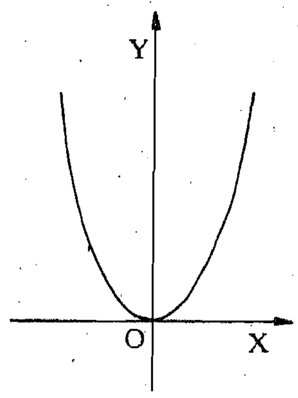


Fig. 4: $x^2 = 4ay$, $a > 0$.

'Foci' is the plural of 'focus'

Why don't you trace some parabolas yourself now?

E2) Trace the standard forms (5) and (6) of a parabola. Explicitly state the coordinates of their vertices and foci.

So far we have considered parabolas whose vertices are at (0, 0) and foci lie on one of the coordinate axes. In Unit 3 you will see that by applying the changes in axes that we have discussed in Sec. 1.4, we can always obtain the equations of a parabola in one of these standard forms, In this section we shall keep our discussion to parabolas in standard form.

Now let us look at a simple mechanical method of tracing a parabola. On a sheet of paper draw a straight line L and fix a point F not on L. Then, as in Fig. 5, fix one end of a piece of string with a drawing pin to the vertex A of a set-square. The length of the string should be the length of the side AD of the set-square.

Fix the other end of the string with a drawing pin at the point F. Now slide the other leg of the set-square along a ruler placed on the line L (as in Fig. 5), and keep a pencil point P pressed to the side AD so that the string stays taut. Then PD = PF. Thus, as P moves, the curve that you draw will be part of a parabola with focus F and directrix L.

Why don't you try this method for yourself? Instead of a set-square you could simply cut out a right-angled triangle from a piece of cardboard.

E3) Use the mechanical method to trace the parabola $x^2 + 8y = 0$

So far we have expressed any point on a parabola in terms of its Cartesian

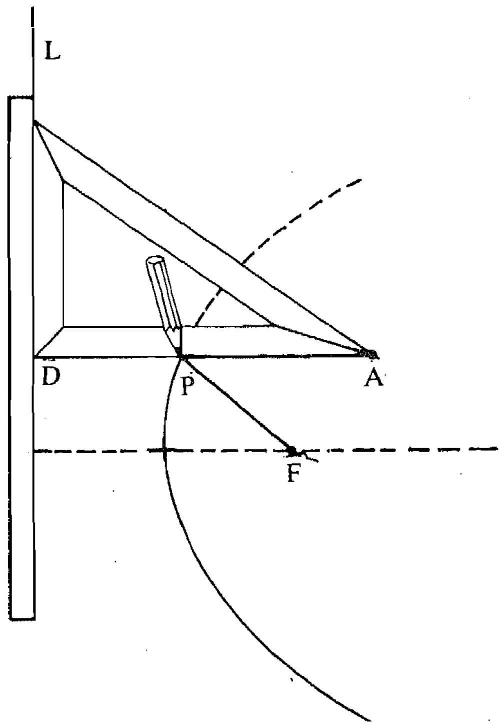


Fig. 5: Mechanical method for tracing a parabola.

coordinates x and y. But sometimes it is convenient to express it in terms of only one variable, or parameter, say t. You can check that the point $(at^2, 2at)$ lies on the parabola $y^2 = 4ax$, for all $t \in R$. Further, any point (x, y) on this parabola is of the form $(at^2, 2at)$ where $t = y/2a \in \mathbf{R}$. Thus, a point lies on the parabola $y^2 = 4ax$ iff it can be represented by $(at^2, 2at)$ for some $t \in \mathbf{R}$. In other words,

> the parametric representation of any point on the parabola $y^2 = 4ax$ is $x = at^2$, $y = 2at$, where $t \in \mathbf{R}$.

And now let us look at the intersection of a line and a parabola.

2.3.2 Tangents and Normals

The line segment PQ is called a chord of the parabola.

Let us consider the parabola $y^2 = 4ax$. What is the equation of the line joining two distinct points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on it (see Fig. *6)?*

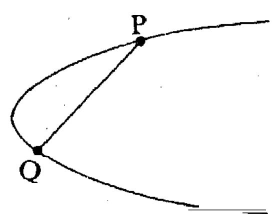


Fig. 6: PQ is a chord of the parabola.

From Unit 1 we know that the equation of PQ is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

Since P and Q lie on the parabola,

$y_1^2 = 4ax_1$ and $y_2^2 = 4ax_2$.

So we can write the equation of PQ as

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{\frac{1}{4a}(y_2^2 - y_1^2)}$$

$\Leftrightarrow (y - y_1)(y_1 + y_2) = 4a(x - x_1)$ (Note that $y_1 \neq y_2$, since P and Q are distinct.)

$\Leftrightarrow y(y_1 + y_2) - y_1y_2 = 4ax + y_1^2 - 4ax_1$

$\Leftrightarrow y(y_1 + y_2) = 4ax + y_1y_2$, since $y_1^2 = 4ax_1$. ...(7)

This is the equation of any line passing through two distinct points on the parabola.

In particular. the equation of the line joining A(a, 2a) and B(a, **-2a)** is $x = a$, which is parallel to the directrix of the parabola.

The chord AB has special significance. It is called the **latus rectum** of the parabola $y^2 = 4ax$, and its length is 4a. Note that the focus lies on the latus rectum. Thus, the latus rectum is the chord of the parabola which corresponds to the line through its focus and perpendicular to its axis (see Fig.-7).

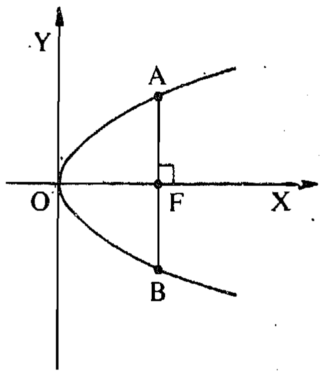


Fig. 7: **AB** is the **latus** rectum of the parabola.

Note that **the length of the latus rectum is the coefficient of x** in the equation of the parabola.

Similarly, the length of the latus rectum of $x^2 = 4ay$ is the coefficient of y.

Now for an exercise.

E4) Find the equation of the latus rectum of $x^2 + 2y = 0$.

Now, let us go back to Equation (7). Suppose we take the point Q closer and closer to P, that is, **Q tends to** P. Then x_2 tends to x_1 and y_2 to y_1. In this limiting case the line PQ is given a special name.

Definition: Let P and Q be any two points on a curve C which are close to each other. Then the line segment PQ is called a **secant** of C. The position of the line PQ when the point Q is taken closer and closer to P, and ultimately coincides with P, is called the **tangent to the curve C at P.** P is called the **point of contact** or **point of tangency.**

Thus, in Fig. 6, as Q moves along the curve towards P, the line PQ becomes a tangent to the parabola at the point $P(x_1, y_1)$ (see Fig. 8). So, from (7) we see that the equation of the tangent at P is

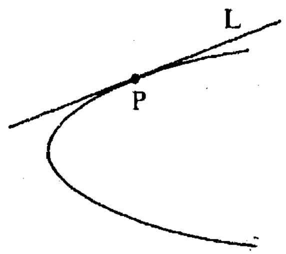


Fig. 8: The line L is the tangent at P to the parabola.

$y \cdot 2y_1 = 4ax + y_1^2$

$= 4a(x + x_1)$, since $y_1^2 = 4ax_1$.

$\Leftrightarrow yy_1 = 2a(x + x_1)$.

So, (8) is the equation of the tangent to the parabola $y_1^2 = 4ax$ at (x_1, y_1).

For example, the tangent to (3) at its vertex will be the y–axis, x = 0.

And what will the equation of the tangent to $y^2 = x$ at (4, 2) be? It will be

$yy_1 = \left(\frac{x + x_1}{2}\right)$ where $x_1 = 4$ and $y_1 = 2$, that is, $4y = (x + 4)$.

Have you noticed how we obtained (8) from (3)? We give you a rule of thumb that we follow in the remark below.

Remark 1: To get the equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1), we replace y^2 by yy_1 and x by $\frac{1}{2}(x + x_1)$. Similarly, the equation of the tangent to $x^2 = 4ay$ at a point (x_1, y_1) lying on it will be $xx_1 = 2a(y + y_1)$.

You may like to try the following exercise now.

E5) Find the equation of the tangent to

a) $x^2 + 2y = 0$ at its vertex, and

b) $y^2 + 4x = 0$ at the ends of its latus rectum.

E6) Give an example of a line that intersects $y^2 = 4ax$ in only one point, but is not a tangent to the parabola.

So, given a parabola and a point on it, you have seen how to find the tangent at that point. But, given a line, can we tell whether it is a tangent to a given parabola? Let us see under what conditions the line $y = mx + c$ is a tangent to $y^2 = 4ax$.

If $y = mx + c$ meets the parabola at (x_1, y_1), then

$y_1^2 = 4ax_1$ and $y_1 = mx_1 + c$.

So $(mx_1 + c)^2 = 4ax_1$, that is,

$$m^2x_1^2 + (2mc - 4a)x_1 + c^2 = 0 \qquad \ldots(9)$$

Now, there are two possibilities—$m = 0$ and $m \neq 0$. Can the first one arise? Can the line $y = c$ be a tangent to $y^2 = 4ax$? Suppose it is a tangent at a point (x_1, y_1). Then $y = c$ is the same as $yy_1 = 2a(x + x_1)$. This is not possible, since $a \neq 0$.

