# UNIT 1 SETS

## Structure



## 1.1 INTRODUCTION

Consider the collection of words that are defined in a given dictionary. A word either belongs to this collection or not, depending on whether it is listed in the dictionary or not. This collection is an example of a set, as you will see in Section 1.2. When you start studying any part of mathematics, you will come into contact with one or more sets. This is why we want to spend some time in discussing some basic concepts and properties concerning sets.

In this unit we will introduce you to various examples of sets. Then we will discuss some operations on sets. We will also introduce you to Venn diagrams, a pictorial way of describing sets.

As mentioned earlier, a knowledge of the material covered in this unit is necessary for studying any mathematics course. So please study this unit carefully.

And now we will list the objectives of this unit. After going through the unit, please read this list again and make sure that you have achieved the objectives.

### Objectives

After studying this unit you should be able to

- identify a set;
- represent sets by the listing method, property method and Venn diagrams;
- perform the operations of complementation, union and intersections on sets;
- prove and apply the distributive laws;
- prove and apply De Morgan's laws;
- obtain the Cartesian product of two or more sets.

## 1.2 SETS

You may have often come across categories, classes or collections of objects. In mathematics, a **well-defined** collection of objects is called a **set.** The adjective 'well-defined' means that given an object it should be possible to decide whether it belongs to the collection or not. For example, the collection of all women pilots is a set. This is because any person is either a woman pilot or not, and accordingly she/he does or does not belong to the collection. On the other hand, the collection of all intelligent human beings is not a set. Why? Because, a particular human being may seem intelligent to one person and not to another. In other words, there is no clear criterion for deciding on who is intelligent and who is not. So, the collection is not well-defined.

Now we give some more examples of sets which you may have come across. We will also use them a great deal in this course.

i) The set of natural numbers, denoted by **N**.

ii) The set of integers, denoted by **Z**.

iii) The set of rational numbers, denoted by **Q**.

iv) The set of real numbers, denoted by **R**.

In the next unit we will be studying another set, namely, the set of complex numbers, denoted by C.

You may like to try this exercise now.

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E1) Which of the collections mentioned below are sets?

a) The collection of all good people in India.

b) The collection of all those people who have been to Mars.

c) The collection of prime numbers.

d) The collection of even numbers.

e) The collection of all rectangles that are not squares.

A **prime number** is a natural number other than one, whose only factors are one and itself.

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An object that belongs to a set is called **an element or member** of that set. For example, 2 is an element of the set of natural numbers.

We normally use capital letters A, B, C, etc., to denote sets. The small letters a, b, c, x, y, etc., are usually used to denote elements of sets.

We symbolically write the statement 'a is an element of the set A' as $a \in A$.

The symbol $\in$ stands for 'belongs to'. It was suggested by the Italian mathematician Peano (1858-1932)

If a is not an element of A or, equivalently, a does not belong to A, we write it as $a \notin A$.

So, for example, if A is the set of prime numbers, then $5 \in A$ and $9 \notin A$.

Try the following exercise now.

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E2) Which of the following statements are true?

a) $0.2 \in \mathbf{N}$

b) $2 \notin \mathbf{N}$

c) $\sqrt{2} \in \mathbf{R}$

d) $\sqrt{2} \in \mathbf{Q}$

e) $\sqrt{-1} \notin \mathbf{R}$

f) Any circle is a member of the set in E1 (e).

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Now, you know that a number is either rational or irrational, but not both. So, what will the set of all numbers that are rational as well as irrational be? It will not have any element.

The set which has no elements is called the **empty set** (or the **void set**, or the **null set**). It is denoted by the Greek letter $\phi$ (phi).

A set, which has at least one element is called a **non-empty set**. We usually describe a non-empty set in two ways—the **listing method** and the **property method.**

In the first method we list all the members of the set within curly brackets. For instance, the set of all natural numbers that are factors of 10 is $\{1, 2, 5, 10\}$.

But what if the set has too many elements to be able to write them all down? In this case we list some of the elements of the set, enough to exhibit some pattern which its elements follow. For example, the set N of natural numbers can be described as

$\mathbf{N} = \{1, 2, 3, \ldots\}$,

and the set of all even numbers lying between 10 and 100 is

$\{12, 14, 16, \ldots, 98\}$.

This method of representing sets is called the **listing method** (or **tabular** method, or roster method).

In the second method of describing a set we describe its elements by means of a property possessed by all of them. As an example, consider the set S of all natural numbers which are multiples of 5. This set S can be written in the form

$$S = \{x \mid x \in N \text{ and } x \text{ is a multiple of } 5\}. \qquad \ldots\ldots\ldots (1)$$

The vertical bar after x denotes 'such that'. (Some authors use : instead of | for such that.)

So (1) states that S is the set of all x such that x is a natural number and x is a multiple of 5.

We can also write this in a slightly shorter form as

$S = \{x \in N \mid x \text{ is a multiple of } 5\}$, or as

$S = \{5n \mid n \in N\}$.

This method of describing the set is called the **property method** or the set - builder method.

In some cases we can use either method to describe the set under consideration.
For instance, the set E, of all natural numbers less than 10, can be described as

$E = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (by the listing method), or
$E = \{x \mid x \text{ is a natural number less than } 10\}$ (by the property method).

You can see that both these sets are the same, since both of them have precisely the same elements. This example leads us to the following definition.

**Definition :** Two sets S and T are called **equal**, denoted by $S = T$, iff every element of S is an element of T and every element of T is an element of S.

Now, while describing a set by the listing method, you must keep the following important remarks in mind.

**Remark 1:** The set $\{1, 2, 3, 3\}$ is the same as $\{1, 2, 3\}$. That is, while listing the elements in a set we do not gain anything by repeating them. By convention, we **do not repeat them.**

**Remark 2:** Consider the sets $\{1, 2, 3\}$ and $\{2, 1, 3\}$. Are they equal or not? You can see that every element of the first set belongs to the second, and vice versa. Therefore, these sets are equal. This example shows that **changing the order in which the elements are listed does not alter the set.**

We would also like to emphasize an observation about the property method.

**Remark 3:** There can be several properties that define the same set. For example,

$\{x \mid 3x - 1 = 5\}$

$= \{x \mid x \text{ is an even prime number}\}$.

