UNIT 1 FUZZY SETS – AN INTRODUCTION

Structure

1.1 INTRODUCTION

Fuzzy sets are a further development of the mathematical concept of a set. Sets were first studied formally by the German mathematician Georg Cantor (1845-1918). Even though the idea of fuzzy sets was introduced in its modern form by Lotfi A. Zadeh in 1965, the idea of multi-valued logic in order to deal with vagueness has been around from the beginning of the century. Fuzzy set theory generalizes classical set theory to allow partial membership in the sense that the membership degree of an object to a set is not restricted to first the integers 0 and 1, but may take on any value in the interval $[0,1]$.

In this unit the basic concepts and the essence of fuzzy sets are presented in Section 1.2. The fuzzy set operations and the fuzzy relation are also discussed. At the end, some applications of the fuzzy sets for creating fuzzy logic systems are presented. To get an idea about the essence of a fuzzy set, a simple example is presented below. The best way to introduce fuzzy sets is to start with a limitation of classical set. A set in classical set theory always has a sharp boundary because membership in a set is a black-and-white concept, i.e., an object either completely belongs to the set or does not belong to the set at all.

For example, suppose we wish to represent the set of high-income families within the framework of classical set theory. This would require us to choose a particular value as a threshold (e.g., yearly income) such that all families with a yearly income greater than the threshold (say, Rs. 2,500,00) are considered to be a member of the set of high-income families, and all families with a yearly income lower than the threshold are not members of the set. However, the fact that a family with a yearly income of Rs. 2,499,99 is not considered high income whereas a family with an extra Rs. 1 in the yearly income is considered in the high income group is counterintuitive, to say the least. Even though unnatural and sometimes disturbing, such artificially created sharp boundaries have often been used in defining classical sets, primarily due to the lack of a better methodology.

Even though some sets do have sharp boundaries (e.g., the set of married people), many others do not have sharp boundaries (e.g., the set of happily married couples, the set of good graduates, the set of tall persons, etc.). Fuzzy set theory directly addresses this limitation by allowing membership in a set to be a matter of degree. The degree of membership in a set is expressed by a number between 0 and 1;0 means entirely not in the set, 1 means completely in the set, and a number in between means partially in the set. This way, a smooth and gradual transition from the region outside the set to those in the set can be described. In Section 1.3 and 1.4, Support and α -cut and hedges are discussed respectively. We shall discuss the operations on fuzzy set in Section 1.5 and fuzzy relation in Section 1.6.

Objectives

After studying this unit, you should be able to

- know the essence of the concept of fuzzy sets
- define the fuzzy set
- differentiate the classical set and the fuzzy set
- apply the theory of fuzzy set to real-life complex problems
- design membership functions for fuzzy sets
- apply operations on fuzzy sets.
- define fuzzy relations

1.2 CLASSICAL SET VS. FUZZY SET

Let us start with the formal definition of classical sets.

Definition 1 (Classical Set): Any collection of objects which can be treated as an entity is known as classical set. Cantor described a set by its members, such that an item from a given universe is either a member or not. The terms *set*, *collection* and *class* are synonyms, just as the terms *item*, *element* and *member are*.
There are three basic methods by which sets can be defined within a given universal set X:

(a) **List method** – This method defines a set by listing all its members. Since, it is not possible to list all the elements of an infinite set; this method is generally confined to define only finite sets. Set A, whose members are $a_1, a_2, \cdots, a_n$, is usually written as

$$A = \{a_1, a_2, \cdots a_n\}$$

(b) **Rule method –** In this method, a set is defined by a property satisfied by its members**.** A common notation expressing this method is

$$A = \{x | P(x)\} \text{ or } A = \{x : P(x)\}$$

Where the symbol '|' or ':' denote the phrase "such that", and $P(x)$ designates a preposition of the form "x has the property P". That is, A is defined in this notation as the set of all elements of χ for which preposition P(x) is true. It is required that the property P be such that for any given $x \in \chi$, the preposition P(x) is either true or false.

(c) **Characteristic function method** – This method defines a set by using a function, called a characteristics function that declares which elements of A are members of the set and which are not. A set A is defined by its characteristics function, χ_A, as follows

$$X_A(x) = \begin{cases} 1 & for\ x \in A \\ 0 & for\ x \notin A \end{cases}$$

That is, the characteristics function maps elements of A to elements of the set $\{0,1\}$, which is formally expressed by

$$\chi_A : A \to \{0,1\}$$

For each $x \in A$, when $\chi_A(x) = 1$, x is declared to be a member of A; when $\chi_A(x) = 0, x$ is declared as a nonmember of A.

Example 1 (classical sets): The following are well defined lists or collections of objects, and therefore entitle to be called sets:

(a) *The set of non-negative integers less than 4. This is a finite set with four members: 0, 1, 2 and 3.*

(b) *The set of live dinosaurs in the basement of the British Museum. This set has no members and is called an empty set or a null set.*

(c) *The set of measurements greater than 10 volts. Even though this set is infinite, it is possible to determine whether a given measurement is a member or not.*

Now, try an exercise.