Thus, for $y = mx + c$ to be a tangent to $y^2 = 4ax$, we must have $m \neq 0$. Then (9) is a quadratic equation in x_1. So it has two roots. Corresponding to each root, we will get a point of intersection of the line and the parabola.

Thus, a line can intersect a parabola in at most two points. If the roots of (9) are real and distinct, the line and parabola have two distinct common points. If the roots of (9) are real and coincide, the line will meet the parabola in exactly one point. And if the roots of (9) are imaginary, the line will not intersect the parabola at all.

So, if $y = mx + c$ is a tangent to $y^2 = 4ax$, the discriminant of (9) must be zero, that is,

$$(2mc - 4a)^2 = 4m^2c^2$$

$$\Rightarrow 4m^2c^2 - 16amc + 16a^2 = 4m^2c^2$$

$$\Rightarrow c = \frac{a}{m}, \text{ since } m \neq 0.$$

Thus,

the straight line $y = mx + c$ is a tangent to $y^2 = 4ax$ if $m \neq 0$ and $c = \frac{a}{m}$.

And then, what will the point of contact be? Since *(9)* has coincident roots, we *see* that

$$x_1 = \frac{4a - 2mc}{2m^2} = \frac{4a - 2m.\frac{a}{m}}{2m^2} = \frac{a}{m^2}; \text{ and then}$$

$$y_1 = mx_1 + c = m\left(\frac{a}{m^2}\right) + \frac{a}{m} = \frac{2a}{m}.$$

Thus, $y = mx + \frac{a}{m}$ will be a tangent to $y^2 = 4ax$ at the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Using the condition for tangency, we can say, for example, that the line $3x + 2y = 5$ is a tangent to $y^2 + 15x = 0$, but not to $y^2 = 15x$.

And now for an exercise.

E7) Under what conditions on m and c, will $y = mx + c$ be a tangent to $x^2 = 4ay$?

A tangent to a parabola has several properties, but one of them in particular has many practical applications. This is the **reflecting property.** According to this, suppose a line L, parallel to the axis of a parabola, meets the parabola at a point P (see Fig. 9). Then the tangent to the parabola at P makes equal

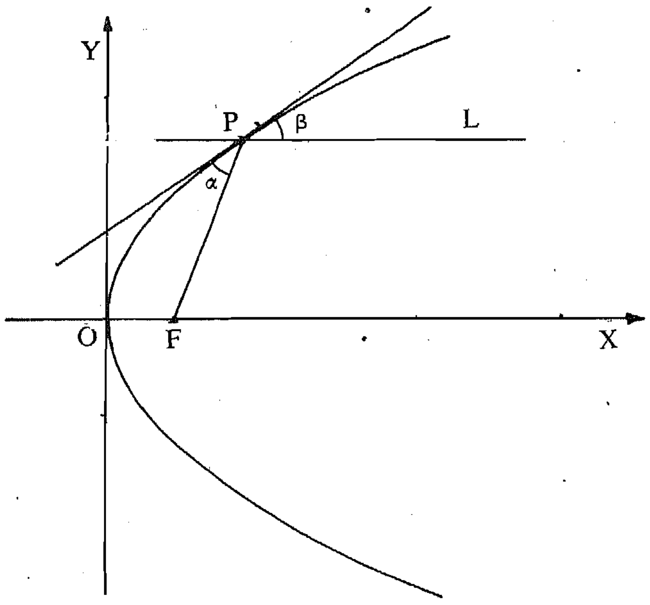


Fig. 9: Reflecting property of a parabola.

The **focal radius** of **a point P** on a parabola with focus F is the line segment PF.

angles with L and with the focal radius PF. That is, $\alpha = \beta$ in Fig. 9.

The reason this property is called the reflecting property is the following application:

Take a mirror shaped like a parabola, that is, a parabolic mirror (see Fig. 10). If a ray of light parallel to the parabola's axis falls on the mirror, then the reflected ray will pass through the focus of the parabola. Thus, a beam of light, parallel to the axis converges to the focus, after reflection. Similarly, the rays of light that are emitted from a source at the focus will be reflected as a beam parallel to the axis. This is why parabolic mirrors are used in car headlights and searchlights.

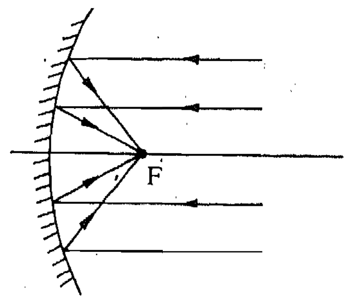


Fig. 10: A parabolic mirror.

'Focus' is Latin for 'fireplace'.

It is also because of this property that the ancient Greek mathematician Archimedes could use parabolic reflectors to set fire to enemy vessels in the harbour! How did he manage this? Archimedes ingeniously thought of applying the following fact:

If a parabolic reflector is turned towards the sun, then the rays of the sun will reflect and converge to the focus and create heat at this point.

This is also the basis of solar-energy collectors like solar cookers.

The reflecting property is also the basis for using parabolic radio and visual telescopes, radars, etc.

The following exercise is about the reflecting property.

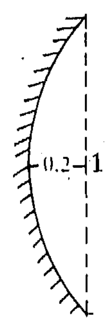


Fig. 11

E8) A parabolic mirror for a searchlight is to be constructed with width 1 metre and depth 0.2 metres. Where should the light source be placed? In Fig. 11 we have given a cross section of the mirror.

(Hint: The parabola is $y^2 = 4ax$, and (0.2, 0.5) lies on it.)

Now let us consider certain lines that are often spoken of along with tangent lines. These are the normals.

Definition: The normal to a curve at a point P on the curve is a straight line which is perpendicular to the tangent at P, and which passes through P (see Fig. 12).

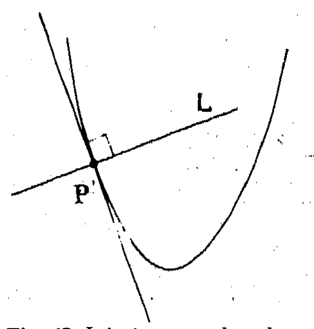


Fig. 12: L is the normal to the parabola at P.

For example, a parabola's axis is the normal at its vertex.

Now, let $P(x_1, y_1)$ be a point on $y^2 = 4ax$. Then, you know that the equation of the tangent at P is

$yy_1 = 2a(x + x_1)$.

If $y_1 = 0$, then $x_1 = 0$ and the normal at (0, 0) is $y = 0$, the axis of the parabola.

On the other hand: if $y_1 \neq 0$, then the slope of the tangent at (x_1, y_1) is $\frac{2a}{y_1}$.

So the slope of the normal will be $-\frac{y_1}{2a}$ (see Equation (13) of Unit 1). Then, from Unit 1 you know that the equation of the normal at (x_1, y_1) is

$y - y_1 = -\frac{y_1}{2a}(x - x_1)$, that is,

$$y = -\frac{y_1 x}{2a} + y_1 + \frac{y_1^3}{8a^2}, \quad \ldots(10)$$

since $y_1^2 = 4ax_1$.

Note that (10) is valid even when $y_1 = 0$.

So, for example, what will the equation of the normal to $y^2 = x$ at (1, 1) be?

Here $a = \frac{1}{4}$, $x_1 = y_1 = 1$. So, by (10) we find that the required equation is

$y = -2x + 1 + 2 = -2x + 3$.

We end this section with some easy exercises.

E9) Find the equation of the tangent and normal at (1, 1) to the parabola $x^2 = 4y$.

E10) What is the normal at the point of contact of the tangent $y = mx + \frac{a}{m}$ to the curve $y^2 = 4ax$?

With this we end our rather long discussion on the standard forms of parabolas. Now let us consider a conic whose focus doesn't lie on the directrix, and whose eccentricity is less than 1.

2.4 ELLIPSE

As the title of this section suggests, in it we shall study an ellipse and its properties. Let us start with a definition.

Definition: An **ellipse** is a set of points whose distance from a point F is e (< 1) times its distance from a line L which does'not pass through F.

Let us find its Cartesian equation. For this we shall return to Equation (1) in Sec. 2.2. As in the case of a parabola, let us start by assuming that F is the origin and L is $x + c = 0$, for some constant c. Then (1) becomes

$x^2 + y^2 = e^2(x + c)^2$,

which is equivalent to

$$\left(x - \frac{ce^2}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2c^2}{(1 - e^2)^2} \text{ (Check!).}$$

If we now shift the origin to $\left(\frac{ce^2}{1 - e^2}, 0\right)$, the equation in the new X'Y'-system becomes

$$x'^2 + \frac{y'^2}{1 - e^2} = \frac{e^2c^2}{(1 - e^2)^2}$$

This is of the form

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$

where $a = \frac{ec}{(1 - e^2)}$ and $b^2 = \frac{(ec)^2}{1 - e^2} = a^2 (1 - e^2)$.

In the X'Y'-systems, the focus is (–ae, 0) and the directrix is $x' + ae + c = 0$,

that is, $x' + \frac{a}{e} = 0$.

Note that $b^2 = a^2 (1 - e^2)$ and $e < 1$. Thus, $b^2 < a^2$.

So, if we simply retrace the steps we have taken above, and find the equation of an ellipse with focus (–ae, 0) and directrix

$x + \frac{a}{e} = 0$, we will get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \ldots(11)$$

where $b^2 = a^2 (1 - e^2)$.