**You may like to do the following exercises now.**

E3) Describe the following sets by the listing method.

a) $\{ x \mid x$ is the smallest prime number $\}$

b) $\{ x \mid x$ is a divisor of 12$\}$

c) $\{ x \in \mathbf{Z} \mid x^2 = 4\}$

d) $\{ x \mid 3x - 5 = 19\}$

e) the set of all letters in the English alphabet.

E4) Describe the following sets by the property method.

a) $\{1, 4, 9, 16, ...\}$

b) $\{2, 3, 5, 7, 11, 13, 17, ...\}$

c) $\{..., -6, -4, -2, 0, 2, 4, 6, ...\}$

d) $\phi$

E5) Give an example of a non-empty set which can be represented only by the property method.

While solving E 3 you would have come across sets consisting of exactly one element. Such a set is called a **singleton.** The singleton with element x is usually written as $\{x\}$.

**Remark 4:** The element x is not the same as the set $\{x\}$. In fact, $x \in \{x\}$.

A set which has a finite number of elements is called a **finite set.** By convention, the empty set is considered to be a finite set.

A set which is not finite is an infinite set.
Some examples of infinite sets are **N**, **Q**, **R** and the set of points on a given line.

The following exercise will help you in getting used to the notion of finite and infinite sets.

E6) Which of the following sets are finite, and which are infinite?

a) $\mathbf{Z}$,

b) $\phi$,

c) the solution set of $2x + 5 = 7$,

The **solution set** of an equation is the set of solutions of the equation.

d) the set of points on the circumference of a circle,

e) the set of stars in the sky.

Now, given any two real numbers a and b, you know that either $a < b$ or $a = b$ or $a > b$. Is there a similar relationship between sets? Let us see.

## 1.3 SUBSETS

In this section we shall see what we mean by the terms 'is contained in' and 'contains'.

Consider two sets A and B, where

A = the set of all students of IGNOU, and

B = the set of all female students of IGNOU.

Every female student of IGNOU is a student of IGNOU. So, each element of B is also an element of A. In such a situation we say that B is contained in A.

Of course, IGNOU also has some male students!
So, there is an element x in A such that x does not belong to B. Mathematically, we write this as

$\exists$ x $\in$ A such that x $\notin$ B.

In this situation we say that B is properly contained in A.

'$\exists$ ' denotes 'there exists'. For a discussion on this, see the appendix to this block.

In general, we have the following definitions.

Definitions : A set A is a subset of a set B if every element of A belongs to B, and we denote this fact by **A $\subseteq$ B.**

In this situation, we also say that A **is contained in** B, or that B **contains** A, denoted by **B $\supseteq$ A.**

If A $\subseteq$ B and $\exists$ y $\in$ B such that y $\notin$ A, then we say that A is a **proper subset** of B (or A **is properly contained** in B). We denote this by **A $\subsetneq$ B.**

If X and Y are two sets such that X has an element x which does not belong to Y, then we say that X **is not contained** in Y. We denote this fact by **X $\not\subseteq$ Y.**

Let us look at a few examples of what we have just defined.

Consider the set A = {1, 2, 3}. Is A $\subseteq$ A ? Since every element of A is in A, we find that A $\subseteq$ A.

In fact, this is true for any set. In other words, any set is a subset of itself. But note that no set is a proper subset of itself.

Now consider the sets A = {1, –1} and B = {0, 1, 2}.

We find that $\exists$ (–1) $\in$ A such that (–1) $\notin$ B. $\therefore$ A $\not\subseteq$ B.

Similarly, B $\not\subseteq$ A.

Note that given any two sets A and B, one and only one of the following possibilities is true.

i) A $\subseteq$ B, or

ii) A $\not\subseteq$ B.

Using this fact we can prove by contradiction (see the appendix to this block) that

**the empty set $\phi$ is a subset of every set.**

$\phi \subseteq$ A for any set A.

To prove this, consider any set A. Suppose $\phi \not\subseteq$ A. Then there must be some element in $\phi$ which is not in A. But this is not possible since $\phi$ has no elements. Thus, we reach a contradiction. Therefore, what we assumed must be wrong. That is, $\phi \not\subseteq$ A is false. Thus, $\phi \subseteq$ A, for any set A.

Try the following exercises now. While doing them remember that to show that A $\subseteq$ B, for any sets A and B, you must show that if a $\in$ A then a $\in$ B,

i.e., a $\in$ A $\Rightarrow$ a $\in$ B.

'$\Rightarrow$' denotes 'implies' (see the block appendix also).

Also, to show that A $\not\subseteq$ B, you must show that $\exists$ x $\in$ A such that x $\notin$ B.

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E7) Write down all the subsets of {1, 2, 3}. How many of these contain

a) no element, b) one element, c) two elements, d) three elements?

The set of all subsets of a set A is called the **power set** of A.

E8) Show that if A $\subseteq$ B and B $\subseteq$ C, then A $\subseteq$ C. This shows that '$\subseteq$' is a **transitive** relation.

E9) Show that '$\not\subseteq$' is not a transitive relation. For this, you need to find an example of three sets A, B and C such that A $\not\subseteq$ B, B $\not\subseteq$ C, but A $\subseteq$ C.

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Now, let us go back for a moment to the point before Remark 1, where we defined equality of sets. Let us see what equality means in terms of subsets. Consider the sets
A = the set of even natural numbers less than 10, and
B = {2, 4, 6, 8 }.

They are equal since every member of A is a member of B, and vice versa. That is, A $\subseteq$ B and B $\subseteq$ A.

Thus, A = B $\Leftrightarrow$ ( A $\subseteq$ B and B $\subseteq$ A ).

Now, for some interesting exercises !

E10) Consider the sets A = { x | x + 1 = 3}, B = {1,2} and C = the set of all even numbers that are prime. What is the relationship between

i) A and B ?

ii) A and C ?

E11) If A = B and B ⊇ C, what is the relationship between A and C ?

So far you have seen two methods of representing sets. There is yet another way of depicting sets and the relationships between them. This is what we discuss in the next section.

## 1.4 VENN DIAGRAMS

It is often easier to understand a situation if we can represent it graphically. To ease our understanding of many situations involving sets and their relationships, we represent them by simple diagrams, called **Venn diagrams.** An English logician, John Venn (1834 - 1923), invented them. To be able to draw a Venn diagram, you would need to know what a universal set is.

A universal set is not unique.

In any situation involving two or more sets, we first look for a convenient large set which contains all the sets under discussion. We call this large set a **universal set** and denote it by U. Clearly, U is not unique. For example, if we are talking about the set D of women directors, and the set S of women scientists, then we can take our universal set U to be the set of all earning women. This is because U contains D as well as S. But, we can also take U to be the set of all women. This would also serve the same purpose.

