UNIT 1 MATHEMATICAL MODELLING – AN OVERVIEW

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1.1 INTRODUCTION

Mathematics is a very effective tool in solving real world problems. The critical step in the use of mathematics for solving real world problems is the building of a suitable mathematical model. A mathematical model is a conversion of a real world problem into an abstract mathematical problem involving mathematical concepts such as constants, variables, functions, equations, inequalities, etc. The process by which a real world problem is represented and interpreted in terms of a mathematical model is called **mathematical modelling**. Our main aim in this unit is to introduce you to the basic concepts of mathematical modelling and discuss the process of development of a mathematical model.

Translation into a mathematical model is one of the many approaches to solving real-world problems. Other approaches include experimentations with physical models or schematic models or with the real world directly. The mathematical approach, which is our main concern throughout this course, has a number of advantages which we shall be illustrating through examples. However, to give you a broader idea about various models, we shall start the unit by discussing model classifications in Sec. 1.2. In Sec. 1.3 we shall discuss the process of translating a real world problem into a mathematical problem. The need for mathematical modelling is also illustrated in this section through various examples. Sec. 1.4 introduces you to different types of modelling.

Objectives

After studying this unit, you should be able to

- define a mathematical model;
- realise the need for developing a mathematical model;
- translate a real world problem into its equivalent mathematical formulation;
- identify the type of modelling to be used for a given problem.

1.2 MODEL CLASSIFICATIONS

A model is an abstraction of reality or a representation of a real object or situation. It could be a simple drawing of office plans or a complicated functional representation of a complex machinery part. A model airplane may be assembled from children's kit, or it may actually contain an engine and a rotating propeller that enable it to fly like a real plane.

Some models are replicas of the physical properties (relative shape, form, and weight) of the object they represent. Some are physical models but do not have the same physical appearance as the object of their representation. Other type of models deal with symbols, expressions, mathematical equations and inequalities. Each of these models can be classified into four main categories: physical models, schematic models, verbal models and mathematical models.

Physical Models

Physical models are prototypes models that look like the objects they represent. They are more or less the exact replicas of the object being modelled. Scale models of Taj Mahal, airplanes, buses, ships, office complexes, shopping centres, homes, etc. look exactly like their counterparts but in much smaller scale. The advantage in having a scaled model is that one can tell exactly what the object under study looks like, in three dimensions, before making a major investment. Also, some of these models can even perform as their counterparts would and this allows you to conduct the study on the model to see how it might perform under actual operating conditions. Scaled models of airplanes can be tested in wind tunnels to determine aerodynamic properties and the effects of air turbulence on their outer surfaces. Models of bridges and dams can be subjected to multiple levels of stress from wind, heat, cold and other factors to test their effects as endurance and safety. Scaled models that behave in a manner similar to the real objects are less expensive to create and test than their actual counterparts.

Schematic Models

Schematic models are more abstract than physical models. They do not look like the physical reality they represent. Graphs and charts are schematic models that provide pictorial representations of mathematical relationships. Mathematical linear relationship between two variables may be indicated by plotting a line on a graph. Pie charts, bar charts and histograms can all model some real situations but do not bear any physical resemblance to them. Diagrams, drawings, blueprints and a flow chart describing a computer program are all examples of schematic models.

Verbal Models

Verbal models use words to represent some object, situation or problem that exists, or could exist, in reality. This could be a simple word presentation of scenery described in a book to a complex business problem (described in words and numbers). Verbal models provide all relevant and necessary information to solve the problem, make recommendations and suggest alternatives. The case studies which you must have studied from management text books are examples of verbal models that expose you to the workings of a business without having to visit the firm's actual premises. Often these verbal models provide enough information to be converted into mathematical models.