(11) is the standard form of the equation of an ellipse. As in the case of a parabola, we can always rotate and translate the axes so that the equation of any ellipse can be put in this form, for some a and b. We call it the standard form because it is a convenient form for checking any geometrical properties of an ellipse, or for solving problems related to an ellipse.

Let us now study (11) carefully, and try to trace it.

2.4.1 Description of Standard Form

Let us start by studying the symmetry of the curve (see Sec. 1.3). Do you agree that the curve is symmetric with respect to the origin, as well as both the coordinate axes? Because of this, it is enough to draw the ellipse in the first quadrant. Why is this so? Well, the portion in the second quadrant will then be its reflection in the y-axis; and the rest of the curve will be the reflection in the x-axis of the portion in these two quadrants.

Next, let us see where (11) intersects the coordinate axis. Putting y = 0 in (11), we get $x = \pm a$; and putting x = 0, we get $y = \pm b$. So, (11) cuts the axes in the four points (a, 0), (–a, 0), (0, b), (0, –b).

Thirdly, let us see in which area of the plane, the ellipse (11) is defined. You can see that if $|x| > a$, y is imaginary. Thus, the ellipse must lie between x = –a and x = a. Similarly, it must lie between y = b and y = –b.

This information helps us to trace the curve, which we have given in Fig. 13.

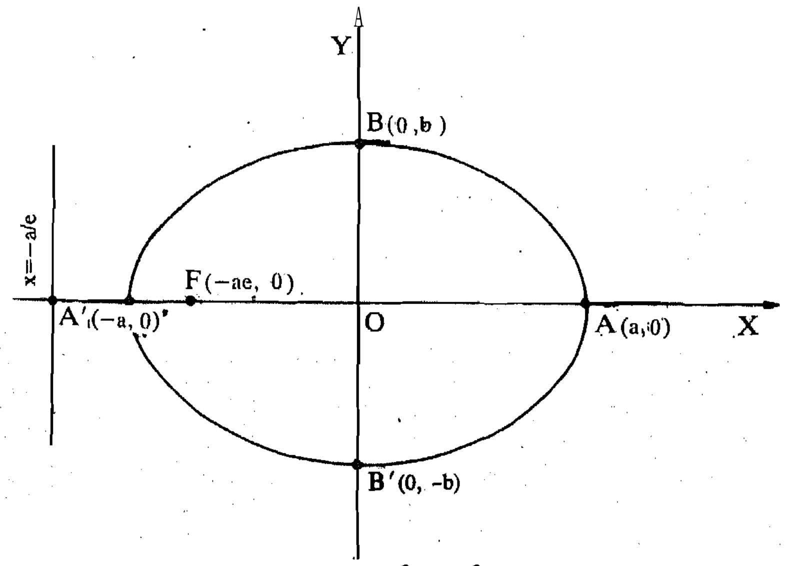


Fig. 13: The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Looking at the symmetry of the curve, do you expect (ae, 0) to be a focus also? If you do, then you are on the right track. (11) has another focus at $F'(ae, 0)$, with corresponding directrix $x = \frac{a}{e}$. Thus, (11) has two foci, namely, $F(-ae,0)$ and $F'(ae, 0)$; and it has two directrices (plural of 'directrix'), namely,

$x = -\frac{a}{e}$ and $x = \frac{a}{e}$.

The chord of an ellipse which passes through the foci is called the major axis of the ellipse. The end points of the major axis are the **vertices** of the ellipse. Thus, in Fig. 13, A and A' are the vertices and the chord A'A is the major **axis.** Its length is 2a.

The midpoint of the major axis is called the centre **of the ellipse.** You can see that the centre of the ellipse (11) is (0, 0).

The chord of an ellipse which passes through its centre and is perpendicular to its major axis is called the minor **axis** of the ellipse. In Fig. 13, the minor **axis** is the line segment B'B. Its length is 2b.

Let us look at an example.

Example 1: Find the eccentricity, foci and centre of the ellipse $2x^2 + 3y^2 = 1$.

Solution: The given equation is $\frac{x^2}{\frac{1}{2}} + \frac{y^2}{\frac{1}{3}} = 1$.

Comparing with (11), we get $a = \frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{3}}$. Since $b^2 = a^2(1 - e^2)$, $\frac{1}{3} = \frac{1}{2}(1 - e^2)$, that is, $e = \frac{1}{\sqrt{3}}$.

The foci are given by $(\pm ae, 0) = \left(\pm \frac{1}{\sqrt{6}}, 0\right)$.

And of course, the centre is (0, 0).

Now let us sketch an ellipse whose major axis is along the y–axis. In this case a and b in (11) get interchanged.

Example 2: Sketch the ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$.

Solution: This ellipse intersects the x–axis in $(\pm 3, 0)$, and the **y–axis** in $(0, \pm 5)$. Thus, its major axis lies along the y–axis, and the minor axis lies along the x–axis. Thus, a and b of (11) have become interchanged. Note that (0, 0) is the centre of this ellipse too.

Also, if e is the eccentricity of this ellipse; then $9 = 25(1-e^2)$. **Therefore,** $e = \frac{4}{5}$. **Thus,** the foci lie at (0, 4) and (0, –4). (Remember that in this case the major axis lies along the y-axis.) The directrices of this ellipse are $y = \pm \frac{25}{4}$.

We sketch the ellipse in Fig. 14.

Here are some exercises now.

E11) Find the length of the major and minor axes, the eccentricity, the coordinates of the vertices and the foci of $3x^2 + 4y^2 = 12$.
Hence sketch it.

E12) Find the equation of the ellipse with centre (0, 0), vertices at $(\pm a, 0)$ and eccentricity 0. Sketch this ellipse. Does the figure you obtain have another name?

E13) The astronomer Johann Kepler discovered in 1609 that the earth and other

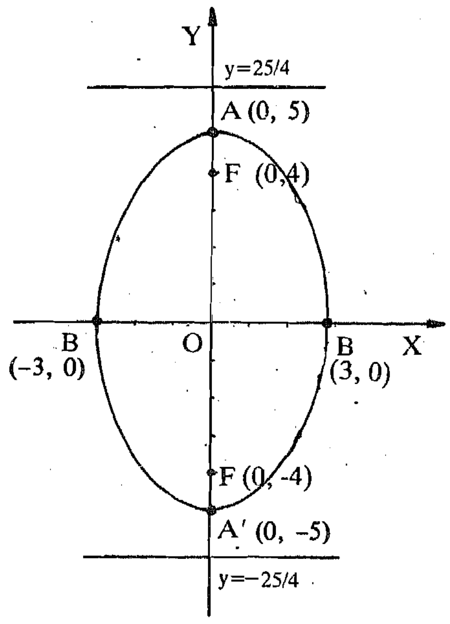


Fig. 14: The ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$.

planets travel in approximately elliptical orbits with the sun at one focus. If the ratio of the shortest to the longest distance of the earth from the sun is 29 to 30, find the eccentricity of the earth's orbit.

E14) Consider the ellipse $\frac{x^2}{4} + \frac{y^2}{4(1-e^2)} = 1$, where e is its eccentricity.

Sketch the ellipses that you get when $e = \frac{1}{4}$, $e = \frac{1}{2}$ and $e = \frac{3}{4}$. Can you find a relationship between the magnitude of e and the flatness of the ellipse?

What E 12 shows you is that a circle is a particular case of an ellipse, and the equation of a circle with centre (0, 0) and radius a is

$$x^2 + y^2 = a^2. \qquad \ldots(12)$$

You may be wondering about the directrices of a circle. In the following note we make an observation about them.

Note: As the eccentricity of an ellipse gets smaller and smaller its directrices get farther and farther away from the centre. Ultimately, when e = 0, the directrices become lines at infinity.

At this point let us mention the parametric representation of an ellipse. As in the case of a parabola, we can express any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in terms of a parameter t. In this case, you can check that any point (x, y) on the ellipse is given by $x = a \cos t$, $y = b \sin t$, where $0 \le t < 2\pi$. Note that the vertices will correspond to $t = 0$ and $t = \pi$.

Let us now look at some important properties of an ellipse.

2.4.2 String Property

In this section we derive a property that characterises an ellipse. Let us go back to Equation (11). Its foci are F(ae, 0) and F'(-ae, 0). Now, take any point P(x, y) on the ellipse. The focal distances of P are PF and PF'. What is their sum? If you apply the distance formula, you will find that PF + PF' = 2a, which is a constant, and is the length of the major axis. This property is true for any ellipse. Let us state it formally.

Theorem 1 a) The sum of the focal distances of any point P on an ellipse is the length of the major axis of the ellipse.

b) Conversely, the set of all points P in a plane such that the sum of distances of P from two fixed points F and F in the plane is a constant, is an ellipse.

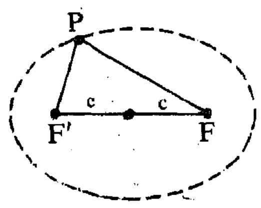


Fig. 15

Proof: We hhve already proved (a). Let us prove (b). We can rotate and translate our coordinate system so that F and F' lie on the x-axis and (0, 0) is the midpoint of the line segment F'F. Then if F has coordinates (c, 0), F' will be given by (–c, 0). Let P(x, y) be an arbitrary point, such that PF + PF' = 2a, where a is a constant (see Fig. 15). Then, by the distance formula we get

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

On squaring and simplifying we get

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

Now, since PFF' forms a triangle, FF' $<$ PF + PF'. Therefore, $2c < 2a$, that is, $c < a$. So we can rewrite the above equation as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b = \sqrt{a^2 - c^2}.$$

Comparing this with (11), we see that the set of all points (x, y) that satisfy the given condition is an ellipse with foci F and F', and majoi axis of length 2a.

