UNIT 5 CONTINUOUS TIME MARKOV PROCESSES-I

Structure

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5.1 INTRODUCTION

In the previous unit, we followed up on the size of the offspring population from one generation to another. Thus, it was as if we were dealing with the number of additions to the population through births over generations. In other words, X_n in the previous unit represented the number of births caused by the X_{n-1} elements of generation $(n-1)$, i.e,. X_n represented the number of descendents over a (generation) time period considered as one unit. It was as if we were counting time on a discrete scale. But there are phenomenon that experience continuous reproduction rendering the concept of a generation in their context inappropriate, and instead of looking at $\{X_n : n \geq 0\}$ one needs to look at $\{X_t : t \geq 0\}$ where X_t can be an alternative notation for $X(t)$ for convenience) is the number of 'births' or 'events' by time t or during the interval $[0, t]$ or alternately (equivalently) the 'population' at time t, depending on the context. This phenomenon can be seen in the following situations.

(i) Suppose, 'customers' arrive at a service point (like a ticket counter, or a doctor's chamber or barber shop, etc.) in a random fashion over time. We may denote $X(t)$ by the number of customers who arrived at the counter for service between time zero and t, i.e. by time t. It may be of interest, for instance, to study if the service facility is adequate or not in matching the arrivals.

(ii) Think of a machine of equipment which while working may fail from time to time. Suppose we have a very efficient maintenance system in place so that the moment it fails, it is repaired instantaneously so that the machine is once again functional. To work out a suitable maintenance strategy, it may be necessary to study the number of failures $X(t)$ that takes place between time zero and t, i.e., by time t.

We shall begin this unit by providing the necessary background on continuous time Markov Chains in Sec. 5.2. In Sec. 5.3, we give the mathematical formulation of continuous time Markov branching process in terms of the generating function and the mean function. We also provide a brief discussion on the extinction probabilities. In Sec. 5.4, we consider growth processes categorised as pure birth processes which are some variation of the branching process in continuous time. Here, we derive population size distribution at time t for Yule and Poisson processes besides some properties of the Poisson processes.

Objectives

After reading this unit, you should be able to:

- get the basic concepts of stochastic process in continuous time;
- use basic Kolmogorov forward/backward equations for obtaining population size distributions of special continuous time processes;
- obtain the generating function, the mean function of generation size at time t, and the extinction probability for a Markov branching process in continuous time;
- define a pure birth process and apply it to derive Poisson process and Yule process.

5.2 PRELIMINARIES

Before, we start continuous time Markov Process, let us discuss the prerequisites.

5.2.1 Continuous Time Markov Chain

Before we proceed to discuss 'Continuous Time Branching Processes" we need to recall certain concepts of Markov processes with discrete state space, which are also referred to as continuous time Markov chains. The objective is to prepare the ground for the introduction of Markovian continuous branching processes.

Consider a continuous time stochastic process $\{X(t): t \geq 0\}$ having a finite or countably infinite state space which we represent without any loss of generality by $S = \{0, 1, 2, \ldots\}$. Such a process is known as a **jump process**. If $X(t)$ **counts** the number of events (like number of births, number of failures of a machine, etc.) by time t, then $X(t)$ is called a **counting process**. Thus, for a process $\{X(t): t \geq 0\}$ to a counting process,

i) $X(t) \geq 0$ for all t

ii) $X(t)$ is integer-valued

iii) $X(s) \leq X(t)$ for $0 \leq s < t$, and

iv) $X(t) - X(s)$ for $0 \leq s < t$ would represent the number of events occurring in time interval $(s, t]$.

If for any process $\{X(t): t \geq 0\}$, $X(t_1), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent random variables for all choices of $n \geq 2$ and $0 \leq t_1 < t_2 < \ldots < t_n$, then the stochastic process is said to have **independent increments**.

Define for $i, j \in S$, the transition probability function

$$P_{ij}(t) = P(X(t) = j \mid X(0) = i),\ t > 0,$$

$$P_{ij}(0) = \delta_{ij},$$

where δ_{ij} is the Kronecker's delta, i.e.

$$\delta_{ij} = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}$$

with $\sum_{j \in S} P_{ij}(t) = 1$. The matrix $(P_{ij}(t))$ is called the transition probability function matrix associated with the stochastic process $X(t), t \geq 0$. For a continuous time

Markov chain, we require

$$P(X(t)=j \mid X(t_1)=i_1,\ X(t_2)=i_2,\ldots,\ X(t_n)=i_n,\ X(s)=i)$$
$$= P(X(t)=j \mid X(s)=i) \qquad (1)$$

for every $n \geq 1$, $0 \leq t_1 < t_2 < \ldots < t_n < s < t$ and $i_1, i_2, \ldots, i_n, i, j \in S$. This is known as the **Markov property**. We shall deal with the case where the transition probability function is stationary with respect to the time, or time homogeneous, i.e., it depends only on the length of the time interval, but not on its location. Thus, in this case the right hand side of Eqn. (1) is essentially $P_{ij}(t-s)$. The Poisson process, or the branching process, that we shall discuss shortly, are examples of such a process.