Again, if we wish to work with the sets of integers and rational numbers, we could take the set of real numbers as our universal set. We could also take **Q** as our universal set, since it contains both **Z** and **Q**.

We usually take our universal set to be just large enough to contain all the sets under consideration.

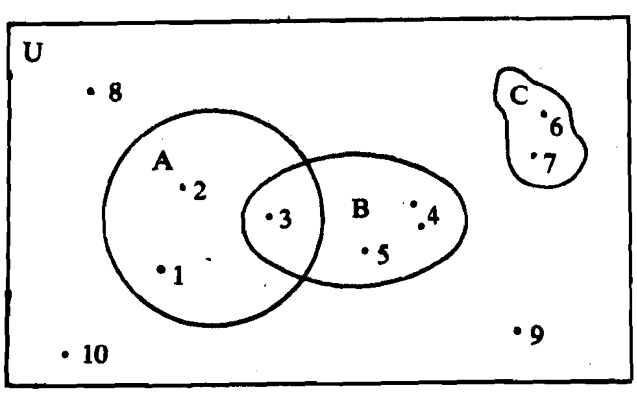
Now, let us see how to draw a Venn diagram. Suppose we are discussing various sets A,B,C, ..... We choose our universal set U. So, A ⊆ U, B ⊆ U, C ⊆ U , and so on. We show this situation in a Venn diagram as follows :

The interior of a rectangle represents U. The subsets A, B, C, etc., are represented by the interiors of closed regions lying completely within the rectangle. These regions may be in the form of a circle, ellipse or any other shape. To clarify what we have just said, consider the following example.

**Example 1 :** Draw a Venn diagram to represent the sets

U = {1,2,...,10}, A = {1,2,3}, B = {3,4,5}, C = {6,7}.

**Solution :** See Figure 1.




$y_2$).

**Figure 1 : A Venn diagram**

We have denoted A by a circle, B by an ellipse which intersects A, and C as another closed region. The points 8, 9 and 10 don't lie in any of A, B or C. All these regions and points lie in the universal set U, which is represented by the outer rectangle.

Note that 3 belongs to both A and B. Therefore, it lies in the circle as well as the ellipse. Also note that A and C do not have any elements in common. Therefore, the regions representing them do not cut each other. For the same reason the regions representing B and C do not cut each other.

Of course, we could have drawn B and C in the shape of circles also.

Now, consider the following situation : A and B are two sets such that $A \subsetneq B$, that is, A is a proper subset of B. What will a Venn diagram corresponding to this situation look like? Well, we can just take B to be our universal set. Then the Venn diagram in Figure 2 is one such diagram. If we take another set U that properly contains B, as our universal set, then we get the Venn diagram in Figure 3.

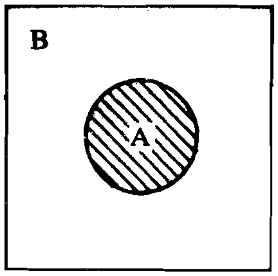



Figure 2

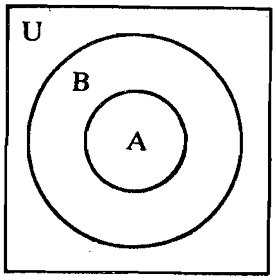



Figure 3

Try this exercise now.

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E12) How would you represent the following situation by a Venn diagram?
The set of all rectangles, the set of all squares and the set of all parallelograms.

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Now that you are familiar with Venn diagrams, let us discuss the various operations on sets. During the discussion we will be using Venn diagrams off and on.

## 1.5 OPERATIONS ON SETS

You must be familiar with the basic operations on real numbers—addition, subtraction, multiplication and division. In using these we combine two real numbers at a time, in different ways, to obtain another real number. In a similar way we can obtain new sets by applying certain operations to two given sets at a time. In this section we shall discuss the operations of complementation, intersection and union.

### 1.5.1 Complementation

Consider the sets **N** and {0, 1}. There are elements of **N** that do not belong to {0, 1}, like 2, 3, etc. The set of these elements is the complement of {0, 1} in **N**, according to the following definition.

**Definition:** Let A and B be two sets. The **complement of A in B**, denoted by $B \setminus A$, is the set $\{x \in B \mid x \notin A\}$.

$A \setminus B \subsetneq A$

Similarly, $A \setminus B = \{x \in A \mid x \notin B\}$.

If B is the universal set U, then $B \setminus A$ is $U \setminus A$. This set is called the **complement of the set A**, and is denoted by $A'$ or $A^c$.

The unshaded area in Figure 2 denotes the set $B \setminus A$ (or $A^c$, since B = U in this case). This diagram shows us that **$x \in A^c$ if and only if $x \notin A$.**

Note that in the situation of Figure 2, $A \setminus B = \phi$, since every point of A is a point of B.

Try this exercise now.

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E13) a) Represent the following sets in a Venn diagram : The set P of all prime numbers, the set **Z** and the set $\mathbf{Q} \setminus \mathbf{Z}$.

b) Is the set $\mathbf{Z} \setminus P$ finite or infinite?

E14) Let A be any set. What will $A \setminus A$, $\phi \setminus A$, $A \setminus \phi$ and $(A^c)^c$ be ?

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Let us now consider another operation on sets, namely, the intersection of two or more sets.

### 1.5.2 Intersection

Let A and B be two subsets of a universal set U. The **intersection** of A and B will be the set of points that belong to both A and B. This is denoted by **A ∩ B**.

Thus, $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$.

To clarify this idea consider the following example.

**Example 2:** Let S be the set of prime numbers and T be the set $\{x \in \mathbf{Z} \mid x \text{ divides } 10\}$. What is $S \cap T$?

**Solution :** We take **Z** to be our universal set. Then $S \cap T$ = set of those integers that are prime numbers as well as divisors of 10 = { 2, 5 }.

You should be able to do the following exercise now.

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E15) Obtain $\mathbf{Z} \cap \mathbf{Q}$, $\mathbf{Q} \cap \mathbf{Z}$, $\mathbf{Z} \cap \mathbf{Z}$ and $\mathbf{N} \cap \phi$.

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While solving E 15 you may have noticed certain facts about the operation of intersection. We explicitly list them in the following theorem.