Mathematicians often use Theorem 1 as the definition of an ellipse. That is,

an ellipse is the set of all points in a plane for which the sum of the distances from two fixed points in the plane is constant.

This property of an ellipse is also called the string property, because it is the basis for the following construction of an ellipse.

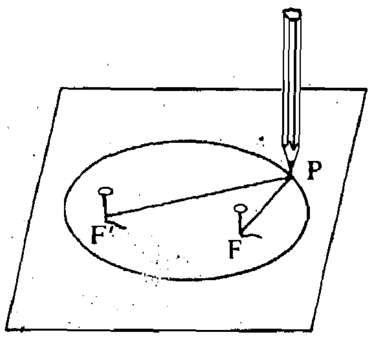


Fig. 16: Sketching an ellipse using n string.

A mechanical method for drawing an ellipse

Take a piece of string of length 2a and fix its ends at the points F and F' (where FF' $<$ 2a) on a sheet of paper (see Fig. 16). Then, with the point of a pencil P, stretch the string into two segments. Now, rotate the pencil point all around on the paper while sliding it along the string. Make sure that the string is taut all the time. By doing this the point P will trace an ellipse with foci F and F' and major axis of length 2a.

Why don't you try this method now?

E15) Use the method we have just given to draw an ellipse with eccentricity $\frac{1}{2}$ and a string of length 4 units. What will the coordinates of its vertices and foci be?

There is another property of an ellipse which makes it very useful in engineering. We shall tell you about it in our discus'sion on tangents.

2.4.3 Tangents and Normals

In Sec. 2.3.2 you studied about a tangent of a parabola. We will discuss the tangents of an ellipse in the same manner.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two distinct points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If $x_1 = x_2 = c$, say, then the equation of PQ is

$$x = c. \quad \ldots(13)$$

Similarly, if $y_1 = y_2 = d$, say, then the equation of PQ is

$$y = d. \quad \ldots(14)$$

If $x_1 \neq x_2$ and $y_1 \neq y_2$, then the equation of PQ is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\Rightarrow \frac{(y - y_1)(y_1 + y_2)}{y_2^2 - y_1^2} = \frac{(x - x_1)(x_1 + x_2)}{x_2^2 - x_1^2}$$

$$\Rightarrow \frac{(y - y_1)(y_1 + y_2)}{b^2} = -\frac{(x - x_1)(x_1 + x_2)}{a^2}, \text{ since } \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2}.$$

$$\Rightarrow \frac{x(x_1 + x_2)}{a^2} + \frac{y(y_1 + y_2)}{b^2} = \frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} + 1, \qquad \ldots(15)$$

since (x_1, y_1) lies on the ellipse.

So, (13), (14) and (15) are the various possibilities for the equation of the line joining P and Q.

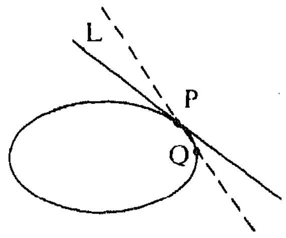


Fig. 17: L is tangent to the ellipse at P.

Now, to get the equation of the tangent at P, we see what happens to the equation of PQ as Q tends to P (see Fig. 17). In this case, can you see from Fig. 13 that we need to consider (15) only? This is because as Q nears P, the line PQ can't be parallel to either axis. Now, as x_2 tends to x_1 and y_2 approaches y_1, (15) becomes, in the limiting case,

$$2\left(\frac{x x_1}{a^2} + \frac{y y_1}{b^2}\right) = 2, \text{ that is,}$$

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1. \qquad \ldots(16)$$

Thus, (16) is the equation of the tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).

Remark 1 may have already suggested this equation to you. The same rule of thumb works here too, that is, replace x^2 by xx_1 and y^2 by yy_1.

For example, the tangent to the ellipse in Example 1 at $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ is

$$2x\left(\frac{1}{2}\right) + 3y\left(\frac{1}{\sqrt{6}}\right) = 1, \text{ that is, } x + \sqrt{\frac{3}{2}}\, y = 1.$$

Now try this exercise on tangents.

E16) Find the equations of the tangents at the vertices and ends of the minor axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let us now find the equation of the normal to (11) at any point (x_1, y_1). If $y_1 = 0$, from E15 you know that the slope of the tangent will be $\pi/2$; and hence, at these points the normal is just the x–axis, that is, $y = 0$. Similarly, you can see that the normal at the points at which $x_1 = 0$ is the y–axis.

Now suppose $x_1 \neq 0$, $y_1 \neq 0$. What is the slope of the normal at (x_1, y_1)? By (16) you know that the slope of the tangent is $-\frac{b^2 x_1}{a^2 y_1}$. Thus, the slope of the normal at (x_1, y_1) to the ellipse is $\frac{a^2 y_1}{b^2 x_1}$ (see (13) of Unit 1). Thus, the equation of the normal at (x_1, y_1) is,

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1), \text{ that is,}$$

$$\frac{y - y_1}{y_1/b^2} = \frac{x - x_1}{x_1/a^2}. \qquad \ldots(17)$$

A diameter of an ellipse is a chord that passes through the centre.

Why don't you try these exercises now?

E17) Find the equation of the tangent and normal at (2, 1) to $x^2 + 4y^2 = 8$.

E18) Show that the tangents at the extremity of a diameter of an ellipse are parallel.

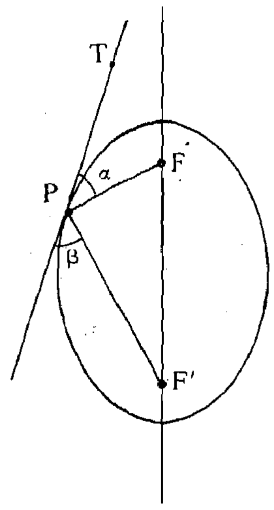


Fig. 18: $\alpha = \beta$

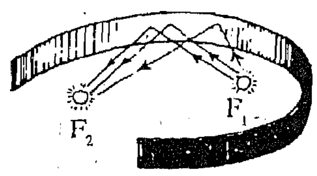


Fig. 19 : Reflected wave property.

Now, in Sec. 2.3.2, we discussed the reflecting property of a parabola. Do you expect it to hold for an ellipse too? The same property is not satisfied, but something like that is.

Reflecting property: The tangent to an ellipse at a point makes equal angles with the focal radii from that point.

That is, if you take the tangent PT at a point P on an ellipse (see Fig. 18), then it makes equal angles with the lines PF and PF'.

We shall not prove this here. We leave the proof to you as an exercise (see Miscellaneous Exercises).

Because of the reflecting property, a ray of light (or sound, or any other type of wave) that is emitted from one focus of a polished elliptical surface is reflected back to the other focus (see Fig. 19). One of the applications of this fact is its use for making whispering galleries

Why don't you try and apply this property now?

E19) An elliptic reflector is to be designed so as to concentrate all the light radiated from a point source on to another point 6 metres away. If the width of the reflector is 10 metres, how high should it be?

Let us now see under what conditions a given line will be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let the line be $y = mx + c$.

Substituting for y in the equation of the ellipse we get

$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$, that is,

$x^2(b^2 + a^2m^2) + 2mca^2x + a^2(c^2 - b^2) = 0.$

$y = mx + c$ will be a tangent to the ellipse if this quadratic equation in x has equal roots. This will happen if its discriminant is zero, that is,

$4m^2c^2a^4 = 4(b^2 + a^2m^2)\, a^2(c^2 - b^2)$

$\Rightarrow c^2 = a^2m^2 + b^2.$ ...(18)

So (18) is the condition that $y = mx + c$ is a tangent to

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Here's an opportunity for you to use this condition.

E20) Check whether $y = x + 5$ touches the ellipse $2x^2 + 3y^2 = 1$.

We shall now stop our discussion on ellipses, and shift our focus to another standard conic.

2.5 HYPERBOLA

Let us now consider the conic we get if $e > 1$ in Equation (1), namely, a hyperbola. Let us define it explicitly.

Definition: A hyperbola is the set of points whose distance from a fixed point F is e (> 1) times its distance from a fixed line L which doesn't pass through F.

There is a similarity between the derivation of the standard equation of an ellipse and that of a hyperbola. In the following exercise we ask you to obtain this equation for a hyperbola.

E21) a) Show that the equation of a conic with focus at (0, 0), directrix x + c = 0 and eccentricity e > 1 is

$$\left(x + \frac{ce^2}{e^2 - 1}\right)^2 - \frac{y^2}{e^2 - 1} = \frac{e^2c^2}{(e^2 - 1)^2}.$$

b) Shift the origin suitably so as to get the equation

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1,$$

where $a = \frac{ec}{e^2 - 1}$, $b = a\sqrt{e^2 - 1}$.

c) What are the coordinates of the focus and the equation of the directrix in the X'Y'-system?