E1) List the elements of the following sets described by rule method:

(i) $A = \{n \in N : n \text{ is an odd prime less than } 150\}$

(ii) $B = \{x \in R : x^2 - 8x + 15 = 0\}$

(iii) $C = \{\max\{a, b\} : a, b \text{ are twin primes } \leq 150\}$

[Note: two consecutive primes are said to be twin if their difference is 2, e.g., <3, 5>, <5, 7>, <11, 13> are all twin-primes.**]**

Now let us define the fuzzy set formally.

Fuzzy Sets: According to Zadeh, many sets have more than an either-or criterion for membership. Take for example the set of *young people*. A one year old baby will clearly be a member of the set, and a 100 years old person will not be a member of this set, but what about people at the age of 20, 30, or 40 years? Another example is a weather report regarding high temperatures, strong wind, or nice days. In other cases a criterion appears non-fuzzy, but is perceived as fuzzy: a speed limit of 60 kilometers per hour, a check-out time at 12 noon in a hotel, a 50 years old man. Zadeh proposed a *grade of membership*, such that the transition from membership for all its members thus describes a fuzzy set. An item's grade of membership is normally a real number between 0 and 1, often denoted by the Greek letter μ. The higher the number, the higher the membership. Zadeh regards Cantor's set a special case where elements have full membership, i.e., $\mu = 1$. He nevertheless called Cantor's sets *non-fuzzy*; today the term *crisp* set is used, which avoids that little dilemma. Notice that Zadeh does not give a formal basis for how to determine the grade of membership. The membership value for a 50 year old in the set *young* depends on one's own view. The grade of membership is a precise, but subjective measure that depends on the context.

Following the above discussions, a fuzzy set is thus defined using a function that maps objects in a domain of concern to their membership value in the set. Such a function is called the *membership function* and is usually denoted by the Greek symbol μ for ease of recognition and consistency. For example, a more realistic representation of the high-income family can now be expressed by the membership function shown in figure 1. The membership function of a fuzzy set A is denoted as μ_A, and the membership value of x in A is denoted by $\mu_A(x)$. The domain of the membership function, which is the domain of concern from which the elements of the set are drawn, is called the *universe of discourse* and is denoted by U. For instance, the universe of discourse of the fuzzy set high-income can be the positive real line $[0, \alpha)$.

In addition to membership functions, a fuzzy set is also associated with a linguistically meaningful term. For instance, the fuzzy set in figure 1 is associated with the linguistic term “high”. Associating a fuzzy set to a linguistic term offers two important benefits. First, the association makes it easier for human experts to express their knowledge using the linguistic terms. Second, the knowledge expressed using linguistic term is easily comprehensible. This benefit often results in significant savings in the cost of designing, modifying, and maintaining a fuzzy logic system. An important concept in fuzzy set that enables these two benefits is that of the linguistic variable which is defined in the following paragraphs.

In summary, a fuzzy set has a dual representation: a qualitative description using a linguistic term and a quantitative description through a membership function, which maps elements in a universe of discourse (i.e., a domain of interest) to their membership degree in the set. The above two are represented by the concepts of “granulation” and “graduation”, respectively.

Figure 1 – Membership function of high annual income

Figure 2 – Fuzzy set to represent “*comfortable house for a 4-person family*”

Example 2 (fuzzy set)**:** Classifying houses problem – A realtor wants to classify the house he offers to his clients. One indicator of comfort of these houses is the number of bedrooms in them. Let the available types of houses of represented by the following set.

$$U = \{1,2,3,4,5,6,7,8,9,10\}$$

The houses in this set are described by μ number of bedrooms in a house. The realtor wants to describe a “*comfortable house for a 4-person family,*” using a fuzzy set.

Solution**:** The fuzzy set to describe *“comfortable house for a 4-person family,”* may be defined in the following manner: *comfortable house for a 4-person family* = 0.2/1+0.5/2+0.8/3+ $\frac{1}{4}$ +0.7/5+0.3/6, which means that the level of comfort for a 1-bedroom house is 0.2, a 2-bedroom house is 0.5 and so on.

The equivalent graphical representation of it is shown in figure 2.

Try the following exercises.

E2) Describe the concept of a fuzzy set in your own words.

E3) Find at least five examples of fuzzy sets in newspaper articles or magazines.

Now let us discuss the various component related to fuzzy sets.

(i) **Linguistic Variables:** A linguistic variable is a variable whose values are an expression involving fuzzy sets. In other words, a linguistic variable is like a composition of a symbolic variable (a variable whose value is a symbol) and a numeric variable (a variable whose value is a number). An example of a symbolic variable is "Shape = Rectangular" where "Shape" is a variable indicating the shape of an object. An example of numeric variable "Height = 5'', where "Height" is a variable representing the height of a person. Numeric variables are frequently used in science, engineering, mathematics, medicine and many other disciplines, while symbolic variables play an important role in artificial intelligence and decision sciences.