Consider a Markovian stochastic process $\{X(t): t \geq 0\}$ with stationary transition probability function $(P_{ij}(t))$. Then, for $0 \leq t_1 < t_2 < \ldots < t_n$,

$$P(X(t_1)=i_1,\ X(t_2)=i_2,\ \ldots,\ X(t_n)=i_n)$$
$$= P(X(t_n)=i_n \mid X(t_{n-1})=i_{n-1})P(X(t_{n-1})=i_{n-1},\ X(t_{n-2})=i_{n-2},\ \ldots,\ X(t_1)=i_1)$$
$$= Pi_{n-1}i_n(t_n - t_{n-1})Pi_{n-2}i_{n-1}(t_{n-1} - t_{n-2})\ldots Pi_1i_2(t_2 - t_1)P(X(t_1)=i_1).$$

thus,

$$P(X(t)=j,\ X(t+h)=k \mid X(0)=i) = P(X(t)=j,\ X(t+h)=k,\ X(0)=i)/P(X(0)=i)$$
$$= P(X(t+h)=k \mid X(t)=j)\ P(X(t)=j \mid X(0)=i)$$
$$= P_{jk}(h)\ P_{ij}(t),$$

thus, for any $t, h \geq 0$, we have

$$P_{ij}(t+h) = P(X(t+h)=j \mid X(0)=i)$$
$$= \sum_{k \in S} P(X(t+h)=j, X(t)=k \mid X(0)=i)$$
$$= \sum_{k \in S} P_{ik}(t)\ P_{kj}(h). \qquad (2)$$

This is the **Chapman-Kolmogorov** equation. In matrix notation, this equation can be represented as

$$\mathbf{P}(t+h) = \mathbf{P}(t)\mathbf{P}(h) \qquad (3)$$

where $\mathbf{P}(t) = (P_{ij}(t))$ and $P(0) = (S_{ij}) = 1$ by definition. Notice that any t can be expressed in relation to another number $t_0 < t$ as $t = r\ t_0 + s$, where r is a positive integer and $0 \leq s < t_0$, so that

$$\mathbf{P}(t) = \mathbf{P}(r\ t_0 + s)$$
$$= \mathbf{P}(t_0)\ \mathbf{P}((r-1)t_0 + s)$$
$$= \mathbf{P}^2(t_0)\ \mathbf{P}((r-2)t_0 + s)$$
$$= \ldots$$
$$= \mathbf{P}^r(t_0)\ \mathbf{P}(s). \qquad (4)$$

This result has a deep significance; it shows that once $\mathbf{P}(t)$ is specified for $0 < t \leq t_0$, however small t_0 may be, we can use the Chapman-Kolmogorov equation to know $\mathbf{P}(t)$ for all $t > 0$.

Now, let us discuss Kolmogorov forward and backward equations in detail.

5.2.2 Kolmogorov Forward and Backward Equations

Let us assume that, as is logical, for all $i, j \in S$

$$\lim_{(t \to 0)} P_{ij}(t) = \delta_{ij}$$

In matrix notation,

$$\lim_{(t \to 0)} P(t) = I$$

Note that we have also defined

$$P(0) = I.$$

Also, let us assume that $P_{ij}(t)$ are continuous for $t > 0$. Then, under the above assumptions, it can be shown that $P(t)$ is differentiable at all $t \geq 0$ (of course, the derivative at $t = 0$ is the right hand derivative). Define for all $i, j \in S$

$$Q_{ij} = P_{ij}(0) = \lim_{(t \to 0)} \frac{\{P_{ij}(t) - P_{ij}(0)\}}{t}.$$

These are well-defined. The quantities Q_{ij} are called the **infinitesimal transition rates** and $Q = (Q_{ij})$ is called the **infinitesimal generator or infinitesimal rate matrix.**

Whenever the state space S is finite, then, differentiating both sides of Eqn. (2) with respect to t at $t = 0$, we get

$$P'_{ij}(h) = \sum_{k \in S} P'_{ik}(0)\, P_{kj}(h)$$

$$= \sum_{k \in S} Q_{ik}\, P_{kj}(h),$$

so that for all $t \geq 0$.,

$$P'(t) = Q\,P(t) \tag{5}$$

This is called **Kolmogorov's Backward Equation.** Again, differentiating Eqn. (2) with respect to h and setting $h = 0$, we get

$$P'_{ij}(t) = \sum_{k \in S} P_{ik}(t)\, P'_{kj}$$

$$= \sum_{k \in S} P_{ik}(t)\, Q_{kj}$$

so that for all $t \geq 0$,

$$P'(t) = P(t)Q, \tag{6}$$

which is **Kolmogorov's Forward Equation.**

It is interesting to note here that whenever S is finite and $P(0) = I$, the backward Eqn.(5) with the initial condition $P(0) = 1$, has a unique solution $P(t) = \exp(tQ)$, where we define

$$ext(tQ) = I + \sum_{n=1}^{\infty} t^n Q^n / n!.$$

as a format matrix series.

Now we are ready to discuss continuous time Markov branching process in this section.

5.3 CONTINUOUS TIME MARKOV BRANCHING PROCESSES

We shall start out by a explaining a popular notation regarding a real-valued function $f(.)$ defined on the real line. We shall write "$f(x) = 0(x)$ as $x \to 0$", read as "the function $f(x)$ is lower order as x goes to zero" to mean that $f(x)$ is such that $f(x)/x \to 0$ as $x \to 0$. For instance, $f(x) = x^{1.2} = 0(x)$ as $x \to 0$ while $f(x) = x \neq 0(h)$ as $x \to 0$. It is not hard to see that:

- If $f(x)=0(x)$ as $x \to 0$ and $g(x)=0(x)$ as $x \to 0$, then $f(x)+g(x)=0(x)$ as $x \to 0$.
- If $f(x)=0(x)$ as $x \to 0$, then for any scalar γ, $\gamma f(x)=0(x)$ as $x \to 0$.

We will have to make use of these two trivial results in our succeeding computations.

Let us formulate the Markov branching process in continuous time mathematically.