**Theorem 1:** For any two sets A and B,

a) $A \cap B \subseteq A$

b) $A \cap B \subseteq B$

c) $A \subseteq B \Rightarrow A \cap B = A$

d) $A \cap A = A$

e) $A \cap \phi = \phi$

f) $A \cap B = B \cap A$

(f) says that the operation of intersection of sets is commutative.

g) $A \setminus B = A \cap B^c$.

h) if C is a set such that $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.

**Proof :** We will prove (a) and (b), and leave you to check the rest (see E 16).

So, let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Thus, $A \cap B \subseteq A$ and $A \cap B \subseteq B$. So we have proved (a) and (b).

Now, using (a) and (b), try to do the following exercise.

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E16) Prove (c) – (h) of Theorem 1.

E17) Is Theorem 1 (h) true if we replace '$\subseteq$' by '$\subsetneq$' everywhere? Why ?

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Now, consider the set $\mathbf{Q}^-$ of negative rational numbers and the set $\mathbf{Q}^+$ of positive rational numbers. Then $\mathbf{Q}^- \cap \mathbf{Q}^+ = \phi$. This shows that $\mathbf{Q}^-$ and $\mathbf{Q}^+$ are disjoint sets, a term we now define.

**Definition :** Let A and B be two sets such that $A \cap B = \phi$. Then we say that they are **mutually disjoint** (or **disjoint**).

Now let us represent the intersection of sets by means of Venn diagrams. The shaded region in Figure 4 represents the set $A \cap C$, which is non-empty, as you can see. Also note that the regions representing A and B do not overlap, that is, $A \cap B = \phi$. From this diagram we can also see that neither is $A \subseteq C$, nor is $C \subseteq A$. Further, both $C \setminus A$ and $A \setminus C$ are non-empty sets. See how much information a Venn diagram can convey!

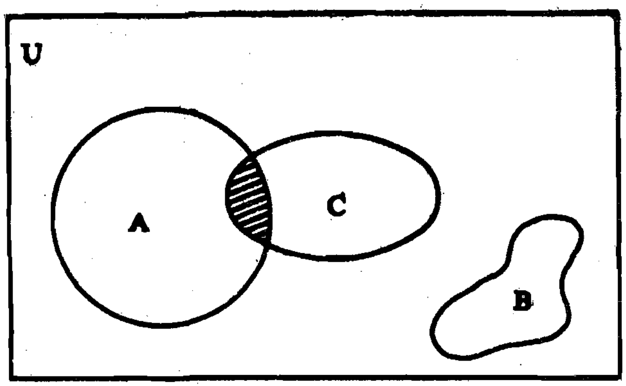


Figure 4

Now, go back to Figure 2 for a moment. What situation does it represent? It shows two sets A and B, with $A \subsetneq B$, that is, A is a proper subset of B. Then the shaded area shows $A \cap B = A$.

Try the following exercise now.

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E18) Let U = Z, A = {1, 3, 5} B = the set of odd integers. Draw the Venn diagram to represent this situation, and shade the portion $A \cap B$.

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So far we have considered the intersection of two sets. Now let us define the intersection of 3, 4 or more sets.

**Definition: The intersection of n** sets $A_1, A_2, \ldots, A_n$ is defined to be the set

$\{x \mid x \in A_i \text{ for every } i = 1, \ldots, n\}$.

This is denoted by $A_1 \cap A_2 \cap \ldots \cap A_n$ or $\bigcap_{i=1}^{n} A_i$.

Let us look at an example involving the intersection of 3 sets.

**Example 3 :** Let A, B and C be the sets of multiples of 3, 6 and 10 in N, respectively. Obtain $A \cap B \cap C$.

**Solution :** $A \cap B \cap C$ will consist of all those natural numbers that belong to A, B and C. Thus,

$$A \cap B \cap C = \{x \in N \mid 3, 6 \text{ and } 10 \text{ divide } x\}$$
$$= \{x \in N \mid 30 \text{ divides } x\}$$
$$= \{30n \mid n \in N\}.$$

Note that 30 is the lowest common multiple (*l* c m) of 3, 6 and 10.

Try the following exercise now.

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E19) Let A = { 1, 2, 3, 4 }, B = { 3, 4, 5, 6 } and C = { 1, 4, 7, 8 }. Determine $A \cap B \cap C$. Also verify that

a) $A \cap B \cap C = (A \cap B) \cap C$,

b) $A \cap B \cap C = A \cap (B \cap C)$.

E20) If $A = \{6n \mid n \in N\}$ and $B = \{15n \mid n \in N\}$, then define $A \cap B \cap A$.

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What you have shown in E 19 is not only true for those sets. It is true for any three sets A, B and C. This property of $\cap$ is called **associativity**. Using this property we can obtain the intersection of any n sets by intersecting any two adjacent sets at a time.

For example, if A, B, C, D are 4 sets, then

$$A \cap B \cap C \cap D = [(A \cap B) \cap C] \cap D$$
$$= A \cap [(B \cap C) \cap D]$$
$$= (A \cap B) \cap (C \cap D).$$

Let us now look at another operation on sets.

### 1.5.3 Union

Suppose we have two sets $A = \{x \in \mathbf{R} \mid x \le 10\}$ and $B = \{x \in \mathbf{R} \mid x \ge 10\}$. Then any element of **R** belongs to either A or B, because any real number will be either less than or equal to 10 or greater than or equal to 10; and 10 will belong to both A and B. In this case we will say that **R** is the union of A and B.

'≤' denotes 'less than or equal to' and '≥' denotes 'greater than or equal to'.

In general, we have the following definition.

**Definition :** Let A and B be two sets. The set of all those elements which belong either to A or to B or to both A and B is called the **union** of A and B. It is symbolically written as $A \cup B$.

Thus

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

Before going further we make a remark.

**Remark 5 :** Since $A \cup B$ contains all the elements of A as well as B, it follows that

$A \subseteq A \cup B$, $B \subseteq A \cup B$.

In fact, $A \cap B \subseteq A \subseteq A \cup B$ and $A \cap B \subseteq B \subseteq A \cup B$.

Using E 8, this shows that $A \cap B \subseteq A \cup B$.

Now let us look at an example.

**Example 4:** Find $\mathbf{N} \cup \mathbf{Z}$.

**Solution :** $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $\mathbf{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

We want to find

$\mathbf{N} \cup \mathbf{Z} = \{x \mid x \in \mathbf{N} \text{ or } x \in \mathbf{Z}\}$.