A linguistic variable enable its value to be described both qualitatively by a linguistic term (i.e., a symbol serving as the name of a fuzzy set) and quantitatively by a corresponding membership function (which expresses the meaning of the fuzzy set). The linguistic term is used to express concepts and knowledge in human communication, whereas membership function is useful for processing numeric input data.

Using the notion of the linguistic variable to combine these kinds of variables into a uniform framework is, in fact, one of the main reasons the fuzzy logic has been successful in offering intelligent approaches in engineering and many other areas that deal with continuous problem domains.

***Example 3 (linguistic variable)*:** Consider the sentence "The amount of trading is heavy" which uses a fuzzy set "Heavy' to describe the quantity of the stock market trading in one day. More formally, this is expressed as:

Trading Quantity is Heavy

The variable *Trading Quantity,* which is a linguistic variable, in this example demonstrates an important concept in fuzzy logic.

(ii) **Universe:** Elements of a fuzzy set are taken from a *universe of discourse* or *universe* for short. The universe contains all elements that can come into consideration. Even the universe depends on the context, as the following example shows.

***Example 4 (Universe)*: (a)** The set of *young people* could have all human beings in the world as its universe. Alternatively it could be the numbers between 0 and 100; these would then represent age.
(b) The set x > 10 could have as a universe all positive measurements.

An application of the universe is to suppress faulty measurement data, for example negative values for the age in the above example. In case we are dealing with a non-numerical quantity, for instance *taste*, which cannot be measured against a numerical scale, we cannot use a numerical universe. The elements are then said to be taken from a *psychological continuum*; an example of such a universe could be {*bitter, sweet, sour, salt, hot, …*}.

(iii) **Membership Function:** Every element in the universe of discourse is a member of the fuzzy set to some grade, maybe even zero. The set of elements that have a non-zero membership is called the *support* of the fuzzy set. The function that ties a number to each element x of the universe is called the *membership function* $\mu(x)$.

A fuzzy set is represented through its membership function. Depending on the nature of the fuzzy set it can be defined in two ways: (i) by enumerating membership values of those elements in the set (completely or partially), or (ii) by defining the membership function mathematically. Obviously, the first approach is possible only if the set is discrete, because a continuous fuzzy set has an infinite number of elements. Generally speaking, a fuzzy set A can be defined through enumeration using the expression given in Eqn.(1)

$$A = \sum_i \mu_A(x_i)|x_i \qquad (1)$$

Where the summation operator refers to the union (disjunction) operation and the notation $\mu_A(x_i)|x_i$ refers to a fuzzy set containing exactly one (partial) element x_i with a membership value $\mu_A(x_i)$. For brevity, we do not list those elements x_i whose membership degree in set A is zero.

***Example 5 (fuzzy set through enumeration)*:** Let us consider a subset of natural numbers; say from 1 to 20, as the universe of discourse, U. We may define the fuzzy sets "small" and "medium" by enumeration as follows:

Small $\equiv 1/1 + 1/2 + 0.9/3 + 0.6/4 + 0.3/5 + 0.1/6$

Medium $\equiv 0.1/2 + 0.3/3 + 0.7/4 + 1/5 + 1/6 + 0.7/7 + 0.5/8 + 0.2/9$

***Example 6 (fuzzy set through defining membership function mathematically)*:** Alternatively, the fuzzy sets "small" and "medium" can be defined by explicitly describing their characteristics as shown in Eqns.(2) and (3) respectively.

$$\mu_{small}(x) = \begin{cases} 1 & x < 2 \\ \dfrac{7-x}{5} & 2 \le x \le 7 \\ 0 & x > 7 \end{cases} \qquad (2)$$

$$\mu_{medium}(x) = \begin{cases} 0 & x < 1 \\ \dfrac{x-1}{4} & 1 \le x < 5 \\ 1 & 5 \le x < 6 \\ \dfrac{10-x}{4} & 6 \le x < 10 \\ 0 & x \ge 10 \end{cases} \qquad (3)$$

Try the following exercises now.

E4) Let x be a linguistic variable that measures a university's academic excellence, which takes values from the universe of discourse $U = \{1,2,3,4,5,6,7,8,9,10\}$.

Suppose the term set of x includes *Excellent, Good, Fair,* and *Bad*. Express these fuzzy sets through enumeration.

E5) Order the fuzzy sets defined by the following membership grade functions (assuming $x \ge 0$) by the inclusion (subset) relation:

$$A(x) = \frac{1}{1+10x}, \quad B(x) = \left(\frac{1}{1+10x}\right)^{1/2}, \quad C(x) = \left(\frac{1}{1+10x}\right)^{1/2}$$

Now let us define the types of membership functions

Whereas there exist numerous types of membership functions, the most commonly used in practice are triangles, trapezoids, Gaussain, and Bell-shape functions. In the following these membership functions are briefly introduced:

(i) **Triangular Membership Function:** A triangular membership function is specified by three parameters $\{a,b,c\}$ as follows:

$$
triangle(x : a,b,c) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ \dfrac{c-x}{c-b} & b \le x \le c \\ 0 & x > c \end{cases} \tag{4}
$$

The precise appearance of the function is determined by choice of parameters a, b, and c.