Consider a continuous time Markovian process $\{X(t): t \geq 0\}$, where $X(t)$ denotes the number of elements at time t given $X(0)=1$. Each element, after surviving for a random amount of time, splits upon producing a random number of elements (offspring) of identical type; we assume the process to be time homogeneous in the sense that an element splits upon producing j elements over a time duration $(t, t+h], h>0$ with probability

$$\delta_{1j} + \alpha_j h + 0(h), \quad j = 0, 1, 2, \ldots \tag{7}$$

as $h \to 0$, where δ_{1j} is the Kronecker delta, i.e., $\delta_{1j}=1$, or 0 accordingly as $j=1$ or $j \neq 1$. It is assumed that $\alpha_j \geq 0$ for $j \neq 1$, and $\sum_{j=0}^{\infty} \alpha_j = 0$. We further assume that individual elements act independently of each other in accordance with probabilities given in Eqn. (7). We observe that for $j \neq 1$

$$\begin{aligned} P_{n\, n+j-1}(h) &= P\{X(t+h)=n+j-1 \mid X(t)=n\} \\ &= n\{\alpha_j h + 0(h)\}\{1+\alpha_1 h + 0(h)\}^{n-1} + 0(h) \\ &= n\,\alpha_j h + 0(h), \end{aligned} \tag{8}$$

while

$$\begin{aligned} P_{nn}(h) &= P\{X(t+h)=n \mid X(t)=n\} \\ &= \{1+\alpha_1 h + 0(h)\}^n + 0(h) \\ &= 1 + n\alpha_1 h + 0(h). \end{aligned} \tag{9}$$

Let us pause for a while to ponder how these expressions could be arrived at from Eqn. (7). At time point t, $X(t)=n$, i.e., the number of elements is n at time t, let us explore the ways the event that the same can be $n+j-1$ after h units of time. It can happen in many ways, for instance, suppose one of the elements splits into j elements and each of the remaining $(n-1)$ elements is replaced by exactly one offspring so that at time $t+h$ the number of elements would be $j+n-1=n+j-1$. Since each element behaves independently of the other, we are in a binomial situation and the probability of this particular possibility is exactly equal to the first expression in Eqn. (8). But the same event can happen also if, say, one of them is replaced by $j-1$ elements, another by 2 elements, and that each of the remaining $(n-2)$ elements is replaced by exactly one offspring, so that at time $t+h$, $X(t+h)=(j-1)+2+(n-2)=n+j-1$. But the probability of the later will be, by independence again, is given by $n(n-1)\{\alpha_{j-1} h + 0(h)\}\{\alpha_2 h + 0(h)\}\{1+\alpha_1 h + 0(h)\}^{n-2}$ the whole of which is $0(h)$ as $h \to 0$. In the same way, all other possibilities can be shown to have probabilities which together will be of $0(h)$ as $h \to 0$. This explains the expression given in Eqn. (8).

Let us now examine Eqn. (9). One of the ways in which the number of elements remain the same at n both at time points t and $t+h$ is when each of the n elements at time point t is replaced by an element produced by it and the relevant probability is $\{1+\alpha_1 h+0(h)\}^n = 1+n\alpha_1 h+0(h)$. Any other possibility of achieving the event of the number of elements to be pegged at n at both time points t and $t+h$ will be of $0(h)$ as $h \to 0$.

Let

$$\phi(t;\, s) = \sum_{j=0}^{\infty} s^j\, P_{1j}(t) = E\, s^{x(t)} \tag{10}$$

the probability generating function of $P_{1j}(t) = P\{X(t)=j \mid X(0)=1\}$, $j = 0, 1, 2, \ldots$; trivially, $\phi(0;\, s) = s$. Also, because of independence amongst the elements, using arguments as in Unit 4, for $i \geq 0$, the probability generating function of $P_{ij}(t)$ can be seen to be

$$\sum_{j=0}^{\infty} s^j\, P_{ij}(t) = \{\phi(t;\, s)\}^i\,. \tag{11}$$

At this stage, let us recall the Chapman-Kolmogorov Eqn. (2) as follows

$$P_{ij}(t+h) = \sum\nolimits_{k \in S} P_{ik}(t)\, P_{kj}(h)\,.$$

Now, from Eqn. (11)

$$\begin{aligned}
\{\phi(t+h;\, s)\}^i &= \sum_{j=0}^{\infty} s^j\, P_{ij}(t+h) \\
&= \sum_{j=0}^{\infty} s^j \sum_{k=0}^{\infty} P_{ik}(t)\, P_{kj}(h) \\
&= \sum_{k=0}^{\infty} P_{ik}(t) \sum_{j=0}^{\infty} s^j\, P_{kj}(h) \\
&= \sum_{k=0}^{\infty} P_{ik}(t)\, \{\phi(h;\, s)\}^k \\
&= \{\phi(t;\, \phi(h;\, s))\}^i\,,
\end{aligned} \tag{12}$$

and with $i = 1$,

$$\phi(t+h;\, s) = \phi(t;\, \phi(h;\, s))\,, \tag{13}$$

a result equivalent to Eqn. (4) in Unit 4 on the discrete case.