Mathematical Models

Mathematical models are the most abstract of the four classifications. These models do not look like their real-life counterparts at all. Mathematical models are built using numbers, symbols, variables, empirical laws related together by means of equations or equalities. Mathematical models can take many forms like statistical models, optimization models, algebraic/differential equations or game theoretic models, etc. In this unit and units to follow, we shall concentrate on mathematical models. We shall be discussing in detail the process and need of mathematical modelling, types of mathematical models and we shall formulate some models with the contexts taken from biology, physics, economics, finance, medicine, etc. Before we discuss basic concepts of mathematical modelling in this unit, you may try the following exercise.

E1) Give two examples each of the physical, schematic, verbal and mathematical models.

Let us now discuss the process of mathematical modelling.

1.3 MATHEMATICAL MODELLING – WHAT AND WHY?

Mathematics is a rich and interesting discipline. It provides a set of ideas and tools effective in solving problems which arise in other fields like history, philosophy, sciences, sociology, political science, life sciences, medical sciences, engineering, etc. It also provides concepts useful for theoretical approach in other fields. Mathematics may be applied to specific problems already posed in mathematical form, or it may be used to formulate such problems. In theory construction, mathematics provides abstract structures which may be used as tools in understanding problems arising in other fields. Problem formulation and theory construction involve a process known as **mathematical modelling** or, mathematical model building. A mathematical model, as we have already mentioned, is an abstract model that uses mathematical language to describe a system. It can be viewed as a representation of the essential aspects of an existing system (or a system to be constructed) which presents knowledge of that system in useable form.

A 'system' is a collection of one or more related objects i.e., physical entities with specific characteristics or attributes.

Mathematical modelling usually begins with a situation in the real world. These situations may arise in different disciplines like engineering, physics, physiology, psychology, ecology, wildlife management, chemistry, economics, sports etc. A psychologist, for example, observes certain types of behaviour in rats running in a maze, a wildlife ecologist notes the number of eggs laid by endangered sea turtles, or an economist records the volume of international trade under a specific tariff policy. Each tries to observe and predict future behaviour. Their efforts may be based completely on intuition, but often they are the result of detailed study, experience, and observing the similarities between the current situation and other situations which are better understood. This study of the system, the accumulation and organisation of information and stating the problem to be studied is in fact the **first step** in model building.

The **next step** is to make the problem simpler by making certain assumptions and approximations. It requires to identify and select the concepts/information to be retained in the problem. For example, the psychologist studying rats in a maze may decide that it makes no difference that all the rats are black or that the maze is constructed of wood. On the other hand, it may be significant for her that all the rats are siblings and the maze is divided into compartments. He may also assume that a rat is always in exactly one compartment and never half in one and half in another compartment.

The **third step** in modelling is to replace the real quantities and processes by mathematical symbols, a set of variables and a set of equations/inequalities that establish relationships between these variables. The values of the variables can be practically anything; real or integer numbers, Boolean values or strings, for example. The variables represent some properties of the system, for example, we may measure system outputs often in the form of signals, timing data, event occurrence (yes/no). The actual model is the set of functions that describe the relations between the different variables.

After the problem is formulated, the **fourth step** is the study of the resuiting mathematical system using appropriate mathematical tools and techniques. This may involve a calculation, solving an equation, proving a theorem, etc. The motivation is to produce new information about the problem being studied. It is likely that new information can be obtained by using well-known mathematical concepts and techniques. If not, we may need to develop new techniques or adopt tested methods from other disciplines.

The **final step** in the model-building process is the evaluation of a mathematical model i.e., comparison of the results predicted on the basis of the mathematical analysis with the real world. It is important to know whether or not our model gives reasonable answers i.e., Does our model reflect all the important aspects of the real world problem? If a model is not accurate enough, then, we need to refine our model. We may need a new formulation, a new mathematical analysis and hence a new evaluation. It usually happens that the model-building process proceeds through several iterations, each a refinement of the preceding, until, finally, an acceptable one is found. Broadly, we can divide the modelling process as follows:

- **Formulation** which involves the following three steps:

  i) **Studying the problem:** Accumulating and organising the information about the real world problem. Describing the context of the problem and stating the problem within this context.

  ii) **Identifying relevant information:** Identifying and selecting the concepts/information which are significant and to be retained in the problem.

  iii) **Mathematical representation:** Replacing real quantities and processes by mathematical symbols, a set of variables and a set of equations/inequalities that establish relationship between variables.