E22) What is the equation of the conic with a focus at (–ae, 0) and directrix $x = -\frac{a}{e}$, where e (> 1) is the eccentricity?

As in the case of the other conics, we can always translate and rotate our coordinate axes so as to get the focus of any given hyperbola as (–ae, 0), and its directrix as $x = -\frac{a}{e}$. Thus, we can always reduce, the equation of any hyperbola to the equation that you obtaihed in E21, namely,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \ldots(19)$$

where $b^2 = a^2(e^2 - 1)$.

So (19) is the standard form of the equation of a hyperbola.

Let us now trace this curve.

2.5.1 Description of Standard Form

Let us study (19) for symmetry and other properties.

Firstly, if $-a < x < a$, then there is no real value of y which satisfies $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Thus, no part of the curve lies between the lines x = –a and x = a.

Secondly, it is symmetric about both the axes, as well as (0, 0). So it is enough to trace it in the first quadrant.

Thirdly, the points (±a, 0) lie on it, and it does not intersect the y–axis.

Finally, since $\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2}$, as x increases so does y. Thus, the hyperbola extends to infihity in both the x and y directions. We have sketched this conic in Fig. 20. You can see that it has two disjoint branches, unlike the other conics.

Looking at the curve's symmetry, do you feel that it has another focus and directrix? You can check that (ae, 0) is another focus with corresponding directrix $x = \frac{a}{e}$

A hyperbola intersects the line joining its foci in two points. These points are called its vertices. The line segment joining its vertices is called its transverse axis. (Some people call the line joining the vertices the transverse axis.) Thus, the points

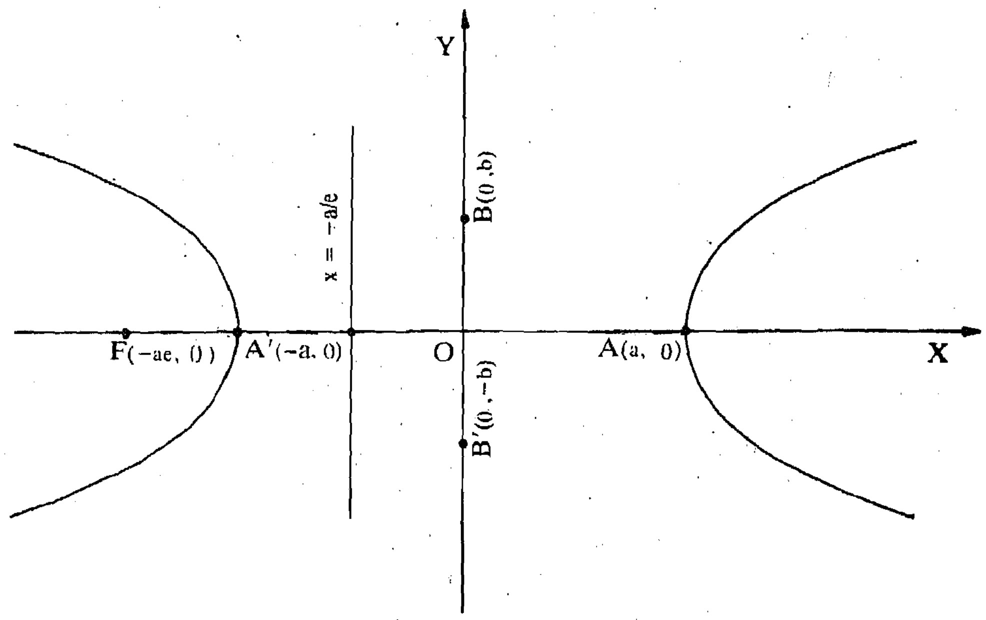


Fig. 20: The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

A' (-a, 0) and A(a, 0) are the vertices of the hyperbola in Fig. 20 and the line segment AA is its transverse axis. The length of this transverse axis is 2a.

The midpoint of the transverse axis is the centre of the hyperbola. In Fig. 20, the line segment BB', where B is (0, b) and B' is (0, -b) is called the conjugate axis of the given hyperbola.

Note that it is perpendicular to the transverse axis and its midpoint is the centre of the hyperbola. The reason it is called the conjugate axis is because it becomes the transverse axis of the conjugate hyperbola of (19), $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$. (We shall not discuss conjugate hyperbolas in this course. If you would like to know more about them, you can look up 'A Textbook of Coordinate Geometry' by Ramesh Kumar).

Let us consider an example of a hyperbola.

Example 3: For the hyperbola $4x^2 - 9y^2 = 36$ find the vertices, eccentricity, foci and the axes.

Solution: We can write the equation in the standard form as $\frac{x^2}{9} - \frac{y^2}{4} = 1$.

Comparing this with (19), we find that a = 3, b = 2.
Therefore, the vertices are $(\pm 3, 0)$.
Now, since $b^2 = a^2(e^2 - 1)$, we find that $e^2 = \frac{13}{9}$. Thus, the eccentricity is $\frac{\sqrt{13}}{3}$. Then the foci are $(\pm ae, 0)$, that is, $(\pm\sqrt{13}, 0)$. The transverse axis is the line segment joining (3,0) and (-3,0), and the conjugate axis is the line segment joining (0, 2) and (0, -2).

Why don't you try some exercises now?

E23) Find the standard equation of the hyperbola with eccentricity $\sqrt{2}$. (Such a hyperbola is called a rectangular hyperbola.)

E24) Find the equation of the hyperbola with centre (0, 0), axes along the coordinate axes, and for which

a) a vertex is at (0, 3) and the transverse axis is twice the length of the conjugate axis;

b) a vertex is at (2, 0) and focus at F $(\sqrt{13}, 0)$.

E25) a) Show that the lengths of the focal radii from any point P(x, y) on the hyperbola (19) are $|ex + a|$ and $|ex - a|$.

b) What is the analogue of (a) for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$?

E26) The more eccentric a hyperbola, the more its branches open out from its transverse axis. True or false? Why?

As in the case of the other conics, we can give a parametric representation of any point on a hyperbola. What do you expect it to be? Does the equation $\sec^2 t - \tan^2 t = 1$, $\forall\, t \in R$ help? Using this, we'can give the parametric form of any point on (19) by $x = a \sec t$, $y = b \tan t$, for $t \in R$ such that $0 \le t < 2\pi$.

Let us now look at some properties of a hyperbola.

2.5.2 String Property

In Theorem 1 you saw that an ellipse is the path traced by a point, the sum of the distances of which from two fixed points is a constant. A similar property is true of a hyperbola. Only, in this case, we look at the difference of the distances.

Theorem 2: a) The difference of the focal distances of any point on a hyperbola is equal to the length of its transverse axis.

b) Conversely, the set of points P such that $|PF_1 - PF_2| = 2a$, where F_1 and F_2 are two fixed points, a is a constant and $F_1F_2 > 2a$, is a hyperbola.

Proof: a) As you know, we can always assume that the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Let P(x, y) be a point on it, and let its foci be F_1 and F_2. Further, let D_1 and D_2 be the feet of the perpendiculars from P on the two directrices (see Fig. 21). In the figure you can see both the cases – when P is on one branch of the hyperbola or the other.

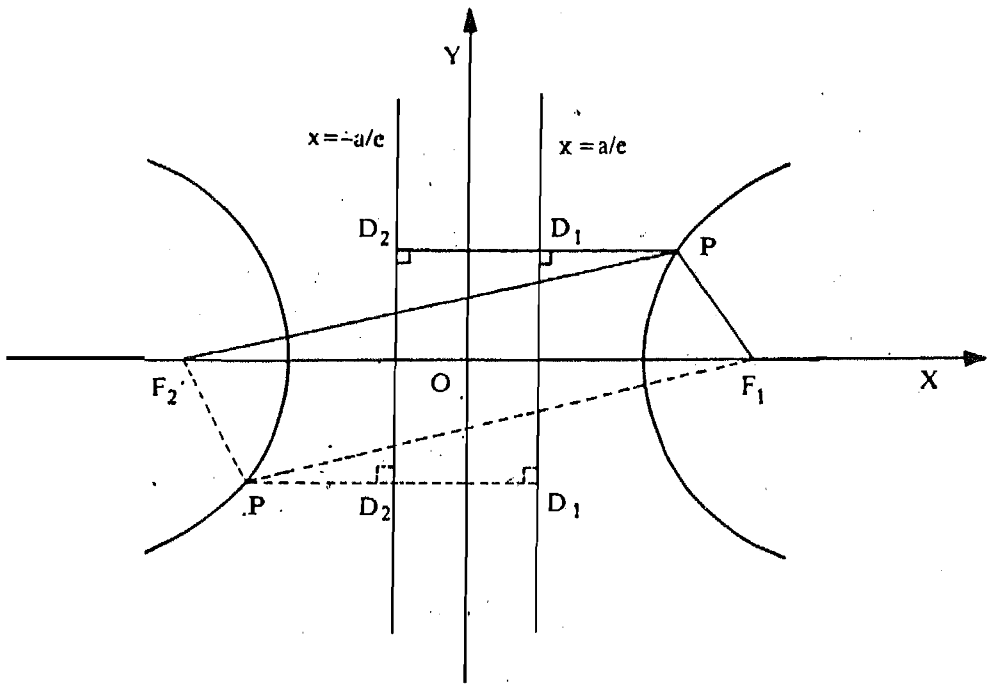


Fig. 21: $|PF_1 - PF_2|$ is conslant.