Let us now specialize these results in the context of our present model as specified by Eqn. (7). Let us denote it by $u(s) = \sum_{j=0}^{\infty} \alpha_j\, s^j$. Then

$$\begin{aligned}
\phi(h;\, s) = \sum_{j=0}^{\infty} s^j\, P_{1j}(h) &= \sum_{j=0}^{\infty} s^j\, (\delta_{1j} + \alpha_j\, h + 0(h)) \\
&= s + h\, u(s) + 0(h)\,,
\end{aligned} \tag{14}$$

so that from Eqn. (13),

$$\phi(t+h;\, s) = \phi(t;\, s + h\, u(s) + 0(h))\,.$$

Now, we shall use the Taylor's expansion of the bivariate function ϕ as below

$$\begin{aligned}
\phi(t+h;\, s) &= \phi(t;\, s + h\, u(s) + 0(h)) \\
&= \phi(t;\, s) + h\, u(s)\, \{d\phi(t;\, s)/ds\} + 0(h),
\end{aligned}$$

and, thus,

$$\{\phi(t+h;\, s) - \phi(t;\, s)\}/h = \{d\phi(t;\, s)/ds\; u(s) + 0(h)/h\,.$$

Taking the limit of both sides as $\lambda \to 0$, we have,

$$\partial\phi(t;\, s) = \{\partial\phi(t;\, s)\}\ u(s)\,. \qquad (15)$$

This partial differential equation can be solved for $\phi(t;\, s)$ subject to the initial condition $\phi(0;\, s) = s$.

Again, from Eqn. (13) and Eqn. (14), we get

$$\phi(t+h;\, s) = \phi(t;\, \phi(h;\, s))$$
$$= \phi(h;\, s) + t\ u(\phi(h;\, s)) + 0(t)$$

Now interchanging t and λ, the same equation can be written as

$$\phi(t+h;\, s) = \phi(t;\, s) + h\ u(\phi(t;\, s)) + 0(h)\,.$$

Hence,

$$\{\phi(t+h;\, s) - \phi(t;\, s)\}/h = u(\phi(t;\, s)) + 0(h)/h$$

so that by taking the limit as t tends to zero, we have another equation, this time an ordinary differential equation

$$\frac{\partial}{\partial t}\phi(t;\, s) = u(\phi(t;\, s)) \qquad (16)$$

which again can be solved under the initial condition $\phi(0,\, s) = s$. The Eqn. (15) and Eqn. (16) may be referred to as forward and backward equations, respectively.

We wish to evaluate the mean function, $m(t) = EX(t)$, under the above set up. As we already know, a convenient way to compute this would be to differentiate the probability generating function $\phi(t;\, s)$ with respect to s, and to evaluate the same at $s = 0$. In this context, we first differentiate both sides of Eqn. (15) with respect to s, and interchange the order of differentiation, to get

$$\frac{\partial}{\partial s}\left\{\left(\frac{\partial}{\partial t}\right)\phi(t;s)\right\} = \frac{\partial}{\partial s}\left\{u(s)\left(\frac{\partial}{\partial s}\right)\phi(t;s)\right\}$$

i.e. $$\frac{\partial}{\partial t}\left\{\left(\frac{\partial}{\partial s}\right)\phi(t;s)\right\} = u(s)\frac{\partial^2}{\partial s^2}\{\phi(t;s)\} + \left(\frac{\partial}{\partial s}\right)\{\phi(t;s)\}\, u'(s).$$

For $s = 1, u(1) = 0$, we have

$$\frac{\partial}{\partial t}\ m(t) = u'(1)\ m(t)\,.$$

Under the initial condition $X(0) = 1$, this can be easily solved for m(t) to yield

$$m(t) = e^{u'(1)\, t}\,. \qquad (17)$$

Now, we shall discuss the probability of extinction of this process.

The process cannot be extinct unless $\alpha_0 > 0$. And, as such it will make sense to consider only the case $\alpha_0 > 0$. Also, we take $X(0) = 1$ as this has been argued earlier in Eqn. (11)

$$\sum_{j=0}^{\infty} s^j\, P_{ij}(t) = \{\phi(t;\, s)\}^i = \left\{\sum_{j=0}^{\infty} s^j\, P_{1j}(t)\right\}^i$$

so that, $P_{i0}(t) = \{P_{10}(t)\}^i$. Here $P_{i0}(t)$ is the probability of a population of size i dying out by time t. Notice that Eqn. (13) and the fact that $\phi(t;s)$ is non-decreasing for fixed t show that

$$P_{i0}(t+h) = \{P_{10}(t+h)\}^i = \{\phi(t+h;\, 0)\}^i = \{\phi(t;\, \phi(h;\, 0))\}^i \geq \{\phi(t;\, 0)\}^i = P_{i0}(t)\,,$$

so that $P_{i0}(t)$ is non-decreasing in t, which is intuitively clear as well. Given this background, the "extinction probability", the probability that a family caused by an individual will eventually die out, can be defined as $\pi_0 = \lim_{t\to\infty} P_{10}(t)$.

We shall close this section after quoting the following theorem without proof which is a bit involved, but not hard:

Theorem 1: Suppose $\alpha_0 > 0$. Then, π_0 the smallest non-negative root of the equation $u(s) = \sum_{j=0}^{\infty} \alpha_j s^j = 0$. Further $\pi_0 = 1$ if and only if $u'(1) \le 0$.

Now, let us apply this theorem in the following example:

Example 1: Consider a continuous time Markov branching process with $\alpha_0 = a, \alpha_1 = -(a+b), \alpha_2 = b$, and $\alpha_j = 0$ for $j \ge 3$.

Then

$$\phi(s) = \sum_{j=0}^{\infty} \alpha_j s^j$$

$$\text{or } \phi(s) = bs^2 - (a+b)s + a$$

$$= b(s - a/b)(s-1)$$

Here, it is clear that $\pi_0 = 1$ iij $b \le a$

or $\pi_0 = 1$ iij $\phi'(1) \le 0$.