- **Mathematical Analysis** which studies the formulated problem using appropriate mathematical tools and techniques.

- **Evaluation** which decides whether the model and its analysis explain the phenomenon/problem we are interested in. If it is not a good model then we need to refine it. There is no unique or correct model; but there are good models and bad models. The skill of modelling lies in being able to judge which is which.

Pictorially, we can represent the process of mathematical modelling as shown in Fig. 1 below.

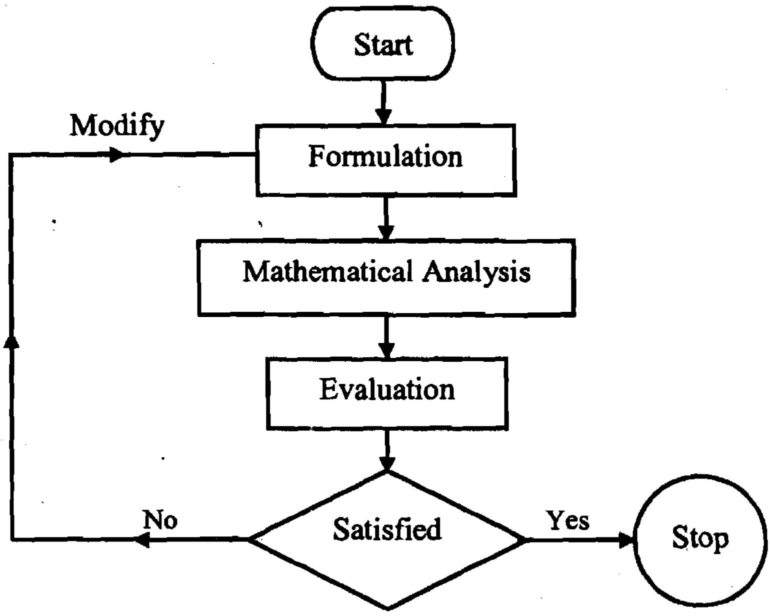

Fig. 1

After a model is evaluated and found good, there are several uses of the model: i) it helps in our better understanding of the real physical system, ii) it can serve as a tool in the prediction of the future state of the system which is currently unknown, iii) it can help in doing trial experiments by changing the parameter values or in perturbing the system to produce a desirable condition, i.e., many unknown parameter values can be estimated.

Let us now look at some simple mathematical models. We shall not go into the details of their formulations here since you must be already familiar with them.

Example 1: *Lotka-Volterra Equations*

The simplest model of interacting populations was proposed independently by Lotka in 1925 in the United States and by Volterra in 1926 in Italy. If x and y are respectively, the predator and prey populations, then their model is in terms of the following equations:

$$\frac{dx}{dt} = \alpha xy - \beta x$$

$$\frac{dy}{dt} = \gamma y - \delta xy,$$

$\alpha, \beta, \gamma, \delta$ are positive constants.

γ and β represent constant specific birth rate of the prey and mortality rate of the predator respectively.

The predator's specific growth rate αy depends on availability of the prey as food source, whereas, the prey death rate δx depends on the number of predators. These equations have oscillatory solutions but have the unsatisfactory feature of forming a conservative system, and oscillations of any magnitude are possible. More realistic versions of this model remove this degeneracy. This model is discussed in detail in Unit 9, Block-3 of our undergraduate course MTE-14 on 'mathematical modelling'.

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Example 2: *Model of a Particle in a Box*

In physics, the **particle in a box** (also known as the **infinite potential well** or the **infinite square well**) is a problem consisting of a single particle inside an infinitely deep potential well, from which it cannot escape, and which loses no energy when it collides with the walls of the box. In one dimensional case, the problem can be described as a single point particle enclosed in a box inside which it experiences no force i.e., it is at zero potential energy. At the walls of the box, the potential rises to infinity, forming an impenetrable wall.