But $\mathbf{N} \subseteq \mathbf{Z}$. Thus, $x \in \mathbf{N} \Rightarrow x \in \mathbf{Z}$. This immediately tells us that $\mathbf{N} \cup \mathbf{Z} = \{x \mid x \in \mathbf{Z}\} = \mathbf{Z}$.

'⇔' denotes 'equivalence' or 'implies and is implied by' (see appendix to block).

Example 4 is a particular case of the general fact that $A \subseteq B \Leftrightarrow A \cup B = B$.

You can use this fact while solving the following exercises.

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E21) For any three sets A, B and C, show that

a) $A \cup A = A$,

b) $A \cup B = B \cup A$, that is, the operation of union is commutative.

c) $A \cup \phi = A$.

d) if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

E22) Let U be the real line **R**, $A = \{x \in \mathbf{R} \mid 0 \le x \le 1\}$ and $B = \{x \in \mathbf{R} \mid 1 \le x \le 3\}$. Determine $A \cup B$.

E23) What can you say about A and B if $A \cup B = \phi$ ?

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It is easy to visualise unions of sets by Venn diagrams. Consider Figure 5. In this diagram we see four sets A, B, C and D, and the universal set U. The shaded area represents $A \cup B$.

$C \cup D$ is the area enclosed by both C and D, which is just D, since $C \subseteq D$. Can you find the area in Figure 5 that represents $A \cup B \cup D$ ? You may be able to, once we have defined the union of 3 or more sets.

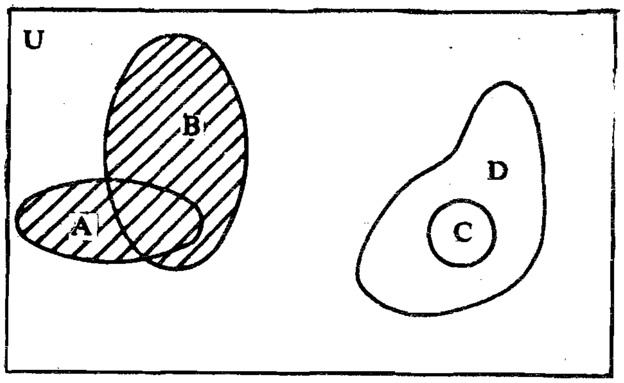


Figure 5

**Definition :** The **union of n sets** $A_1, A_2, ..., A_n$ is the set $\{ x \mid x \in A_i \text{ for some i such that } 1 \le i \le n \}$.

This is denoted by $A_1 \cup A_2 \cup ... \cup A_n$ or $\bigcup_{i=1}^{n} A_i$.

So now you can see that $A \cup B \cup D$ is represented in Figure 5 by the shaded area along with the area enclosed by D.

Now, let us consider $A \cup B \cup C$, where $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$, $C = \{1\}$. Then $A \cup B \cup C = \{1, 2, 3, 4, 5\}$.

You will find that this is the same as $(A \cup B) \cup C$ as well as $A \cup (B \cup C)$.

You may also like to verify that

$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$.

These statements are particular cases of the general facts that we ask you to prove in the following exercise.

E24) a) For the sets A, B and C in Example 3, show that

$$A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$$

b) For any two sets A and B, show that $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$. (We depict this situation in the Venn diagram in Fig.6.)

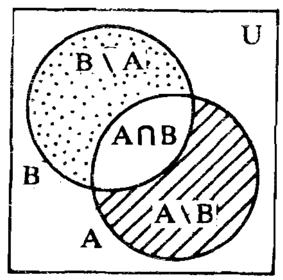


Figure 6

What you have shown in E 24 (a) is true for any three sets, that is, the operation of union of sets is associative. Consequently, we can obtain the union of any number of sets by taking the union of any two adjacent ones at a time. For example, if A, B, C, D are four sets, then

$$A \cup B \cup C \cup D = [(A \cup B) \cup C] \cup D$$
$$= A \cup [(B \cup C) \cup D]$$
$$= (A \cup B) \cup (C \cup D).$$

By now you must be familiar with the operations that we have discussed in this section. Now let us try and prove some laws that relate them.

## 1.6 LAWS RELATING OPERATIONS

In this section we shall discuss two sets of laws that relate unions with intersections, Cartesian products with unions and Cartesian products with intersections. We start with the distributive laws.

### 1.6.1 Distributive Laws

You must be familiar with the law of distributivity that connects the operations of multiplication and addition of real numbers. It is

$a \times (b + c) = a \times b + a \times c \;\; \forall \; a, b, c \in \mathbf{R}.$

Whereas we have only one law for numbers, we have two laws of distributivity for sets, which we will now state.

**Theorem 2 :** Let A, B and C be three sets. Then

a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

**Proof :** We will prove (a) and ask you to prove (b) (see E 25).

a) We will show that

$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and ......... (2)

$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C).$ ......... (3)

Now, $x \in A \cap (B \cup C)$

$\Rightarrow x \in A$ and $x \in B \cup C$

$\Rightarrow x \in A$ and ($x \in B$ or $x \in C$)

$\Rightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

$\Rightarrow x \in A \cap B$ or $x \in A \cap C$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

So, we have proved the first inclusion, (2). To prove (3), let

$x \in (A \cap B) \cup (A \cap C)$.

$\Rightarrow x \in A \cap B$ or $x \in A \cap C$

$\Rightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

$\Rightarrow x \in A$ and ($x \in B$ or $x \in C$)

$\Rightarrow x \in A$ and $x \in B \cup C$

$\Rightarrow x \in A \cap (B \cup C)$

So we have proved (3).

Note that our argument for proving (3) is just the reverse of our argument for proving (2). In fact, we could have combined the proofs of (2) and (3) as follows :

$x \in A \cap (B \cup C)$

$\Leftrightarrow x \in A$ and $x \in B \cup C$

$\Leftrightarrow x \in A$ and ($x \in B$ or $x \in C$)

$\Leftrightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

$\Leftrightarrow x \in A \cap B$ or $x \in A \cap C$

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

This proves (a).

Now try to solve the following exercise, using a two - way implication.

E25) Prove (b) of Theorem 2.

Let us verify Theorem 2 in the following situation.

**Example 5 :** Verify the distributive laws for the sets **N**, **Q** and **R** in place of A, B and C.

**Solution :** We first show that $\mathbf{N} \cap (\mathbf{Q} \cup \mathbf{R}) = (\mathbf{N} \cap \mathbf{Q}) \cup (\mathbf{N} \cap \mathbf{R})$.