5.4 PURE BIRTH PROCESSES

We will now consider a variation of the Markov branching process discussed in the previous section. In that process, we postulate as follows:

$$P\{X(t+h) = k+1 \mid X(t) = k\} = k\alpha h + g_1(h)$$

$$P\{X(t+h) = k \mid X(t) = k\} = 1 - k\alpha h + g_2(h)$$

and for $j \ne 0, 1$

$$P\{X(t+h) = n+j \mid X(t) = n\} = g_j(h)$$

for some $\alpha > 0$, where $g_j(h)$ are $o(h)$ as $h \to 0$, $j = -1, 0, 1, 2, \ldots$. Note the characteristics of this process. Here, it is clear that the chances of a 'birth' in the time interval $[t, t+h]$ depends on the size of the population as well as on the length of the time interval and further, those of more than one 'birth' in the time interval $[t, t+h]$ is negligible. No death is allowed.

To generalize further, suppose $\{X(t) : t \ge 0\}$ is a Markov process, and for series of positive numbers λ_k, the following postulates are satisfied as $h \to 0$.

1. $P_{kk+1}(h) = P\{X(t+h) = k+1 \mid X(t) = k\} = \lambda_k h + g_1(h)$

2. $P_{kk}(h) = P\{X(t+h) = k \mid X(t) = k\} = 1 - \lambda_k h + g_2(h)$

3. $P\{X(t+h) = k+j \mid X(t) = k\} = g_j(h)$, for all $j = -1, 2, 3, \ldots$ where $g_j(h) = 0(h)$ as $h \to 0$.

   In addition, sometimes, to be able to handle the consequent algebra and also to be in tune with the reality of the process being modeled, when $\lambda_0 > 0$ we may postulate the following:

4. $X(0) = 0$, for convenience.

This is required to be used as the initial condition at times and also postulate 4 may be changed if felt appropriate in a given situation. Notice that in view of the last postulate, $X(t)$ can be interpreted as the number of births in $[0, t]$ and not the size of the population at time t. Also, postulates 1 and 2 together imply that the chance of more than one birth over a period of duration h is of lower order h. Such processes can be referred to as pure birth processes. Let $P_n(t) = P(X(t) = n)$. Then, for $n \ge 1$,

$$P_n(t+h) = P(X(t+h)=n) = \sum_{j=0}^{\infty} P(X(t+h)=n,\ X(t)=j)$$

$$= P(X(t+h)=n,\ X(t)=n-1) + P(X(t+h)=n,\ X(t)=n) + 0(h)$$

$$= P(X(t+h)=n \mid X(t)=n-1)\,P(X(t)=n-1)$$

$$+ P(X(t=h)=n \mid X(t)=n)P(X(t)=n + 0(h)$$

$$= \{\lambda_{n-1}h + g_1(h)\}\,P_{n-1}(t) + \{1-\lambda_n h + g_2(h)\}\,P_n(t) + 0(h)$$

so that

$$\{P_n(t+h) - P_n(t)\}/h = \lambda_{n-1}P_{n-1}(t) - \lambda_n P_n(t) + \frac{0(h)}{\lambda}.$$

Taking the limit as $h \to 0$, we get

$$P_n'(t) = \lambda_{n-1}\,P_{n-1}(t) - \lambda_n\,P_n(t) \tag{18}$$

On the other hand,

$$P_0(t+h) = P(X(t+h)=0) = P(X(t+h)=0,\ X(t)=0)$$

$$= P(X(t+h)=0 \mid X(t)=0)\ P(X(t)=0)$$

$$= \{1-\lambda_0 h + g_2(h)\}\ P_0(t)$$

so that, as before, we now get

$$P_0'(t) = -\lambda_0\,P_0(t) \tag{19}$$

Solving Eqn. (19) with the initial condition $X(0)=0$, i.e., $P_0(0)=1$,

$$P_0(t) = \exp(-\lambda_0 t) \tag{20}$$

Notice that since $X(0)=0$ as per our postulate 4, the event $\{X(t)=0\} = \{T_0 > t\}$, where T_0 is the random variable denoting the time to the first birth; according to Eqn. (20) then, $P_0(t) = P(X(t)=0) = P\{T_0 > t\}$, so that the time to the first event is exponentially distributed with the mean $(1/\lambda_0)$.

Now, from Eqn. (17), solving successively, for $n = 1, 2, \ldots$

$$P_n(t) = \lambda_{n-1}\exp(-\lambda_n t)\int_0^t \exp(\lambda_n u)\ P_{n-1}(u)\,du\,. \tag{21}$$

Example 2 (Yule Process): Consider a pure birth process that has exactly one element originally (originator). It and its successive offspring each has a probability $\alpha h + 0(h)$ of reproducing a similar element over a period of infinitesimal length $h(\alpha > 0)$; as usual, we assume that the elements reproduce independently of each other. Remember that we are not bringing in the concept of "death" and as such the value of $X(t)$ can only go up through births at random points of time. Also, note that given that the probability of birth over a duration of length h is $\alpha h + 0(h)$, the probability of no birth is $(1-\alpha h + 0(h))$ as more than one birth is not permitted. Suppose that the population has achieved size k at time t, i.e., $X(t)=k$, and after h units of time its size has to become, say, $k+1$, there being no 'deaths', the only way it can happen would be when exactly one out of k elements present must have reproduced an offspring (the probability of which for each of them is $\alpha h + 0(h)$) while the others did not (the probability of which for each of them is $1-\alpha h + 0(h)$). Thus, because of independence of the behaviour of the k elements, we are in a binomial situation, and obviously,

$$P(X(t+h)=k+1 \mid X(t)=k) = ({}^kC_1)\{\alpha h + 0(h)\}^1\{1-\alpha h + 0(h)\}^{k-1}$$

$$= k\lambda h + 0(h).$$

Arguing in a similar fashion, for $j > 1$,

$$P(X(t+h)=k+j \mid X(t)=k) = ({}^kC_j)\ \{\alpha h + 0(h)\}^j (1-\alpha h + 0(h))^{k-j}$$
$$= 0(h).$$