In classical mechanics, the problem can be modelled using Newton's laws of motion and the solution to the problem is trivial: The particle moves in a straight line, always at the same speed, until it reflects from a wall. A quantum-mechanical solution of the problem becomes very interesting and reveals some decidedly quantum behaviour of the particle that agrees with observations but contrasts sharply with the predictions of classical mechanics. According to quantum theory, particle has no definite position or velocity. Rather, a probabilistic interpretation is given to the state of the particle in terms of a time-independent wave function $\psi(x)$. The square of the wave function $|\psi|^2$, is a probability density.

The problem of a particle situated in a 1-dimensional infinite square well with momentum only in the direction of quantum confinement (the x-direction) is described by the **schrödinger equation**, a basic equation of quantum mechanics, given by

$$\frac{-h^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\,\psi(x) = E\,\psi(x)$$

where h is the reduced planck constant,

m is the mass of the particle,

$\psi(x)$ is the complex-valued stationary time-independent wave function,

$V(x)$ is the spatially varying potential and

E is the energy, a real number.

For the case of the particle in a 1-dimensional box of length L, the potential is zero inside the box, but rises to infinity at $x = 0$ and $x = L$. Thus, the equation reduces to

$$\frac{-h^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\,\psi(x),\ 0 < x < L.$$

the solution to which can be found easily under appropriate boundary conditions. One of the possible choices is

$$\psi(0) = \psi(L) = 0.$$

The motivation for this choice is that the particle is unlikely to be found at a location with a high potential (the potential repulses the particle), thus the probability of finding the particle, $|\psi(x)|^2$, must be small in these regions and decreases with increasing potential. For the case of an infinite potential, $|\psi(x)|^2$ must be infinitesimally small or 0, thus $\psi(x)$ must be zero in this region. Thus, we get $\psi(0) = \psi(L) = 0$.

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Example 3: *Model of Rational Behaviour for a Consumer*

In this model, we assume that a consumer faces a choice of n commodities labelled $1, 2, \ldots, n$ each with a market price $p_1, p_2, \ldots, p_n$. The consumer is assumed to have a cardinal utility function U (cardinal in the sense that it assigns numerical values to utility), depending on the amounts of commodities $x_1, x_2, \ldots, x_n$ consumed. The model further assumes that the consumer has a budget M which she uses to purchase $x_1, x_2, \ldots, x_n$ in such a way as to maximise $U(x_1, x_2, \ldots, x_n)$. The problem of rational behaviour in this model then becomes an optimization problem, that is:

$$\max U(x_1, x_2, \ldots, x_n)$$

$$\text{subject to: } \sum_{i=1}^{n} p_i x_i \le M,\ x_i \ge 0\ \forall\ i = (1, 2, \ldots, n).$$

This model is used in general equilibrium theory, particularly to show existence and optimality of economic equilibria. However, the fact that this particular formulation assigns numerical values to levels of satisfaction is the source of criticism. However, it is not an essential ingredient of the theory, only an assumption.

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And now an exercise for you.

E2) Give at least two examples of the formulas you are already familiar with as the mathematical models of the real situations.

Why formulate a Mathematical Model?

As we mentioned earlier, the use of mathematics is one of many approaches for understanding and solving a real world problem. Others include experimentation with scaled physical models in a laboratory on a smaller scale simulating all the conditions of the real situation, or with the real world directly. But these may be highly risky and costly as they may involve the use of explosive and expensive chemicals or materials. On the other hand, mathematical approach has many advantages. Mathematical modelling is very inexpensive and provides systematic approach to problem solving. Once developed properly, a great deal can be learnt about the real-life situation by manipulating a model's variables and analysing the results. Mathematical modelling requires users to accumulate and organize information and, in the process, to indicate areas where additional information is needed, and hence increase the understanding of the problem.