Now $\mathbf{Q} \cup \mathbf{R} = \mathbf{R}$, since $\mathbf{Q} \subseteq \mathbf{R}$.

Therefore, $\mathbf{N} \cap (\mathbf{Q} \cup \mathbf{R}) = \mathbf{N} \cap \mathbf{R} = \mathbf{N}$, since $\mathbf{N} \subseteq \mathbf{R}$.

Also $\mathbf{N} \cap \mathbf{Q} = \mathbf{N}$ and $\mathbf{N} \cap \mathbf{R} = \mathbf{N}$.

Therefore, $(\mathbf{N} \cap \mathbf{Q}) \cup (\mathbf{N} \cap \mathbf{R}) = \mathbf{N} \cup \mathbf{N} = \mathbf{N}$.

Thus, $\mathbf{N} \cap (\mathbf{Q} \cup \mathbf{R}) = (\mathbf{N} \cap \mathbf{Q}) \cup (\mathbf{N} \cap \mathbf{R})$.

Now, to verify that $\mathbf{N} \cup (\mathbf{Q} \cap \mathbf{R}) = (\mathbf{N} \cup \mathbf{Q}) \cap (\mathbf{N} \cup \mathbf{R})$, note that both sides are equal to **Q**. Hence the law holds.

**Remark 6 :** Theorem 2 (a) says that $\cap$ distributes over $\cup$, and Theorem 2 (b) says that $\cup$ distributes over $\cap$.

Let us now consider another set of laws.

### 1.6.2 De Morgan's Laws

We will now discuss two laws that relate the operation of finding the complement of a set to that of the intersection or union of sets. These are known as De Morgan's laws, after the British mathematician Augustus De Morgan (1806 - 1871).

**Theorem 3 :** For any two sets A and B in a universal set U,

a) $(A \cap B)^c = A^c \cup B^c$

b) $(A \cup B)^c = A^c \cap B^c$.

**Proof :** As in the case of Theorem 2, we will prove (a), and ask you to prove (b).

a) Let $x \in (A \cap B)^c = U \setminus (A \cap B)$

$\Leftrightarrow x \notin A \cap B$

$\Leftrightarrow x \notin A$ or $x \notin B$ (because, if $x \in A$ and $x \in B$, then $x \in A \cap B$ )

$\Leftrightarrow x \in A^c$ or $x \in B^c$

$\Leftrightarrow x \in A^c \cup B^c$

So $(A \cap B)^c = A^c \cup B^c$.

Now try the following exercises.

E26) Prove (b) of Theorem 3.

E27) Verify De Morgan's laws for A and B, where $A = \{1, 2\}$, $B = \{2, 3, 4\}$.

(You can take $U = \{1, 2, 3, 4\}$, i.e., $U = A \cup B$. Of course, the laws will continue to hold true with any other U.)

So far we have discussed operations on sets and their inter-relationship. We will now talk of a product of sets, of which the coordinate system is a special case.

## 1.7 CARTESIAN PRODUCT

**Figure 7: Rene Descartes**

An interesting set that can be formed from two given sets is their Cartesian product, named after the French philosopher and mathematician Rene Descartes (1596 – 1650). He also invented the Cartesian coordinate system. Let us see what this product is.

Let A and B be two sets. Consider the pair (a, b), in which the first element is from A and the second element is from B. Then (a, b) is called an **ordered pair.** In an ordered pair, the order in which the two elements are written is important. **Thus, (a, b) and (b, a) are different ordered pairs.** We call two ordered pairs (a, b) and (c, d) **equal** (or the same) if $a = c$ and $b = d$.

Using ordered pairs, we give the following definition.

**Definition :** The **Cartesian product A × B**, of the sets A and B, is the set of all possible ordered pairs (a, b) where $a \in A$, $b \in B$.

That is, $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

For example, if $A = \{1, 2, 3\}$, $B = \{4, 6\}$, then

$A \times B = \{(1,4), (1,6), (2,4), (2,6), (3,4), (3,6)\}$.

Also note that

$B \times A = \{(4,1), (4,2), (4,3), (6,1), (6,2), (6,3)\}$.

You can see that $(1,4) \in A \times B$, but $(1,4) \notin B \times A$.

Therefore, $A \times B \neq B \times A$.

Try these exercises now.

---

E28) If $A = \{2, 5\}$ and $B = \{2, 3\}$, find $A \times B$, $B \times A$, $A \times A$.

E29) If $A \times B = \{(7,2), (7,3), (7,4), (2,2), (2,3), (2,4)\}$, determine A and B.

---

Now that we have defined the Cartesian product of two sets, let us extend the definition to any number of sets.

∀ denotes 'for every'

**Definition :** Let $A_1, A_2, \ldots, A_n$ be n sets. Then their Cartesian product is the set

$A_1 \times A_2 \times \ldots \times A_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in A_i \;\; \forall\, i = 1, 2, \ldots, n\}$.

For example, if **R** is the set of real numbers, then $\mathbf{R} \times \mathbf{R} = \{(a_1, a_2) \mid a_1 \in \mathbf{R}, \; a_2 \in \mathbf{R}\}$,

$\mathbf{R} \times \mathbf{R} \times \mathbf{R} = \{(a_1, a_2, a_3) \mid a_i \in \mathbf{R} \;\forall\, i = 1, 2, 3\}$, and so on.

It is customary to write $\mathbf{R}^2$ for $\mathbf{R} \times \mathbf{R}$ and $\mathbf{R}^n$ for $\mathbf{R} \times \ldots \times \mathbf{R}$ (n times).

In your earlier mathematical studies you must have often used the fact that **R** can be geometrically represented by a line. Are you also familiar with a geometrical representation of $\mathbf{R} \times \mathbf{R}$?

You know that every point in a plane has two coordinates, x and y, and every ordered pair (x, y) of real numbers defines the coordinates of a point in the plane. Thus, $\mathbf{R}^2$ is the Cartesian product of the x-axis and the y-axis and hence, $\mathbf{R}^2$ represents a plane. In the same way $\mathbf{R}^3$ represents three-dimensional space, and $\mathbf{R}^n$ represents n-dimensional space, for any $n \geq 1$.

Try the following exercise now.