Thus, comparing this with the model discussed just before Example 2, we find that $\lambda_k = k\lambda$. The birth process with parameters $\lambda_k = k\lambda$ is called a **Yule Process**. The solutions of Eqns. (19) and (20) through routine computations with the initial conditions, $X(0)=1$ (i.e. $P(X(0)=j)=\delta_{1j}$, $j=0, 1, 2, \ldots$) yields

$$P_k(t) = e^{-\alpha t}(1-e^{-\alpha t})^{k-1},\ k=1, 2, \ldots \tag{21}$$

We note that for each $t > 0$, $X(t)$ has a geometric distribution with success probability $e^{-\alpha t}$. Therefore, $EX(t) = (1/e^{-\alpha t}) = e^{\alpha t}$, i.e. on the average, the process grows exponentially.

Example 3 (Poisson Process): The postulates of a pure birth process may be recounted as follows:

In the pure birth process we allowed the probability of a birth over a small interval of time to depend on the number of births k. But, suppose that, in a special situation, the λ_k's are identical and equal to a constant λ. For instance, $X(t)$ can be the number of occurrences of an event of a certain kind taking place in the time interval $[0, t]$; e.g. $X(t)$ can be the number of accidents taking place in a city over time, or number of telephone calls passing through a PABX board, or number of times a particular component of a machine is replaced following failures over time, or even the number of customers arriving at a store over a time, the number of cars that pass a particular bus-stop by time t, etc. In such cases, obviously, no event lies in the root of the occurrence of a previous one (i.e. unlike fathering the "next generation") so that the number of events that have taken place until the recent time will not have any apparent influence whatsoever on the occurrence of a subsequent one; at the same time it is logical to argue that the chance of occurrence of event in a time interval of small length is proportional to its length. We will later approach the Poisson process from this aspect as well.

We define a Poisson process as a birth process satisfying the following postulates:

1. $P_{k\,k+1}(h) = P\{X(t+h)=k+1 \mid X(t)=k\} = \lambda h + g_1(h)$

2. $P_{kk}(h) = P\{x(t+h)=k \mid X(t)=k\} = 1-\lambda h + g_2(h)$

3. $P\{X(t+h)=k+j \mid X(t)=k\} = g_j(\lambda)$, for all $j \neq 0, 1$

4. $X(0)=0$

where $g_i(h) = 0(h)$ as $h \to 0$.

Then, from Eqn. (17) and Eqn. (18), we have

$$P_0'(t) = -\lambda P_0(t)$$
$$P_n'(t) = \lambda P_{n-1}(t) - \lambda P_n(t),$$

the initial conditions being, because of postulate 4 given above, $P_0(0)=1, P_n(0)=0$ for $n \geq 1$. Then from Eqn. (19), we will have

$$P_0(t) = \exp(-\lambda t) \tag{22}$$

and for $n \geq 1$

$$P_n(t) = \lambda \exp(-\lambda t) \int_0^t \exp(\lambda u)\ P_{n-1}(u)\ du \tag{23}$$

which can be recursively solved to yield

$$P_n(t) = P(X(t) = n) = \frac{(\lambda t)^n}{n!}\exp(-\lambda t),$$

So that for $t > 0$, $X(t)$, the number of event occurrences of events that take place up to time t, have a Poisson distribution with mean λt. Is there an alternate set of postulates to identify a Poisson process? In fact, we shall now show that a counting process, $\{X(t): t \geq 0\}$ ($X(t)$, that often denotes the number of births in the time interval $[0, t]$) is a Poisson process with rate $\lambda > 0$, if

1. $X(0) = 0$
2. $\{X(t); t > 0\}$ has independent increments
3. for any $t > 0$, $X(t+h) - X(t)$, which is the number of events (births) in the interval $(t, t+h]$ of length h follows a Poisson distribution with mean λh.

Notice that because of 2 and 3,

$$\begin{aligned} P_{kk+1}(h) = P(X(t+h) = k+1 \mid X(t) = k\} &= P\{X(t+h) - X(t) = 1 \mid X(t) = k\} \\ &= P\{X(t+h) - X(t) = 1\} \\ &= \lambda h e^{-\lambda h} \\ &= \lambda h\{1 - \lambda h + (\lambda h)^2/2 - \ldots\} \\ &= \lambda h + g_1(h) \\ &= \lambda h + 0(h) \end{aligned}$$

since

$|g_1(h)| \leq (\lambda h)^2\{1 + (\lambda h)/2 + (\lambda h)^2/3! + (\lambda h)^3/4! + \cdots\} \leq (\lambda h)^2\{1 + (\lambda h) + (\lambda h)^2/2! + (\lambda h)^3/3! + \cdots\} = (\lambda h)^2 e^{\lambda h}$ so that $|g_1(h)/h| \leq (\lambda^2 h)e^{\lambda h}$; thus $g_1(h) = 0(h)$ as $h \to 0$. This shows that the postulate 1 of the Poisson process is satisfied, i.e., $P_{kk+1}(h) = P\{X(t = h) = k+1 \mid X(t) = k\} = \lambda h + 0(h)$.