Let us look at some of the problems which illustrate the use of mathematical modelling.

For meteorological purposes, rockets are launched so that they reach a certain altitude and record atmospheric conditions such as temperature and pressure. In such cases we are confronted with the following problem:

Example 4: What is the rocket thrust level and duration necessary to ensure that the rocket reaches the desired altitude?

For this problem, a solution, based on experimentation involving rocket launches with different thrust-time, is unacceptable due to the cost and the uncertainty of success. Further, the solution to this problem cannot be obtained using scale-model experiments. For this kind of study, a mathematical approach is preferred.

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In large supermarkets you must have seen a number of checkout counters. Customers prefer to stand in a queue which is the shortest. The problem to be sorted out by the management is the following:

Example 5: What is the optimum number of check out counters that the supermarket should have in order to maximize the expected profit?

Information of this kind is frequently needed for planning purposes. Fewer counters imply long queue lengths and can affect customer goodwill. On the other hand, if there are too many checkout counters, manning them reduces the total profit. There is really no scientific alternative to a mathematical treatment for problems of this kind.

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You may now think of more situations like this where mathematical treatments of the problem become necessary.

E3) Give two situations from the field of management where mathematical treatment of the problem is necessary to get the required solution.

Mathematical models can be classified into different categories depending on the mathematical structure of the underlying formulation. We shall discuss these classifications in the next section.

1.4 CLASSIFYING MATHEMATICAL MODELS

According to the structure of the models we can classify mathematical models into the following four types:

(i) **Linear vs. Nonlinear**

Mathematical models are usually composed by variables, which are abstractions of quantities of interest in the described systems, and operators that act on these variables, which can be algebraic operators, functions, differential operators, etc. If all the operators in a mathematical model present linearity, the resulting mathematical model is defined as **linear**. A model is considered to be **nonlinear** otherwise.

For example, consider the equation

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial C}{\partial x}\right). \qquad (1)$$

Eqn. (1) is a one-dimensional diffusion equation (also known as Fick's II law) where $C(x, t)$ is the concentration of diffusing substance, x is the space coordinate and D is the diffusion coefficient. This equation represents the diffusion of a substance in the x direction due to a concentration gradient in that direction. In some cases, e.g., diffusion in dilute solutions, D can be approximated by a constant, while in other cases, e.g., diffusion in high polymers, D depends on concentration C.

For the case when D is a constant, Eqn. (1) becomes

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}. \qquad (2)$$

which is a linear differential equation and hence is said to be a **linear model** of the problem. You may note that Eqn. (1) is a second order linear partial differential equation with C as dependent variable and t and x as independent variables. In general Eqn. (2) along with appropriate initial and boundary conditions is solvable. You must have also come across this equation and solved it in your differential equation course at the undergraduate level (Ref. Unit 17, Block-4, MTE-08).

If, on the other hand, D is not a constant but depends on C, say, for example, if $D = D_0 \exp[\beta\,(C - C_s)]$ where D_0, β, C_s are constants then Eqn. (1) reduces to

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left[D_0 \exp[\beta(C - C_s)]\,\frac{\partial C}{\partial x}\right]. \qquad (3)$$

Eqn. (3) is a non-linear PDE and hence the model obtained is a **non-linear** one.

You may **notice** that if the model tries to replicate more closely the real situation (i.e., D is a function of the concentration in this case), the corresponding mathematics involved is more challenging. (solving a non-linear PDE, in this case).

(ii) **Static vs. Dynamic**

A **static** model does not account for the element of time and hence the variables and relationships describing the system are time-independent. Consider, for instance, the transportation problem generally associated with industries:

Suppose there are m origins O_i, $i = 1, 2, \ldots, m$ in an industry, where various amount of a commodity are produced or stored for transportation to n destinations D_j, $j = 1, 2, \ldots, n$. It is then required to transport all units of the commodity from all O_i, exhaustively, to all D_j, exactly satisfying all their requirements in such a way that the total transportation cost becomes minimum. The following assumptions are made:

i) The i^{th} origin can supply exactly a_i unit of the commodity.

ii) The j^{th} destination can accept exactly b_j unit.

iii) The cost of transportation of one unit from O_i to D_j is C_{ij}.