(ii) **Trapezoidal Membership Function:** A trapezoidal membership function is specified by four parameters $\{a,b,c,d\}$ as follows:

$$
trapezoid(x : a,b,c,d) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x < b \\ 1 & b \le x < c \\ \dfrac{d-x}{d-c} & c \le x \le d \\ 0 & x \ge d \end{cases} \tag{5}
$$

The triangular membership function is a special case of the trapezoidal membership function. Due to their simple formulas and computational efficiency, both triangular and trapezoidal membership functions have proven popular with fuzzy logic practitioners and been used extensively, particularly in control.

(iii) **Gaussian Membership Function:** A Gaussian membership function is specified by two parameters $\{m,\sigma\}$ as follows:

$$
Gaussian(x : m,\sigma) = e^{\left(\frac{(x-m)^2}{\sigma^2}\right)} \tag{6}
$$

Where m and σ denote the centre and width of the function, respectively. We can control the shape of the function by adjusting the parameter σ**.** A small σ will generate a "thin" membership function, while a big σ will lead to a "flat" membership function.

(iv) **Bell-shaped Membership Function:** A Bell-shaped membership function is specified by three parameters $\{a,b,c\}$ as follows:

$$
Bell(x : a,b,c) = \frac{1}{1+\left|\frac{x-c}{a}\right|^{26}} \tag{7}
$$

where the parameter b is usually positive. A desired bell- shaped membership function can be obtained by a proper selection of the parameters a, b, and c. Specifically, we can adjust c and a to vary the center and width of the function and then use b to control the slopes at the crossover points.

Now, we are ready to design Membership Functions

One of the questions you may have by now is, "How do we determine the exact shape of the membership function for a fuzzy set?" What is important about membership function is that it provides gradual transition from regions completely outside a set to regions completely in the set.
A membership function can be designed in three ways:

(i) Interview those who are familiar with the underlying concept and later adjust it based on a tuning strategy.

(ii) Construct it automatically from data.

(iii) Learn it based on feedback from the system performance.

The first approach was the main approach used by fuzzy logic researchers and practitioners until the late 80s. Due to the lack of systematic strategies in those days, most fuzzy systems were tuned through trial-and-error process. This has become one of the main criticisms of fuzzy logic technology. Fortunately, many techniques in the second two categories have been developed since the late 80s using statistical techniques, neural networks, and genetic algorithms.

Even though one may attempt to define a membership function of arbitrary shape, it is generally recommended to use parametrerizable functions that can be defined by a small number of parameters. Using parameterized membership functions can not only reduce the system design time, it can also facilitate the automated tuning of the system because desired changes to the membership function can be directly related to corresponding changes in the related parameters.

The parameterizable membership function most commonly used in practice is the triangular membership function and the trapezoidal membership function. The former has three parameters and the latter has four parameters. Simplicity is the main advantage of triangular and trapezoidal membership functions.

To summarize we present the following guidelines for membership function design:

(i) Always use parameterizable membership functions. Do not define a membership function point by point.

(ii) Use a triangular or trapezoidal membership function, unless there is a good reason to do otherwise.

(iii) If you want to learn the membership function using neural network learning techniques, choose a differentiable (or even continuous differentiable) membership function (i.e., Gaussian).

So far, we discussed the fuzzy set with its membership function.

1.3 SUPPORT AND ALPHA-CUTS

In the following section, we shall discuss support and alpha-cuts.

The support of a fuzzy set A is the set of elements whose degree of membership in A is greater than 0. For example, the support of the fuzzy set in figure 3 is the open interval (10, 20). We can formally define support as follows:
Let A be a fuzzy set in the universe of discourse U. The support in A is defined as:

$$Spt(A) = \{x \in U \mid \mu_A(x) > 0\} \qquad (8)$$

Figure 3 – A fuzzy set with support set (10, 20)

The notion of $\alpha-\text{cut}$ (also called α-level) is more general than that of support. Let α_0 be a number between 0 and 1. The $\alpha-\text{cut}$ of a fuzzy set A at α_0, denoted as $A_{\alpha 0}$, is the set of elements whose degree of membership in A is no less than α_0. Mathematically, the $\alpha-\text{cut}$ of a fuzzy set A in U is defined as:

$$A_{\alpha 0} = \{x \in U \mid \mu_A(x) \geq \alpha_0\} \qquad (9)$$

Figure 4 – The fuzzy membership function of *moderately approved* presidential candidate

***Example 7 (α-cut)*:** Let us consider the concept "moderately approved" regarding the public's opinion of a presidential candidate. The universe of discourse is the percentage of those people supporting the candidate in a national poll. To simplify our discussion we use a discrete universe of discourse $U = \{0\%, 10\%, 20\%, \cdots, 100\%\}$. The membership function of the moderately fuzzy set approved is shown in figure 1.4. We can construct the $\alpha-\text{cut}$ of the fuzzy set at 0.7 as:
Moderately Approved$_{0.7}$ = {40%, 50%, 60%}
Similarly, we can construct the following $\alpha-\text{cut}$:
Moderately Approved$_{0.2}$ = {30%, 40%, 50%, 60%, 70%, 80%}
Moderately Approved$_{0.6}$ = {40%, 50%, 60%, 70%}
Obviously, as the α value increases, the set generated by α-cutting, the fuzzy set becomes smaller.