Next, let us note that

$$P\{X(t+h) - X(t) \geq 2\} = \sum_{n=2}^{\infty} e^{-\lambda h}(\lambda h)^n / n! \leq e^{-\lambda h}(\lambda h)^2 \sum_{m=0}^{\infty} (\lambda h)^m / m! \leq (\lambda h)^2$$

which is $0(h)$ as $h \to 0$ justifying postulate 3. Finally

$$\begin{aligned} P_{kk}(h) &= P\{X(t+h) - X(t) = 0\} = P(X(t+h) - X(t) = 0\} \\ &= 1 - P\{X(t+h) - X(t) = 1\} - P(X(t+h) - X(t) \geq 2\} = 1 \\ &\quad -\{(\lambda h) + 0(h)\} - 0(h) = 1 - (\lambda h) + 0(h), \end{aligned}$$

so that postulate 2 is also satisfied. Infact, note that postulate 3 is automatically satisfied since we are dealing with a counting process. And, postulate 4 is there in the form of condition 1. Thus, a counting process satisfying conditions 1-3 is a Poisson process.

From now onwards, therefore, we shall take conditions 1-3 to define a Poisson process.

Example 4: Let us, again, consider a Poisson process $\{X(t), t \geq, 0\}$ of parameter λ.

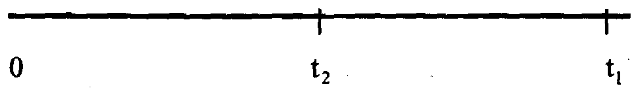
Solution: For $k = 1, 2, \ldots, h$ let T_k be the time between the $(k-1)$ th and k-th event, then $\{T_k : k \geq 1\}$ is the sequence of "inter-arrival times" in the Poisson process. Note that for $n = 1, 2, \ldots$, S_n time $T_1 + T_2 + \cdots + T_n$ is the time instant of occurrence of the nth event. Note that by Eqn. (22), the probability that the waiting time T_1 for the first occurrence of the of event will exceed t turns out to be

$$P(T_1 \geq t) = P(X(t) = 0) = P_0(t) = e^{-\lambda t}$$

which means that T_1 is distributed exponentially with mean $\frac{1}{\lambda}$. Notice, then, that roughly speaking, λ is the rate of occurrence of the event. Again, by independence of occurrence of events over disjoint intervals, by postulate 2, for $0 \leq t_1 \leq t_2 < \infty$ we have

$$\begin{aligned} P(T_1 \leq t_1, S_2 > t_2) &= P(X(t_1) \geq 1, \text{ second occurrence of event after } t_2) \\ &= P(X(t_1) = 1, X(t_2) - X(t_1) = 0) \\ &= P(X(t_1) = 1)P(X(t_2 - t_1) = 0) \\ &= (\lambda t_1)e^{-\lambda t_1}e^{-\lambda(t_2 - t_1)} = (\lambda t_1)e^{-\lambda t_2}. \end{aligned}$$

Consider the case, $0 \leq t_2 \leq t_1 < \infty$, i.e.,



$$\begin{aligned} P(T_1 < t_1, S_2 > t_2) &= P(X(t_1) \geq 1, \text{ second event after } t_2) \\ &= P(\text{ both first and second occurrence of event after } t_2\text{, but the first event before } t_1) \\ &\quad + P(\text{first occurrence of event before } t_2\text{, and the second after } t_2) \\ &= P(X(t_2) = 0, X(t_1) - X(t_2) \geq 1) + P(X(t_2) = 1) \\ &= e^{-\lambda t_2}(1 - e^{-\lambda(t_1 - t_2)}) + (\lambda t_2)e^{-\lambda t_2}. \end{aligned}$$

Thus, the joint probability density function of T_1 and S_2 is given by

$$-\frac{\delta}{\delta t_1 \, \delta t_2} P(T_1 < t_1, S_2 > t_2) = \begin{cases} \lambda^2 e^{-\lambda t_2} & \text{for } 0 \leq t_1 \leq t_2 < \infty \\ 0 \text{ for} & 0 \leq t_2 \leq t_1 < \infty \end{cases}$$

Now, by transforming as $X = T_1$ and $Y = S_2 - T_1$, we obtain the joint probability density of T_1 and T_2, which is given by

$$\lambda^2 e^{-\lambda(x+y)}, \; x \geq 0 \text{ and } y \geq 0.$$

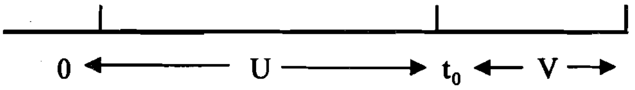
It is, thus, seen that the inter-arrival waiting times T_1 and T_2 are i.i.d with each exponentially distributed mean λ^{-1}.

In Example 3 we showed that the first two inter-arrival times are i.i.d., with each having exponential distribution. Proceeding in the same manner, one can prove a more general property of Poisson process as stated in the theorem given below.

Theorem 2: If $\{X(t) : t \geq 0\}$ is a Poisson process with rate λ, then the inter-arrival times $T_1, T_2, \ldots$ are independently distributed each having exponential distribution with mean λ^{-1}. Conversely, suppose $\{T_n : n \geq 1\}$ is a sequence of independently and identically distributed random variables each having the exponential distribution with mean λ^{-1}; define $S_0 = 0$ and for $n = 1, 2, \ldots, S_n = T_1 + T_2 + \cdots + T_n$. For $t \geq 0$, let $X(t) := \max\{n \geq 0 : S_n \leq t\}$. Then $\{X(t) : t \geq 0\}$ is a Poisson process.

The above theorem, which we will not prove here, gives a simpler insight into the structure and profile of a Poisson process.