This transportation problem, therefore, is essentially a minimization problem. Mathematically, if x_{ij} = number of units transported from O_i to D_j, then we want to

$$\text{minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} x_{ij} \quad (4)$$

subject to $$\sum_{j=1}^{n} x_{ij} = a_i, \; i = 1, 2, \ldots, m \quad (5)$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \; j = 1, 2, \ldots, n \quad (6)$$

where $x_{ij} \geq 0 \; \forall \; i, j$ and $a_i, b_j > 0 \; \forall \; i, j$. (7)

Eqns.(4)-(7) define the transportation problem where all variables are independent of time. Such a system is a **static** system. We shall discuss this model in detail in Unit 5 of this course.

A system is open/closed if the objects in the system do/do not interact with objects of the super-system which do not belong to the system.

In contrast to the static system, in **dynamic** system, time plays a very important role. The relationship between the variables describing the system changes with time. If you look at the problem of rocket launch then it can be described in terms of a closed system consisting of two objects-the rocket and the earth. The variables describing the rocket are its position and velocity relative to some fixed point on the earth, and the interaction between the two objects is given by the theory of dynamics. In this description we may as well include the influence of other planets in the solar system by treating the planets as belonging to the environment of the system and the system being open. In either case, the variables- i.e. position and velocity of the rocket-change continuously with time. Hence the system is a dynamic system.

(iii) Discrete vs. Continuous

Mathematical model may be discrete or continuous according as the variables involved are discrete or continuous. In models involving changes over time, the changes may take place continuously with time and the variables of the system are described for all time instants over the interval of interest. On the other hand, the changes may occur at discrete instants of time and the variables described only for the relevant time instants. Thus, one should be clear whether to treat the time element as continuous or discrete. In certain instances, the data available are such that a discrete treatment of time is more appropriate as opposed to continuous. For instance, if a soft drink manufacturing company is interested in estimating the weekly demand of the soft drink, then the time element is a week and hence it has to be treated as a discrete variable.

Models involving continuous variables are often expressed through differential equations-ordinary or partial, whereas, those involving discrete characterisation result in difference equations. Consider for instance, the mathematical model, referred to as the classical Malthusian population scheme for population growth given by Thomas R. Malthus (1766-1834). The model is based on the idea that the population size for one generation depends on the size of the previous generation. This is expressed mathematically by the following equation

$$p_{t+1} = r \times p_t \tag{8}$$

where:

- t : represents the time period (which could be minutes, hours, weeks, years, etc.) depending on the species being considered
- p_t : represents the population size at time t. The units of time could be hours, days, years, etc.
- p_{t+1} : represents the population size at the next time period. Again it could be the next hour, next day, next year, etc.; and
- r : referred to as the Malthusian factor, is the multiple that determines the growth rate.

The model given by Eqn. (8) which is a difference equation, is a **discrete model**. It allows you to find out the value of p at different discrete time intervals say, at years $3, 4, 5$, etc. You couldn't use this model to find out the size of p when $t = 3.578$ because $t = 3.578$ does not represent a prescribed discrete time. The value of r in Eqn. (8) has a strong impact on how fast the population will grow. We shall be discussing this model and other discrete and continuous population models in detail in Unit 4. Analogous **continuous** model of the population growth resulting in differential equation is given by the equation

$$\frac{dx(t)}{dt} = r\,x(t),\ x(0) = x_0. \tag{9}$$

where $x(t)\ (> 0)$ is the size of the population at time t, x_0 is the size of the population at the initial time and r represents the net growth rate. For details of this model refer to Unit 8, Block-3, MTE-14.