---

E30) Which of the following belong to the Cartesian product $\mathbf{Q} \times \mathbf{Z} \times \mathbf{N}$? Why?

a) $(3, 0)$, b) $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, c) $(1, 1, 1)$, d) $\left(\frac{1}{2}, -5, \sqrt{2}\right)$, e) $\{-2, 2, 3\}$.

E31) Give an example of a proper non-empty subset of $\mathbf{R} \times \mathbf{R}$.

E32) Prove that for any 3 sets A, B and C,

a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

× distributes over ∪ as well as over ∩.

---

We will end our discussion on sets here. Let us summarise what we have covered in this unit.

## 1.8 SUMMARY

In our discussion on sets we have brought out the following points.

1) A set is a well-defined collection of objects.
2) The listing method and property method for representing sets.
3) Given two sets A and B, what we mean by $A \subseteq B$, $A \supseteq B$ and $A = B$.
4) The pictorial representation of sets and their relationships by Venn diagrams, and its utility.
5) The operations of complementation, intersection and union of sets, and their properties.
6) The distributive laws : For any three sets A, B and C,

   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
7) De Morgan's laws : For any two sets A and B,

   $(A \cap B)^c = A^c \cup B^c$

   $(A \cup B)^c = A^c \cap B^c$.
8) The Cartesian product of sets.

Now, we suggest that you **go back to the objectives given in Sec.1.1,** and see if you have achieved them. One way of checking this is to solve all the exercises in the unit. If you would like to know what our solutions are, we have given them in the next section.

## 1.9 SOLUTIONS/ANSWERS

E1) (b), (c), (d), (e)

E2) (c), (e)

E3) a) {2}

b) {1, 2, 3, 4, 6, 12}

c) {2, – 2}

d) {8}

e) {a, b, c, ......, x, y, z}

E4) a) $\{x^2 \mid x \in \mathbf{Z}\}$

b) $\{ x \mid x \text{ is a prime number} \}$

c) $\{ x \mid x \text{ is an even integer}\}$

d) We can have several representations (see Remark 2). For example,
$\phi = \{ x \in N \mid x \text{ is both odd as well as even} \}$, or
$\phi = \{ x \in N \mid x < 0 \}$.

E5) **R**

E6) (b), (c), and (e) are finite.
(a) and (d) are infinite.

E7) $\phi$, {1}, {2}, {3}, {1,2}, {1, 3}, {2, 3}, {1, 2, 3}.
If a finite set A has n elements, then its power set has $2^n$ elements.

E8) Let $x \in A$. Then
$A \subseteq B \Rightarrow x \in B$. And then
$B \subseteq C \Rightarrow x \in C$
$\therefore x \in A \Rightarrow x \in C$
$\therefore A \subseteq C$

E9) Consider A = {0}, B = {1}, C = {0}.
Then $A \not\subseteq B$, $B \not\subseteq C$, but $A \subseteq C$.

E10) Note that A = {2} = C, B = {1, 2}.
$\therefore$ A = C and $A \subsetneq B$.

E11) $A \supseteq C$

E12)

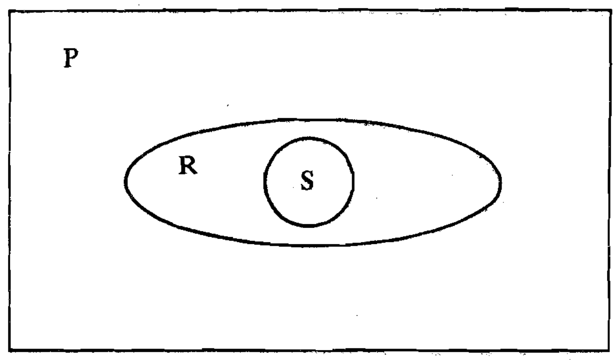


**Figure 8**

S, R, P are the sets of squares, rectangles and parallelograms, respectively. Here we have taken U = P.

E13) a)

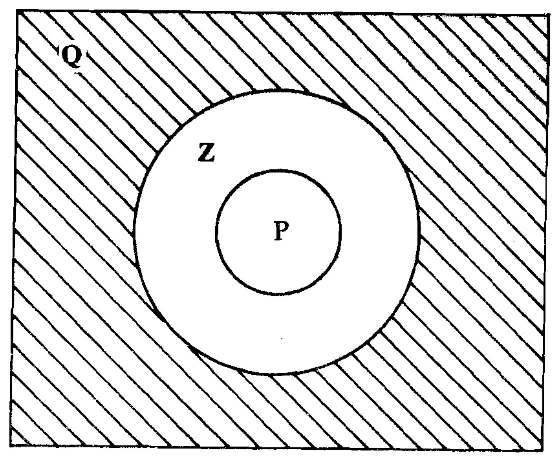


**Figure 9**

The shaded area is $Q \setminus Z$.

b) Infinite

E14) $\phi$, $\phi$, A and

$(A^c)^c = A$, since $x \in A \Leftrightarrow x \notin A^c \Leftrightarrow x \in (A^c)^c$.

E15) Z, Z, Z, $\phi$.

E16) c) From (a) we know that $A \cap B \subseteq A$. We need to prove that $A \subseteq A \cap B$. For this, let $x \in A$. Then, since $A \subseteq B$, $x \in B$

Thus, $x \in A \cap B$. $\therefore$ $A \subseteq A \cap B$.

$\therefore$ $A = A \cap B$.

d) $A \cap A \subseteq A$, applying (a).
$A \subseteq A \cap A$, as in proof of (c).

$\therefore$ $A = A \cap A$.

e) $A \cap \phi \subseteq \phi$, applying (F)
Also $\phi \subseteq A \cap \phi$, since $\phi$ is a subset of every set.
$\therefore A \cap \phi = \phi$.

f) $A \cap B \subseteq B$ and $A \cap B \subseteq A$. $\therefore A \cap B \subseteq B \cap A$.
Similarly, $B \cap A \subseteq A \cap B$.
$\therefore A \cap B = B \cap A$.

g) $x \in A \setminus B \Leftrightarrow x \in A$ and $x \notin B$
$\Leftrightarrow x \in A$ and $x \in B^c$
$\Leftrightarrow x \in A \cap B^c$.
$\therefore A \setminus B = A \cap B^c$.

h) Let $x \in C$. Then $C \subseteq A \Rightarrow x \in A$. Similarly, $x \in B$. Therefore, $x \in A \cap B$. Hence, $C \subseteq A \cap B$.