Now, try these exercises.

E6) Construct the $\alpha-\text{cut}$ at $\alpha = 0$ and $\alpha = 1$ for the fuzzy set moderately approved shown in figure 1.3.

E7) Construct the $\alpha-\text{cut}$ at $\alpha = 0.4$ for the fuzzy sets defined in exercise E4.

E8) If $\alpha_1 < \alpha_2$ then prove that $A_{\alpha 1} \supseteq A_{\alpha 2}$, where $\supseteq$ denotes a crisp superset (i.e., suppositions) relation.

So far, we discussed classical set, fuzzy set, support and α-cut. In the following section, we shall discuss hedges.

1.4 HEDGES

Hedges are also called fuzzy set transformers and are used to model adjectives or adverbs, whose role in natural language is to modify the semantics of an adjective or noun on which they operate. Hedges are used to intensify or dilute the membership function of a fuzzy set. The shape of a fuzzy set can be changed through the use of contrasts and restrictions. When operating on fuzzy numbers, hedges can increase or decrease the expectancy of a fuzzy number. On the basis of underlying actions the following categories of hedges are defined in literature.

- ***Approximation hedges*** – Such hedges are used to increase or decrease the expectancy of a fuzzy number. Some of the linguistic qualifiers modeled using approximation hedges are *about, around, near, close to,* and *in vicinity of.* The approximation hedges allow fuzzy models to treat all numeric values as fuzzy numbers, thus reducing complexity and allowing a uniform approach to knowledge acquisition and management.

- ***Contrast hedges* –** Such hedges are used to increase or decrease the membership function's central measure of membership by focusing in the area around 50% membership space. Example hedges of this category are *almost, definitely, positively, generally,* and *usually*.
- ***Dilution hedges* –** Hedges belonging to this category softens the membership function over the fuzzy set. Dilution hedges include *quite, rather, slightly,* and *somewhat*.
- ***Intensification hedges* –** Hedges belonging to this category hardens the membership function over the fuzzy set. Intensification hedges include *very,* and *extremely*.
- ***Negation hedges*** – Negation hedges reverse the truth membership of the fuzzy set. "Not" is a primary negation hedge that produces the complement of a fuzzy set.
- ***Restriction hedges*** – Hedges belonging to this category restricts the membership function relative to the shape of the underlying fuzzy set. Restriction hedges include above, below, more than, and less than.

Most hedges involve complex algorithmic processing of the membership function. The basic intensification and dilution hedges, however, simply modify each point on the fuzzy set by applying a power function. Eqn.(10) shows the power function for the intensification hedge.

$$\mu_{intesify(A)}(x_i) = \mu_A^n(x_i) \quad (10)$$

The "very" hedge has n = 2 and so simply squares the membership value; "slightly" has n = 1.2 and other intensification hedges have stronger or weaker exponents on the membership function. Eqn.(11) shows the power function for the dilution hedge.

$$\mu_{dilute(A)}(x_i) = \mu_A^{1/n}(x_i) \quad (11)$$

The "somewhat" hedge has n = 2 and so simply takes the square root of the membership values. Other forms of the dilution hedge class supply different exponent values and hence take different roots.
Hedges are used in fuzzy models to dynamically create new fuzzy sets and change the meaning of linguistic variables. This enables the modification of existing fuzzy sets temporarily to provide different meaning to the underlying linguistic variable. The modified shapes of a fuzzy number "*around 50,000*" after applying "*somewhat*" and "*very*" hedges on it are shown in figure 5.

Figure 5 – A fuzzy number "around 50,000" and its modified shape after applying "somewhat" and "very" hedges.

Try an exercise.

E9) Apply the "very" hedge on the fuzzy sets defined in exercise E4 to get the new modified fuzzy sets. Sow the modified fuzzy sets through numeration.

In this section, we shall discuss various operations on fuzzy sets.

1.5 OPERATIONS ON FUZZY SETS

The three fundamental operations in classical sets are union, intersection, and complement. The union of two sets A and b (denoted as $A \cup B$) is the collection of those objects that belongs to either A or B. The intersection of A and B (denoted as $A \cap B$) is the collection of those objects that belong to both A and B. The complement of a set A (denoted as A^c or $\overline{A}$) is the collection of objects not belonging to A.
Like classical sets, the operations, union, intersection, and complement can also be defined for the fuzzy sets. The membership function is obviously a crucial component of a fuzzy set. It is therefore natural to define operations on fuzzy sets by means of their membership functions. Since membership in a fuzzy set is a matter of degree, set operations should be generalized accordingly. The fuzzy intersection operation turns out to be mathematically equivalent to the fuzzy conjunction (AND) operation, because they have identical desired properties. Similarly, the fuzzy union operation is mathematically equivalent to the fuzzy disjunction (OR) operation and the fuzzy complement is equivalent to the fuzzy negation (NOT) operation.