(iv) **Deterministic vs. Stochastic**

A system is said to be deterministic if the values assumed by the variables or the changes in the variables are known with certainty. Consider for instance, the problem of rocket launch considered in Example 4. The variables of the system are the position and velocity of the rocket. The laws of classical dynamics can be used to describe the motion fairly accurately and the changes in position and velocity can be predicted with a high degree of certainty. Hence, in this case we can view the system as being **deterministic**. On the other hand, in a **stochastic** model, randomness is taken into account and variables are described by probability distributions instead of unique values. For example, a manufacturer, having trouble deciding whether to build a large or small facility knows that the solution to this capacity problem depends upon the volume of demand. High demand would require a large facility while low demand would require a small facility. While the manufacturer has no way of knowing with certainty what the exact demand will be, he can, at least, determine the probability of the occurrence of each (high or low). For example, if the manufacturer estimates that the probability of the occurrence of high demand is 70 percent and the occurrence of low demand is 30 percent, he can use this information along with the monetary value (expected pay-off) of each situation to construct mathematical models such as **pay-off matrices** to find an optimal decision (see Table-1)

Table-1

	High demand (70%)	**Low demand (30%)**
Large facility	Rs.240, 000	–Rs.60,000
Small facility	Rs.120, 000	Rs.100,000

This type of model is said to be a **stochastic optimization model**. We shall be discussing some optimization models in Unit 5. Some models of this type are also discussed in Unit 12, Block-4 of MTE-14.

Once again look at the problem of supermarket operation considered in Example 5. The problem is to determine the optimum number of checkout counters in a supermarket to maximize the expected profit. The model can be successful once a relationship between some measure of queue (e.g. average queue length or average time period for which the customer has to wait before being served) and the number of checkout counters is obtained. The variables characterizing the system are the number of customers, their arrival rate, their departure rate, service time, peak period, etc. Here the arrival of customers, departure of customers and their service time are all random. They cannot be determined uniquely, rather they are given by certain probability distributions. Stochastic models based on fitting these probability distributions to the arrival, departure and service time can be obtained. For the details of some of these models you can refer to Unit 14, Block-4 of Mathematical Modelling course (MTE-14).

In real life, there is always uncertainty. If the uncertainty is insignificant then it can be ignored and the system can be treated as a deterministic system. This is a process of simplification. If the uncertainty is significant then it cannot be ignored and must be taken into account while characterising the system.

As mentioned earlier, most of the discrete and stochastic models lead to difference/algebraic equations whereas the knowledge of algebraic/differential equations is required while dealing with linear/nonlinear, static/dynamic, and continuous models. The skills you have in algebra, calculus, analysis, differential equations, optimization and probability theory will be useful for successfully dealing with mathematical modelling. The type of mathematics required depends on the type of model formulated.

You may now try the following exercises.

E4) State the types of modelling you will use for the following problems. Also give reasons in support of your answer.

a) Human motion (e.g. walking, lifting, jumping) involves dynamic action at one or more joints in the body. If the joints function normally they cause no discomfort otherwise they cause severe pain when in motion. One way of reducing the pain and discomfort is to replace the defective joint with an artificial one. For the artificial joint to be effective it must be capable of executing all the motions of a normal joint. The problem is to **build a model** to describe the functioning of a joint, say, hip joint, so that it is useful to bioengineers when designing hip replacements.

b) Research and development (R&D) is an important activity in any modern industrial organization. The R&D manager is faced with the difficult problem of allocating limited resources (money, manpower, space, etc.) between a number of competing projects. The problem is to **build a model** to help the R&D manager to make right decisions.

c) In a chemical process, the output quality and yield depend critically on the levels assigned to relevant factors (temperature, pressure, concentration, etc.) It is important to optimally select these levels as well as to monitor and control the system to ensure that they stay at the desired levels. The problem is to **build a model** to help achieve this.

d) The formation of sand dunes and their encroachment into de-forested lands near deserts has become a serious problem in many parts of the world. It is needed to predict the spread of desert, as well as to devise policies to control the spread. The problem is to **build a model** to describe the movement of sand dunes.

e) Advertising is a means by which a manufacturer can promote the product and improve sales and hence, the revenue generated. However, advertising costs money and is worthwhile only if the cost of advertising is less than the increase in the sale revenue. The problem is to evaluate the effectiveness of different advertising policies and the selection of the optimal policy. The problem is to **build a model** to help solve this problem.