E17) No. For example, if $A = \{1,2,3\}$, $B = \{1,2,4\}$, $C = \{1,2\}$, then $C \subsetneq A$ and $C \subsetneq B$, but C is not **properly** contained in $A \cap B$ ; it is exactly **equal** to $A \cap B$.

E18)

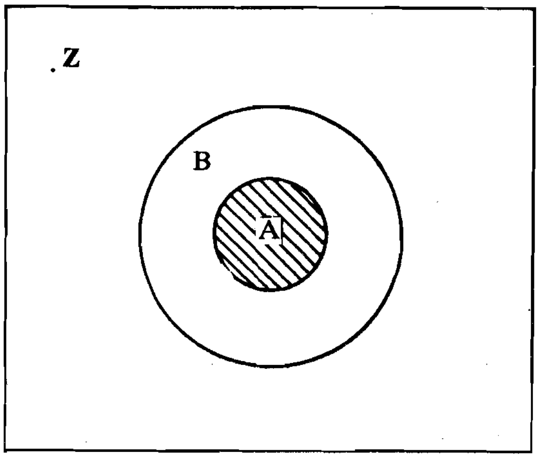


**Figure 10**

E19) $A \cap B \cap C = \{4\} = (A \cap B) \cap C = A \cap (B \cap C)$.

E20) $A \cap B \cap A = A \cap (B \cap A) = A \cap (A \cap B) = (A \cap A) \cap B$.
$= A \cap B = \{30\,n \mid n \in N\}$.

E21) a) $A \subseteq A \Rightarrow A \cup A = A$.

b) $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$
$\Leftrightarrow x \in B$ or $x \in A$.
$\Leftrightarrow x \in B \cup A$.
$\therefore A \cup B = B \cup A$.

c) $\phi \subseteq A \Rightarrow A \cup \phi = A$

d) Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. In either case $x \in C$, since $A \subseteq C$ and $B \subseteq C$. Thus,
$x \in A \cup B \Rightarrow x \in C$.
$\therefore A \cup B \subseteq C$.

E22) $\{x \in \mathbf{R} \mid 0 \le x \le 3\}$

E23) Since $A \subseteq A \cup B = \phi$, we see that $A \subseteq \phi$. Also, $\phi \subseteq A$ always.
$\therefore A = \phi$. Similarly, $B = \phi$.

E24) a) $A = \{3x \mid x \in N\}, B = \{6x \mid x \in N\}, C = \{10x \mid x \in N\}$.
Note that $B \subsetneq A$.
$\therefore A \cup B = A$.

$\therefore (A \cup B) \cup C = A \cup C = \{x \in \mathbf{N} \mid 3 \text{ divides } x \text{ or } 10 \text{ divides } x\}$
Also, $A \cup B \cup C = \{x \in \mathbf{N} \mid 3, 6 \text{ or } 10 \text{ divide } x\}$
$= \{x \in \mathbf{N} \mid 3 \text{ or } 10 \text{ divide } x\}$
$= (A \cup B) \cup C.$
Similarly, $A \cup B \cup C = A \cup (B \cup C)$.

b) $A \setminus B \subseteq A \subseteq A \cup B$, $B \setminus A \subseteq B \subseteq A \cup B$, $A \cap B \subseteq A \cup B$.
$\therefore (A \setminus B) \cup (A \cap B) \cup (B \setminus A) \subseteq A \cup B$.
Conversely, let $x \in A \cup B$. Then $x \in A$ or $x \in B$.

Now, there are only three possibilities for x :

i) $x \in A$ but $x \notin B$, that is, $x \in A \setminus B$, or

ii) $x \in A$ and $x \in B$, that is, $x \in A \cap B$, or

iii) $x \in B$ but $x \notin A$, that is, $x \in B \setminus A$.

Thus, $A \cup B \subseteq (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$.

So we have proved the result.

E25) $x \in A \cup (B \cap C)$
$\Leftrightarrow x \in A$ or $x \in B \cap C$
$\Leftrightarrow x \in A$ or $(x \in B$ and $x \in C)$
$\Leftrightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$
$\Leftrightarrow x \in A \cup B$ and $x \in A \cup C$
$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$.

E26) $x \in (A \cup B)^c \Leftrightarrow x \notin A \cup B$
$\Leftrightarrow x \notin A$ and $x \notin B$
$\Leftrightarrow x \in A^c$ and $x \in B^c$.
$\Leftrightarrow x \in A^c \cap B^c$.

E27) $A \cup B = U$. $\therefore (A \cup B)^c = \phi$
Also $A^c = \{3, 4\}$, $B^c = \{1\}$
$\therefore A^c \cap B^c = \phi$
$\therefore (A \cup B)^c = A^c \cap B^c$
Further $A \cap B = \{2\}$. $\therefore (A \cap B)^c = \{1, 3, 4\}$
$\therefore A^c \cup B^c = (A \cap B)^c$.

E28) $A \times B = \{(2,2), (2,3), (5,2), (5,3)\}$

$B \times A = \{(2,2), (3,2), (2,5), (3,5)\}$

$A \times A = \{(2,2), (2,5), (5,2), (5,5)\}$

E29) $A = \{7, 2\}$, $B = \{2, 3, 4\}$

E30) Only (c). (a) is not, since it is only an ordered pair, and not a triple. (b) is not, since $\frac{1}{2} \notin \mathbf{N} \cup \mathbf{Z}$. (d) is not, since $\sqrt{2} \notin \mathbf{N}$. (e) is not, since it is not an ordered triple; it is a set of three elements.

E31) Any subset is $A \times B$, where $A \subseteq \mathbf{R}$, $B \subseteq \mathbf{R}$. For a proper subset of $\mathbf{R}^2$, either A or B should be a proper subset of $\mathbf{R}$.

E32) a) $(x, y) \in A \times (B \cup C)$
$\Leftrightarrow x \in A$ and $y \in B \cup C$
$\Leftrightarrow x \in A$ and $(y \in B$ or $y \in C)$
$\Leftrightarrow (x, y) \in A \times B$ or $(x, y) \in A \times C$
$\Leftrightarrow (x, y) \in (A \times B) \cup (A \times C)$.

b) $(x, y) \in A \times (B \cap C)$
$\Leftrightarrow x \in A$ and $y \in B \cap C$
$\Leftrightarrow (x, y) \in A \times B$ and $(x, y) \in A \times C$
$\Leftrightarrow (x, y) \in (A \times B) \cap (A \times C)$.