1.3.1 The Fuzzy AND operator

Let A and B be two fuzzy sets on mutual universe of discourse with membership function μ_A and μ_B. The intersection of fuzzy sets A and B is defined as:

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x)) \qquad (12)$$

Figure 6 illustrates the fuzzy region (shown in bold black color) produced by the proposition *Young Adult And Middle-Aged*. This is the region where an age is member of (compatible with) both the *young adult middle-aged* concepts.

Figure 6 – The fuzzy AND operator

1.3.2 The Fuzzy OR Operator

Let A and B be two fuzzy sets on mutual universe of discourse with membership functions μ_A and μ_B. The intersection of fuzzy sets A and B is defined as: $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$ (13)

Figure 1.5 illustrates the fuzzy region produced by the proposition *Young adult Or Middle-Aged*. This is the region where an age is member of (compatible with) either the *young adult* or *middle-aged* concepts. In figure 7, the bold line traces the resulting truth function. The area represented by the twin peaks, spans the width of both fuzzy sets.

Figure 7 – The fuzzy OR operator

1.3.3 The Fuzzy NOT Operator

Let A be a fuzzy set defined on a universe of discourse U with membership function μ_A. The negation of the fuzzy set A is defined as:

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x) \qquad (14)$$

Figure 8 illustrates the fuzzy region produced by the proposition *Not Middle-aged*. This is the region in the linguistics variables supporting Client age that forms the complement of the middle-aged concept. In figure 6, the bold line traces the resulting truth function.

Figure 8 – The fuzzy NOT operator

***Example 8 (Fuzzy operators)*:** A four person family wants to buy a house. An indication of how comfortable they are to be is the number of bedrooms in the house. But they also want a large house. Let $U = \{1,2,3,4,5,6,7,8,9,10\}$ be the set of available houses described by their number of bedrooms. Then the fuzzy set C (for comfortable) may be described as

$$C = \left\{ \frac{0.2}{1}, \frac{0.5}{2}, \frac{0.8}{3}, \frac{1.0}{4}, \frac{0.7}{5}, \frac{0.3}{6}, \frac{0}{7}, \frac{0}{8}, \frac{0}{9}, \frac{0}{10} \right\}$$

Let L be the fuzzy set *Large* defined as $L = \left\{ \frac{0}{1}, \frac{0}{2}, \frac{0.2}{3}, \frac{0.4}{4}, \frac{0.6}{5}, \frac{0.8}{6}, \frac{1}{2}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10} \right\}$

Then, $C \cap L = \left\{ \frac{0}{1}, \frac{0}{2}, \frac{0.2}{3}, \frac{0.4}{4}, \frac{0.6}{5}, \frac{0.3}{6}, \frac{0}{7}, \frac{0}{8}, \frac{0}{9}, \frac{0}{10} \right\}$

To interpret this, five bedrooms is optimal, but only satisfactory to the grade 0.6. The second best solution is four bedrooms.

The union of the fuzzy sets *Comfortable* and *Large* is

$$C \cup L = \left\{ \frac{0.2}{1}, \frac{0.5}{2}, \frac{0.8}{3}, \frac{1}{4}, \frac{0.7}{5}, \frac{0.8}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10} \right\}$$

Here four bedrooms is fully satisfactory (1,0) because it is comfortable, and 7-10 bedrooms also, because that would mean a large house.

The complement of Large is

$$\overline{L} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{0.8}{3}, \frac{0.6}{4}, \frac{0.4}{5}, \frac{0.2}{6}, \frac{0}{7}, \frac{0}{8}, \frac{0}{9}, \frac{0}{10} \right\}$$

Now you can try the following exercises.

E10) Let A and B are two fuzzy sets and $x \in U$, If $\mu_A(x) = 0.3$ and $\mu_B(x) = 0.9$ then find out the following membership values:

(i) $\mu_{A \cup B}(x)$ (ii) $\mu_{A \cap B}(x)$ (iii) $\mu_{\overline{A} \cup \overline{B}}(x)$ (iv) $\mu_{\overline{A} \cap \overline{B}}(x)$ (v) $\mu_{\overline{A \cup B}}(x)$ (vi) $\mu_{\overline{A \cap B}}(x)$

E11) For the above example "buying a house" what will be the elements of the following fuzzy sets

(i) $\overline{C \cup L} = \overline{C} \cap \overline{L}$ (ii) $\overline{C \cap L} = \overline{C} \cup \overline{L}$

E12) By using the membership function defined in exercise E4 construct the membership functions of the following compound sets

(i) Not Bad but Not Very Good

(ii) Good but Not Excellent

In the following section, we shall discuss the fuzzy relation.

1.6 FUZZY RELATION

A fuzzy relation generalizes the notion of a classical black-and-white relation into one that allows partial membership. For example a binary relation "friend" will classify all human relationship into either being friend or not being friend. A fuzzy relation "friend", in contrast, can describe the degree of friendship between two persons.