Now, by definition,

$PF_1 = e\, PD_1$ and $PF_2 = e\, PD_2$. Therefore,

$|PF_1 - PF_2| = e\,|PD_1 - PD_2| = e\left(\frac{2a}{e}\right) = 2a$, the length of the transverse axis.

(You can also prove this by using E25(a).)

We ask you to prove (b) in the following exercise.

E27) Prove Theorem 2(b). Where is the condition $F_1F_2 > 2a$ used?

Theorem 2 is called the string property, for a reason that you may have guessed by now. We can use it to mechanically construct a hyperbola with a string. Since this construction is more elaborate than that of an ellipse, we shall not give it here.

The string property is also the basis for hyperbolic navigation – a system developed during the World Wars for range finding and navigation.

And now we shall see how to find the tangent to a hyperbola.

2.5.3 Tangents and **Normals**

You must have noticed the similarity between the characteristics of an ellipse and a hyperbola. The derivation of the equation of a chord joining two points on a hyperbola and of the equation of a tangent are also obtained as in Sec. 2.4.3. We shall not give the details here. Suffice it to say that all these equations can be obtained from the elliptic case by substituting $-b^2$ for b^2.

Thus, the equation of the tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at a point (x_1, y_1) lying on it is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1. \qquad \ldots(20)$$

Also, the equation of the normal at a point (x_1, y_1) to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$a^2\left(\frac{x - x_1}{x_1}\right) + b^2\left(\frac{y - y_1}{y_1}\right) = 0. \qquad \ldots(21)$$

Similarly, the condition for the straight line $y = mx + c$ to be a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 - b^2$.

Now for a short exercise.

E28) a) Find the tangent and normal to $\frac{x^2}{4} - \frac{y^2}{9} = 1$ at each of its vertices.

b) Is $3y = 2x$ a tangent to this hyperbola? If so, find the point of tangency.

We will now introduce you to some special tangents to a hyperbola. Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the lines $y = \pm \frac{b}{a} x$ (see Fig. 22). These lines satisfy the condition for tangency. They are the pair of tangents to the hyperbola which pass through its centre. Such tangents are called the asymptotes of the hyperbola.

Now, let P (x, y) be a point of the branch of the hyperbola in the first quadrant. Then its distance from $y = \frac{b}{a}x$ is

$$\frac{|ay - bx|}{\sqrt{a^2 + b^2}} = \frac{|a^2y^2 - b^2x^2|}{\sqrt{a^2 + b^2}\,(ay + bx)}$$

$$= \frac{a^2b^2}{\sqrt{a^2 + b^2}\,(bx + b\sqrt{x^2 - a^2})},$$

since $b^2x^2 - a^2y^2 = a^2b^2$ and $y = \frac{b}{a}\sqrt{x^2 - a^2}$.

$$= \frac{a^2b}{\sqrt{a^2 + b^2}\,(x + \sqrt{x^2 - a^2})}$$

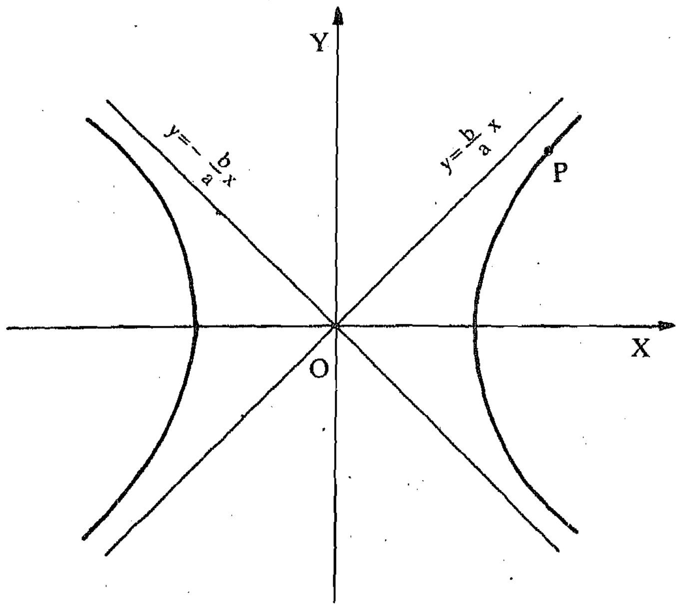


Fig. 22: $y = \pm \frac{b}{a}x$ are the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

As x increases, this distance gets smaller and smaller. Thus, as P tends to infinity along the branch $y = \frac{b}{a}\sqrt{x^2 - a^2}$ of the hyperbola, its distance from $y = \frac{b}{a}x$ tends to zero. But the asymptote never actually intersects the curve. So we say that its point of contact is at an infinite distance.

You can check that the same is true of $y = -\frac{b}{a}x$. In fact, tangents with their points of contact 'at infinity' are called asymptotes.

Try these exercises now.

Asymptotes are discussed in the course MTE-01 in detail.

E29) Find the asymptotes of the rectangular hyperbola $x^2 - y^2 = a^2$. Are they the same for any value of a?

E30) Under what conditions on a and b will the asymptotes of the hyperbola (19) be perpendicular to each other?

So far we have discussed the Cartesian and parametric equations of the conics. But, in some applications the polar equation (see Sec. 1.5) of a conic is more useful. So let us see what this equation is.

2.6 POLAR EQUATION OF CONICS

Consider a conic with eccentricity e. Take a focus F as the pole. We can always rotate the conic so that the corresponding directrix L lies to the left of the pole, as in Fig. 23. Let the line FA, perpendicular to the directrix, be the polar axis and d the distance between F and L.

Let $P(r, \theta)$ be any point on the conic. Then, if D and E are the feet of the perpendiculars from P onto L and FA, we have

$PF = ePD$

$\Rightarrow r = e(d - EF) = e(d - r\cos(\pi - \theta))$

$= e(d + r\cos\theta)$.

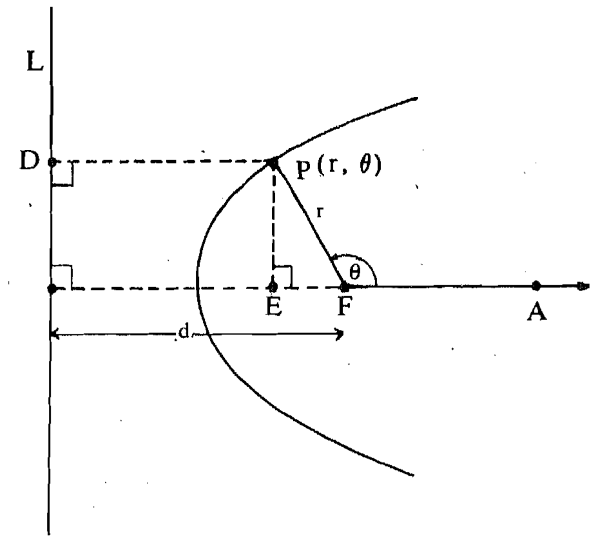


Fig. 23: Obtaining the polar equation of a conic.

$$\Rightarrow r = \frac{ed}{1 - e\cos\theta}, \qquad \text{...(22)}$$

which is the polar equation of a conic.

Can you find the polar equations of the standard conics from this?

For instance, the polar equation of the parabola (3) is $r = \frac{d}{1 - \cos\theta}$, where d = 2a.

Now, suppose you try to derive the polar equation of a conic by taking the directrix L corresponding to a focus F, to the right of F. Will you get (22)? You can check that the equation will now be

$$r = \frac{ed}{1 + e\cos\theta} \qquad \text{...(23)}$$

Let us consider an application of the polar form.

Example 4: In Fig. 24 we show the elliptical orbit of the earth around the sun, which is at a focus F. The point A on the ellipse closest to the sun is called the perihelion; and the point A' farthest from the sun is called the aphelion.

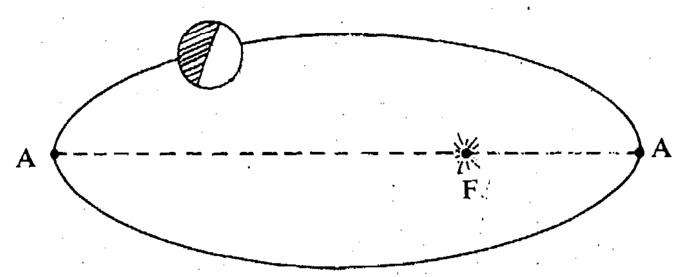


Fig. 24: Aphelion and perihelion on the orbit of the earth around the sun.

Show that the perihelion distance FA and the aphelion distance FA' are given by

$FA = \frac{ed}{1+e}$ and $FA' = \frac{ed}{1-e}$, where d is as given in (22) and (23).

Solution: The polar coordinates of A are (FA, 0) and of A' are (FA', π). Thus, from (23) we find that

$FA = \frac{ed}{1+e}$ and $FA' = \frac{ed}{1-e}$.

You can do the following exercises to see if you have understood what we have done in this section.

E31) Let 2a be the length of the major axis of the ellipse

$r = \frac{ed}{1+e\cos\theta}$. Show that $a = \frac{ed}{1-e^2}$.

E32) A comet is travelling in a parabolic course. A polar coordinate system is introduced in the plane of the parabola so that the sun lies at the focus and the polar axis is along the axis of the parabola, drawn in the direction in which the curve opens. When the comet is 3.0×10^7km from the centre of the sun, a ray from the sun to the comet makes an angle of $\pi/3$ with the polar axis. Find

a) an equation for this parabolic path,

b) the minimum distance of the comet from the sun,

c) the distance between the comet and the sun when $\theta = \frac{\pi}{2}$.