1.5 LIMITATIONS OF A MATHEMATICAL MODEL

Mathematical modelling is a multistage activity involving various concepts and techniques. Ideally, a mathematical model ends by returning to its origin. We need to check whether the model and its analysis explain the phenomenon we are interested in. The whole art of mathematical modelling lies in its self-consistency. A mathematical model is an adequate mathematical model if it captures the salient features of the system associated with the problem and is capable of yielding a meaningful solution to the original problem.

It is rare that we obtain an adequate mathematical model at the first attempt for the problem under consideration. In general, an iterative procedure is needed where improvements are progressively made until an adequate mathematical model is obtained. The adequacy of a model is established by checking the validity of the assumptions made in building the model and by the closeness of the agreement between the behaviour of the model and the system under consideration. For example, while developing a model to describe the motion of a simple pendulum if the time interval of study is sufficiently small, so that the energy loss due to frictional drag is very small, then the assumption that the frictional drag is negligible is valid. However, if the time interval of study is large then the assumption of negligible frictional drag is not valid, since the effect of drag on the pendulum motion is cumulative.

In a rocket launch model, the assumption that the thrust generated by the rocket is an impulse, is valid if the thrust lasts for a very small fraction of the total flight time, something that is not known initially. Only after a first solution, can this be checked and if the need be, the model has to be modified. This emphasizes the iterative nature of modelling.

To end the unit we now give the summary of what we have covered in it.

1.6 SUMMARY

In this unit we have covered the following points:

1) Mathematical model uses mathematical language to describe a real world problem.
2) Mathematical models are built using numbers, symbols, variables related together by means of functions, formulas, empirical laws, equations or equalities and can take many forms like statistical models, optimization model, algebraic, differential equations or game theoretic models, etc.
3) The process of mathematical modelling involves three main steps – formulation, mathematical analysis and evaluation.
4) Depending on the mathematical structure of the underlying formulation, mathematical models can be classified into linear/nonlinear, static/dynamic, discrete/continuous and deterministic/stochastic models.
5) The skills in subjects like algebra, calculus, analysis, differential equations, statistics optimization, matrix theory and probability theory are useful for dealing with mathematical modelling. The type of mathematics required depends on the type of model formulated.
6) A mathematical model is adequate if it captures the salient features of the system associated with the problem and is capable of yielding a meaningful solution to the original problem.
7) Mathematical modelling is an iterative process where improvements are progressively made until an adequate mathematical model is obtained.

1.7 SOLUTIONS/ANSWERS

E1) Physical model – models of comic book super-heroes which look exactly like their counterparts but in much smaller scale.

Schematic model – a diagram showing the sequence of activities that must be maintained in an assembly-line balancing.

Verbal model – word problems given in any mathematics book. Similar examples of various types may be given.

Other similar examples from your surroundings or real life experience may be given.

E2) You may give examples of familiar mathematical models from your own experience.

E3) A firm that assembles computers and computer equipment is to start the production of two new types of computers. Each type will require assembly time, inspection time and storage space. The amounts of each of these resources that can be devoted to the production of the computers is limited. The manager of the firm likes to determine the quantity of each computer to be produced in order to maximize the profit generated by their scale. For this kind of study a mathematical approach is preferred.

Examples of other similar situations may be given.

E4) a) Dynamic, continuous, deterministic.
b) Static, discrete, deterministic.
c) Dynamic, continuous, deterministic.
d) Dynamic, discrete, probabilistic.
e) Static, discrete, probabilistic.

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