The classical notion of relation describes a relationship that holds between two or more objects. A relationship between two objects is represented by a *binary relation*, a relation with two arguments. For example, the parent relationship between a parent and his/her child can be represented as a binary relation. More generally, we can use an n-ary relation, a relation with n arguments, to describe a relationship between n objects. For instance, we can use an n-ary relation to describe that student X took course Y during semester Z in year W. This relation has four arguments such as took_course(student, course, semester, year). An n-ary relation can be formally defined as set of ordered listed of n objects. Each list describes a case in which the relation holds.

A binary relation on variables x and y, whose domains are X and Y respectively, can be defined as a set of ordered pairs in $X \times Y$. For instance, the binary relation "less than" between two real numbers can be formally defined as
$R = \{(x, y) | x < y; x, y \in R\}$

It is easy to see that the relation is a subset of $X \times Y$. In general, an n-ary relation on $x_1, x_2, \cdots, x_n$ whose domains are $X_1, X_2, \cdots, X_n$ is a subset of $X_1 \times X_2 \times \cdots \times X_n$.

Since a relation can be viewed as a set, we can easily generalize the classical notion of relation using fuzzy sets. We will show how to generalize binary relations. First we can represent a binary relation R on x, y with domains X, Y as a function that maps an ordered pair (x, y) in $X \times Y$ to 0 (i.e., $R = X \times Y \to \{0,1\}$.

A fuzzy relation generalizes the classical notion of relation into a matter of degree. As mentioned earlier, the fuzzy relation *friend* describes the degree of friendship between two persons. Similarly, a fuzzy relation *Petite* between height and weight of a person describes the degree by which a person with a specific height and weight is considered petite. Formally, fuzzy relation R between variables x and y, whose domains are X and Y, respectively, is defined by a function that maps ordered pairs in $X \times Y$ to their degree in the relation, which is a number between 0 and 1, i.e., $R = X \times Y \to [0,1]$.

More generally, a fuzzy n-ary relation R in $x_1, x_2, \cdots, x_n$ whose domains are $X_1 \times X_2 \times \cdots \times X_n$, respectively, is defined by a function that maps an n-ary $(x_1, x_2, \cdots, x_n)$ in $X_1 \times X_2 \times \cdots \times X_n$ to a number in the interval [0, 1]. i.e., $R : X_1 \times X_2 \times \cdots \times X_n \to [0,1]$. Just as a classical relation can be viewed as a set, a fuzzy relation can be viewed as a fuzzy subset. From this perspective, the mapping above is equivalent to the membership function of a multidimensional fuzzy set.

If the possible values of x and y are discrete, we can express a fuzzy relation in a matrix form. For example, suppose we wish to express a fuzzy relation Petite in terms of the height and the weight of a female. Suppose the range of the height and the weight of interest to us are {5',5'1", 5'2", 5'3", 5'4", 5'5", 5'6"}, denoted h, and {90, 95, 100, 105, 110, 115, 120, 125} (in lb,) denoted w, respectively. We can express the fuzzy relation in a matrix form as shown in table 1.

Table 1 A fuzzy relation – "Petite"

	90	95	100	105	110	115	120	125
5	1	1	1	1	1	1	0.5	0.2
5′1″	1	1	1	1	1	0.9	0.3	0.1
5′2″	1	1	1	1	1	0.7	0.1	0
5′3″	1	1	1	1	0.5	0.3	0	0
5′4″	0.8	0.6	0.4	0.2	0	0	0	0
5′5″	0.6	0.4	0.2	0	0	0	0	0
5′6″	0	0	0	0	0	0	0	0

Each entry in the matrix indicates the degree a female with the corresponding height (i.e., the row heading) and weight (i.e., the column heading) is considered to be *petite*. For instance, the entry corresponding to a height of 5’3” and a weight of 115 lb. has a value 0.3, which is the degree to which such a female person will be considered a *petite* person; i.e., petite(5’3”, 115 lb) =0.3.

Once we define the Petite fuzzy relation, we can answer two kinds of questions:

- What is the degree that a female with a specific height and a specific weight considered to be petite?
- What is the possibility that a petite person has a specific pair of height and weight measures?

In answering the first question, the fuzzy relation is equivalent to the membership function of a multidimensional fuzzy set. In the second case, the fuzzy relation becomes a possibility distribution assigned to a petite whose actual height and weight are unknown.

Let us now summarize the unit.

1.7 SUMMARY

In this unit, we have discussed the fundamental concepts of the fuzzy set and fuzzy relation. Specifically we have covered the following:

1. Elaborated the essence of fuzzy set to solve real life complex problems.
2. Provided a comparative introduction of the fuzzy set vs. classical set.
3. Defined membership function and its role to represent fuzzy set.
4. Provided details about well-known membership functions along with guidelines to define a membership function.
5. Introduced support and alpha-cut of fuzzy sets.
6. Defined hedges that are used to change the shapes of the fuzzy sets.
7. Introduced basic set theoretic operations that are also applicable on fuzzy sets with examples.
8. Defined fuzzy relation as a generalization of the classical relations.