So in this unit you have seen the Cartesian, parametric and polar representations of the various conics for which the foci do not lie on the corresponding directrices. Such conics are called **non-degenerate conics.**

In case a focus of a conic lies on the directrix corresponding to it, the conic we get is called a **degenerate conic.** We will not go into details about them. But let us list the possible types that there are.

A degenerate conic can be of 5 types:

a point, a pair of intersecting lines, a pair of distinct parallel lines, a pair of coincident lines and the empty set.

Now let us do a brief run-through of what we have covered in this unit.

2.7 SUMMARY

In this unit we have discussed the following points:

1) The focus-directrix defining property of conics.

2) A standard form of a parabola is $y^2 = 4ax$. It's focus is at (a, 0), and directrix is $x = -a$. Its eccentricity is 1. The other standard forms are $x^2 = 4ay$, $x^2 = -4ay$ and $y^2 = -4ax$, $a > 0$.

3) The tangent to $y^2 = 4ax$ at a point (x_1, y_1) lying on it is $yy_1 = 2a(x + x_1)$.

4) $y = mx + c$ is a tangent to $y^2 = 4ax$ if $c = \frac{a}{m}$.

5) The normal at (x_1, y_1) to $y^2 = 4ax$ is $y = -\frac{y_1}{2a}x + y_1 + \frac{y_1^3}{8a^2}$.

6) The standard form of the equation of an ellipse with eccentricity e (< 1) is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2(1 - e^2)$.

Its foci are $(\pm ae, 0)$ and directrices are $x = \pm\frac{a}{e}$.

7) The sum of the focal distances of any point on an ellipse equals the length of the major axis of the ellipse.

8) The tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

9) $y = mx + c$ is a tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

10) The normal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{b^2}{y_1}(y - y_1) = \frac{a^2}{x_1}(x - x_1)$.

11) The standard form of the equation of a hyperbola with eccentricity e (>1) is

11) The standard form of the equation of a hyperbola with eccentricity $e(>1)$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$. Its foci are $(\pm ae, 0)$ and directrices are $x = \pm \frac{a}{e}$.

12) The difference of the focal distances of any point on a hyperbola equals the length of its transverse axis.

13) The tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

14) $y = mx + c$ is a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 - b^2$.

15) The normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{a^2}{x_1}(x - x_1) + \frac{b^2}{y_1}(y - y_1) = 0$.

16) The asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm \frac{b}{a}x$.

17) The parametric representation of any point on

a) the parabola $y^2 = 4ax$ is $(at^2, 2at)$, where $t \in R$;

b) the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $(a \cos \theta, b \sin \theta)$, where $0 \le \theta < 2\pi$;

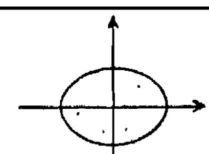
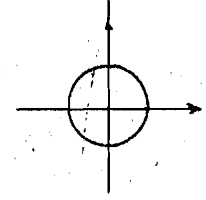
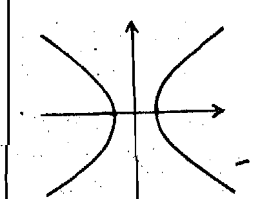
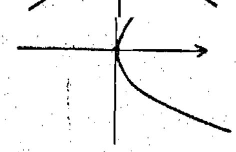
c) the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $(a \sec \theta, b \tan \theta)$, where $0 \le \theta < 2\pi$.

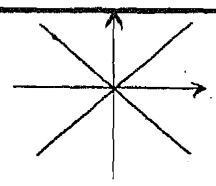
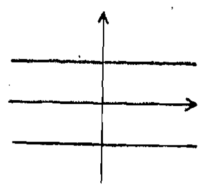
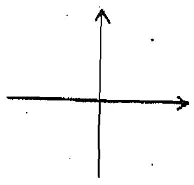
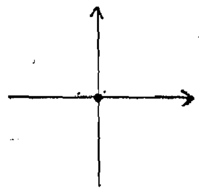
18) The polar equation of a conic with eccentricity e is $r = \frac{ed}{1 - e\cos\theta}$ or $r = \frac{ed}{1 + e\cos\theta}$, depending on whether the directrix being considered is to the left or to the right of the corresponding focus.

Here, d is the distance of the focus from the directix.

19) The list of possible conics is

Table 1: Standard Forms of Conics.

Conic	Standard Equation	Sketch
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a, b > 0$	
Circle	$x^2 + y^2 = a^2,\ a \neq 0$	
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ a, b > 0$	
Parabola	$y^2 = 4px,\ p > 0$	

Pair of intersecting lines	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,\ a, b \neq 0$	
Pair of parallel lines	$y^2 = a^2, a > 0$	
Pair of coincident lines	$y^2 = 0$	
Point conic	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0,\ a, b \neq 0$	

And now you may like to check whether you have achieved the objectives of this unit (see Sec. 2.1). If you'd like to see our solutions to the exercises in this unit, we have given them in the following section.

2.8 SOLUTIONS/ANSWERS

E1) a) The required equation is $(x - 2)^2 + y^2 = 1^2 \frac{(x - y)^2}{2}$

$\Leftrightarrow x^2 + y^2 + 2xy - 8x + 8 = 0.$

b) The required equation is $x^2 + (y - 1)^2 = \frac{1}{4} \frac{(2x + y - 1)^2}{5}$

a $16x^2 - 4xy + 19y^2 + 4x - 38y + 19 = 0.$

E2) In Fig. 25 we have traced the parabola $y^2 = -4ax$, $a>0$. Its vertex is $(0, 0)$ and focus is $(-a, 0)$. In Fig. 26 we have drawn $x^2 = -4ay$, $a > 0$. Its vertex is $(0, 0)$ and focus is $(0, -a)$.

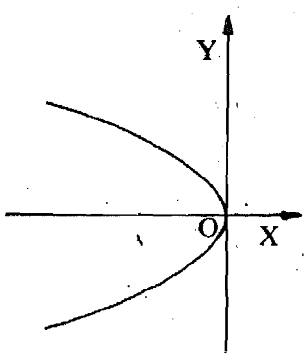


Fig. 25: $y^2 = -4ax$, $a > 0$.

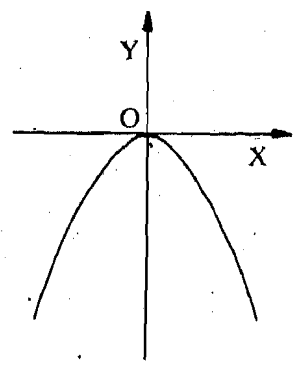


Fig. 26: $x^2 = -4ay$, $a > 0$.

E3) The parabola is similar to the one in Fig. 26.

E4) The parabola is $x^2 = -2y$. Thus, its focus is $(0, -1/2)$, and its latus rectum is $y = -\frac{1}{2}$.

E5) a) The equation is $xx_1 + 2\left(\frac{y + y_1}{2}\right) = 0$, where $x_1 = y_1 = 0$, that is, $y = 0$.

b) The ends of the latus rectum are $(-1, 2)$ and $(-1, -2)$. The tangents at these points are $x + y - 1 = 0$ and $x - y - 1 = 0$.

E6) The axis of the parabola intersects it at the vertex only, but it is not a tangent at the vertex.

E7) The first point to note is that no tangent line can be parallel to the axis of the parabola. For any other m, the line will be a tangent at (x_1, y_1) if $x_1^2 = 4ay_1$, $y_1 = mx_1 + c$, and $x_1^2 = 4a(mx_1 + c)$ has coincident roots. Thus, $y = mx + c$ will be a tangent if $m \neq \tan \pi/2$ and $c = -am^2$.

E8) We have to find the focus of the parabola. We know that $(0.2, 0.5)$ lies on it. Therefore,
$0.25 = 4a(0.2) = 0.8a \Rightarrow a = 0.3125.$

E9) The tangent is $x = 2(y+1)$. Its slope is $\frac{1}{2}$.
Thus, the normal is $y-1 = -2(x-1)$.

E10) The pdint of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

The slope of the normal is $-\frac{1}{m}$.
Thus, its eqbation is

$$y - \frac{2a}{m} = -\frac{1}{m}\left(x - \frac{a}{m^2}\right)$$

$$\Leftrightarrow x + my = a\left(2 + \frac{1}{m^2}\right)$$

E11) The equation can be rewritten as $\frac{x^2}{4} + \frac{y^2}{3} = 1$.
The major axis is of length 4 and lies along the x-axis.
The minor axis is of length $2\sqrt{3}$.
$\therefore (\sqrt{3})^2 = 2^2(1-e^2)$, where e is the eccentricity.

$$\Rightarrow e = \frac{1}{2}$$

The Vertices are $(\pm 2, 0)$ and the foci are $(\pm 1, 0)$. We trace the curve in Fig. 27.

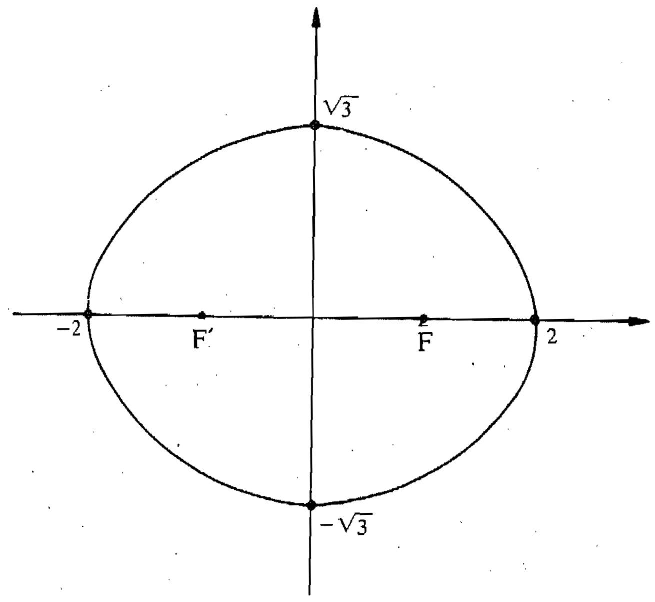


Fig. 27

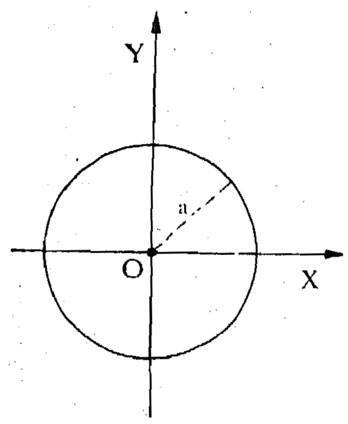


Fig. 28: The circle $x^2 + y^2 = a^2$.

E12) The equation is $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$, that is, $x^2 + y^2 = a^2$. The foci coincide with the centre (0, 0). In this case the ellipse becomes a circle, given in Fig. 28.

E13) The shortest and longest distances will be the distances of the vertices from the focus at which the sun lies.

So, suppose the orbit is $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$ and the sun is at (ae, 0).

The vertices lie at (a, 0) and (–a, 0). Then

$$\frac{a - ae}{a + ae} = \frac{29}{30} \Rightarrow e = \frac{1}{59}.$$

E14) As e grows larger the minor axis becomes smaller, and the ellipse grows flatter. Thus, the eccentricity of an ellipse is a measure of its flatness.

E15) Your ellipse should be similar to the one in Fig. 27. Its vertices will be $(\pm 2,0)$. Its foci will be $(\pm 1,0)$.

E16) The tangent at (a, 0) is $\frac{xa}{a^2} + \frac{y \cdot 0}{b^2} = 1 \Rightarrow x = a$. Similarly, the tangents at $(-a, 0)$, $(0, b)$ and $(0, -b)$ are $x = -a$, $y = b$ and $y = -b$, respectively.

E17) The tangent is $2x + 4y = 8$, that is, $x + 2y = 4$. Therefore, the slope of the normal is 2. Thus, its equation is $y - 1 = 2(x - 2) \Rightarrow y = 2x - 3$.

E18) Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The equation of any diameter will be $y = mx$, for some m, since it passes through (0, 0). Further, if one end of the diameter is (x_1, y_1), then $y_1 = mx_1$. Thus, $(-x_1, -y_1)$ also lies on the ellipse and the line $y = mx$. Thus, it is the other end of the diameter. So, we need to find the tangents at (x_1, y_1) ahd $(-x_1, -y_1)$. They are

$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ and $-\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2}\right) = 1$, respectively. Since both their slopes are $-\frac{b^2x_1}{a^2y_1}$, they are parallel.

E19) We have shown a cross-section of the reflector in Fig. 29. The major axis of this ellipse is 10 metres and its foci lie at $(\pm 3, 0)$. Thus, its eccentricity, $e = \frac{3}{5}$.
So, if h is its height, then $h = 5\sqrt{1 - e^2} = 4$ metres.

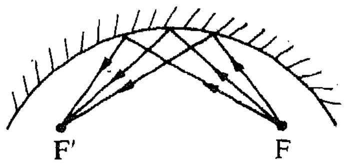


Fig. 29

E20) In this case $a^2 = \frac{1}{2}$, $b^2 = \frac{1}{3}$, $c = 5$ and $m = 1$.

$\therefore c^2 \neq a^2m^2 + b^2$. So the line is not a tangent to the given ellipse.

E21) a) Using (1), we see that

$$x^2 + y^2 = e^2(x + c)^2$$

$$\Rightarrow \left(x + \frac{ce^2}{e^2 - 1}\right)^2 - \frac{y^2}{e^2 - 1} = \frac{e^2c^2}{(e^2 - 1)^2}, \text{ as in Sec. 2.4.}$$

b) Shifting the origin to $\left(-\frac{ce^2}{e^2 - 1}, 0\right)$, the equation in (a) becomes

$$x'^2 - \frac{y'^2}{e^2 - 1} = \frac{e^2c^2}{(e^2 - 1)^2}$$

$$\Rightarrow \frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1, \text{ where } a = \frac{ec}{e^2 - 1} \text{ and } b = a\sqrt{e^2 - 1}.$$

c) In the X'Y'-system the focus is $(-ae, 0)$ and directrix is $x + \frac{a}{e} = 0$.

E22) The required equation is

$$(x + ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

E23) The required equation is

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(2 - 1)} = 1 \Leftrightarrow x^2 - y^2 = a^2.$$

E24) a) The transverse axis lies along the y-axis in this case. Thus, we interchange x and y in (19). Also, the length of the transverse axis is 6. So, the required equation is

$$\frac{y^2}{9} - \frac{x^2}{\left(\frac{3}{2}\right)^2} = 1 \Leftrightarrow y^2 - 4x^2 = 9.$$

b) Here, the transverse axis lies along the x–axis and $a = 2$ and $ae = \sqrt{13}$.

$\therefore e = \frac{1}{2}\sqrt{13}$.

Thus, the required equation is

$$\frac{x^2}{4} - \frac{y^2}{4\left(\frac{13}{4} - 1\right)} = 1 \Leftrightarrow \frac{x^2}{4} - \frac{y^2}{9} = 1$$

E25) a) The foci lie at F(–ae, 0) and F′ (ae, 0). Thus

$PF = \sqrt{(x + ae)^2 + y^2}$.

But, since P lies on the hyperbola.

$y^2 = (x^2 - a^2)(e^2 - 1)$.

$\therefore PF = \sqrt{(ex + a)^2} = |ex + a|$

Similarly, $PF' = |ex - a|$.

b) In this case $PF = ex + a$ and $PF' = a - ex$.
Remember, from Sec. 2.4.2, that $PF + PF' = 2a$.

E26) Suppose you fix the transverse axis and increase the eccentricity of a hyperbola. You will see that the lengths of its latera recta (plural of 'latus rectum') increase. Thus, the given statement is true.

A latus rectum of a hyperbola is the chord through a focus and perpendicular to its transverse axis.

E27) Let $F_1F_2 = 2c$ and let (0, 0) bisect F_1F_2. Then the coordinates of F_1 will be (–c, 0) and of F_2 will be (c, 0). If P is given by (x, y), then $|PF_1 - PF_2| = 2a$

$\Rightarrow |\sqrt{(x + c)^2 + y^2} - \sqrt{(x + c)^2 + y^2}| = 2a$

$= (x^2 + c^2 + y^2 - 2a^2)^2 = \sqrt{(x^2 - c^2)^2 + 2y^2(c^2 + x^2) + y^4}$, on squaring and simplifying.

$\Rightarrow (c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$, again squaring and simplifying.

Now, since $c > a$, we can rewrite this equation as

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1,$$ which is a hyperbola.

E28) a) The vertices are (2, 0) and (–2, 0).
The tangents at these points are $x = 2$ and $x = -2$, respectively. The normals at both these points is the x–axis.

b) Here $a^2 = 4$, $b^2 = 9$, $m = \frac{2}{3}$, $c = 0$.

$\therefore c^2 \neq a^2m^2 - b^2$. $\therefore 3y = 2x$ is not a tangent to the given hyperbola.

E29) $y = \pm x$, which are independent of a: Thus, these are the asymptotes of any rectangular hyperbola.

E30) $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ will be mutually perpendicular iff

$\left(\frac{b}{a}\right)\left(-\frac{b}{a}\right) = -1$, that is, iff $a = b$, that is, iff the hyperbola is rectangular.

E31) As in Example 5, you can show that $FA = \frac{ed}{1 - e}$ and $FA' = \frac{ed}{1 + e}$;
where A and A' are the vertices of the ellipse and F is a focus.

Then $2a = AA' = FA + FA' = \frac{2ed}{1 - e^2}$.

$\therefore a = \frac{ed}{1 - e^2}$.

E32) a) Since the curve is a parabola, its equation is $r = \frac{d}{1 - \cos\theta}$. We also, know that $(3.0 \times 10^7, \pi/3)$ lies on it.

$\therefore d = 3.0 \times 10^7\left(1 - \cos\frac{\pi}{3}\right) = 1.5 \times 10^7$ km.

Thus, the required equation is

$$r = \frac{1.5 \times 10^7}{1 - \cos \theta}$$

b) The minimum distance will be when the comet is at the vertex of the parabola, that is, when $\theta = \pi$.
Thus, the minimum distance

$$= \frac{1.5 \times 10^7}{2} \text{ km.}$$

c) The required distance'is 1.5×10^7 km.