1.8 SOLUTIONS/ANSWERS

E1) A= {3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61,67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149}
B = {3, 5}
C = {5, 7, 13, 19, 31, 43, 61, 73, 103, 109, 139}

E4) Excellent = 0.2/8+0.6/9+1/10

Good = 0.1/6+0.5/7+0.9/8+1/9+1/10

Fair = 0.3/2+0.6/3+0.9/4+1/5+0.9/6+0.5/7+0.1/8

Bad = 1/1+0.7/2+0.4/3+0.1/4

E5) $C(x)=\left(\frac{1}{1+10x}\right)^2$, $A(x)=\frac{1}{1+10x}$, $B(x)=\left(\frac{1}{1+10x}\right)^{1/2}$

E6) Moderately Approved$_{0.0}$ = {30%, 40%, 50%, 60%, 70%, 80%}

Moderately Approved$_{0.2}$ = {50%, 60%}

E7) Excellent $_{0.4}$ = {9, 10}

Good $_{0.4}$ = {7, 8, 9, 10}

Fair $_{0.4}$ = {3, 4, 5, 6, 7, 8}

Bad $_{0.4}$ = {1, 2, 3}

E8) Given that $\alpha_1 < \alpha_2$. We have to prove that $A_{\alpha 1} \supseteq A_{\alpha 2}$.
Let $x \in A_{\alpha 2}$. By definition to alpha-cut we have
$\mu_A(x) \geq \alpha_1$,
$\Rightarrow \mu_A(x) \geq \alpha_1$, since $\alpha_1 < \alpha_2$
$\Rightarrow x \in A_{\alpha 1}$
Hence $A_{\alpha 1} \supseteq A_{\alpha 2}$.

E9) Very Excellent = 0.04/8+0.36/9+1/10

Very Good = 0.01/6+0.25/7+0.81/8+1/9+1/10

Very Fair = 0.09/2+0.36/3+0.81/4+1/5+0.81/6+0.25/7+0.01/8

Very Bad = 1/1+0.49.2/2+0.16/3+0.01/4

E10) (i) $\mu_{A\cup B}(x)=0.9$
(ii) $\mu_{A\cap B}(x)=0.3$
(iii) $\mu_{\overline{A}\cup\overline{B}}(x)=0.6$
(iv) $\mu_{\overline{A}\cap\overline{B}}(x)=0.1$
(v) $\mu_{\overline{A\cup B}}(x)=0.1$
(vi) $\mu_{\overline{A\cap B}}(x)=0.6$

E11) (i) $\overline{C\cup L}=\{0.8,0.5,0.2,0,0.3,0.2,0,0,0,0\}$..............(a)

$\overline{C}\cap\overline{L}=\{0.8,0.5,0.2,0,0.3,0.7,1,1,1,1\}\cap\{1,1,0.8,0.6,0.4,0.2,0,0,0,0\}$

$=\{0.8,0.5,0.2,0,0.3,0.2,0,0,0,0\}$...................(b)

From (a) and (b) we have $\overline{C\cup L}=\overline{C}\cap\overline{L}$

(ii) $\overline{C\cup L}=\{1,10.8,0.6,0.4,0.7,1,1,1,1,\}$..............(a)

$\overline{C}\cap\overline{L}=\{0.8,0.5,0.2,0,0.3,0.7,1,1,1,1\}\cap\{1,1,0.8,0.6,0.4,0.2,0,0,0,0\}$

$=\{1,1,0.8,0.6,0.4,,0.7,1,1,1,1,\}$...................(b)

From (a) and (b) we have $\overline{C\cup L}=\overline{C}\cap\overline{L}$

E12) (i) From the answer of E4 we have,
Bad = 1/1+0.7/2+0.4/3+0.1/4
Therefore, Not Bad = 0.3/2+0.6/3+0.9/4+1/5+1/6+1/7+1/8+1/9+1/10....(a)
Similarly, we have from E4,

Good = 0.1/6+0.5/7+0.9/8+1/9+1/10
Therefore, Very Good = 0.01/6+0.25/7+0.81/8+1/9+1/10
Not Very Good = 1/1+1/2+1/3+1/4+1/5+0.99/6+0.75/7+0.19/8…..(b)
Apply the fuzzy intersection between (a) and (b) we get,
Not Bad but Not Very Good = 0.3/2+0.6/3+0.9/4+1/5+0.99/6 0.75/70. 19/8

(ii) From E4 we have
Good = 0.1/6+0.5/7+0.9/8+1/9+1/10 … (a)
Excellent = 0.2/8+0.6/9+1/10 ... (b)
Applying fuzzy NOT operation on (a) we get,
Not Excellent = 1/1+1/2+1/3+1/4+1/5+1/6+1/7+0.8/8+0.4/9 … (c)
Now, applying fuzzy intersection operation between (a) and (c) we get,
Good but Not Excellent = 0.1/6+0.5/7+0.8/8+0.4/9