We will now discuss one more important property of the Poisson process. Towards that, visualize a situation where events are occurring in the Poisson fashion over a time scale. For instance, the event can be one of arrival of a public bus at a bus stop. Suppose, you are waiting for a bus at a bus stop and you are interested in the time you may have to wait now, or the time when the last bus went past the bus stop before your arrival. Let t_0 is the time when you arrived at the bus stop. The last bus went past U units of time ago and V is the time that you have to wait to get the next bus. Pictorially,



Thus, in respect of the time point t_0, U is the time since the last occurrence of the event, and V is the waiting time for the next occurrence of the event since t_0. In the context of Reliability Theory which deals with event-like failure of systems, the random variable of interest is the life (i.e., the time to failure) of a system, then U is often referred to as the current life (at the present time t_0) and V is called the excess (to t_0) life. For $0 < u < t_0$ and $v > 0$, the joint distribution function of the random variables U and V can be computed as

$$\begin{aligned} &P(U \geq u, V \geq v) \\ &= P(X(t_0) - X(t_0 - u) = 0 = X(t_0 + v) - X(t_0)\} \\ &= P\{X(t_0) - X(t_0 - u) = 0\}\, P\{X(t_0 + v) - X(t_0) = 0\} \\ &= e^{-\lambda u} e^{-\lambda v} = e^{-\lambda(u+v)},\ 0 < u < t_0,\ v > 0 \end{aligned}$$

Also, $P(U \geq u) = P(U \geq u, V \geq 0) = e^{-\lambda u}$ for $0 < u < t_0$,

$\qquad\qquad = 0$ for $u \geq t_0$

and, $P(V \geq v) = e^{-\lambda v},\ v \geq 0$.

Thus, U and V are independently distributed.

We will see a few other interesting results through the exercises given below.

E1) For a Poisson process $\{X(t): t \geq 0\}$, find the conditional probability that there are k events in t units of time, given that there are $k+n$ events in $t+s$ units of time.

E2) For a Poisson process $\{X(t): t \geq 0\}$, let T_m be the random time to the m-th event. Find its distribution. Also, for $n \geq 1$, obtain $P\{T_1 \leq t \mid X(s+t) = n\}$.

E3) Let the Poisson process $\{X(t): T \geq 0\}$ with intensity α be independent of another such process $\{Y(t): t \geq 0\}$ with intensity β. Then $\{Z(t) = X(t) + Y(t): t \geq 0\}$ is also a Poisson process with intensity $\alpha + \beta$.

E4) Suppose that from a bus stand, one person has the option of traveling to a given destination either by Bus No. 2 or Bus No. 9. Bus No. 2 arrives at the stand following a Poisson process with intensity 6 per hour and Bus No. 9 arrives also following a Poisson process with intensity 4 per hour.

a) Let X denotes the extent of time in hours a person has been late in arriving at the bus stop since the last bus went by. What is the distribution of X?

b) Let W denotes the extent of time in hours the person has to wait for a bus. What is the probability that this person has to wait for more than 10 minutes?

E5) For a Poisson process $\{X(t): t \geq 0\}$, define another process $N(t) = X(t+s_0) - X(t)$, where $s_0 > 0$ is a fixed constant. Is $N(t)$ also a Poisson process?

E6) Suppose a system breaks down from time to time because of the failure of one of two components – component A and component B. The number of system breakdowns follow the Poisson process with parameter λ; also, the probability that a breakdown is caused by the failure of component A is π, so that the probability of a breakdown being caused by the failure of component B is $(1-\pi)$. Show that the number of system breakdowns owing to the failure of component A over time is a Poisson process with parameter $\lambda\pi$.

E7) Suppose in a given society, there are N individuals; moreover the population produces no new individuals, i.e., has birth rate zero, but adds to the population through immigration which takes place as a Poisson process with parameter λ. Let $X(t)$ denotes the size of the population at time t and denote, as usual, by $p_k(t) = P(X(t) = k)$. Develop a suitable differential equation to determine the probability generating function of $X(t)$, and discuss what happens as $t \to \infty$. Also, calculate $EX(t)$.

E8) The time to failure of a certain variety of machine is distributed exponentially with mean $\frac{1}{\alpha}$; the time a mechanic would require, then, to restart the machine is also exponentially distributed with mean $\frac{1}{\beta}$. The failure and repair times are independent of each other. Consider two such machines and two mechanics so that each of the machines can get the immediate attention of a mechanic whenever required. The two machines are put into operation at the same time and let $X(t)$ denotes the number of machines out of action at time t. For $k = 0, 1$, let $p_k(t) = P(x(t) = k)$; show that $p_0(t) = \frac{\beta}{\alpha+\beta}(1 - e^{-(\alpha+\beta)})$. Obtain the limiting distribution of $X(t)$ as $t \to \infty$ and show that it has a mean $\frac{2\pi}{(1+\pi)}$, where $\pi = \frac{\alpha}{\beta}$.

Now, we summarise what we have learnt in this unit.

5.5 SUMMARY

In this unit, we covered the following:

1) We learnt a number of concepts that are, in some sense realistic and, hence, are targeted towards modeling processes that we come across in our daily dealings with life.

2) In the previous unit, we dealt with the number of additions to the population through births over generations. Thus, time (generations) was counted on a discrete scale; thus, the first generalization over that concept that one thinks of is to allow time to vary over a continuous scale. Now, more in tune with reality the population is viewed to be growing, or rather changing, over a continuous time scale. We, however, desist at this stage, from even conceiving the process $X(t)$ to be changing continuously itself.

3) We have introduced the Markov property of processes, which, in plain words, is that the future development of a process would depend on the recent-most past.

4) The definition and interesting properties of a Poisson process are discussed, which essentially models occurrences of events that are kind of 'rare' (the probability of occurrence of an event in a time interval is proportional to the length of the said interval, irrespective of its location on the time scale). Obviously, the next behaviour of interest concerning a process has been to study the growth, or otherwise, of a process that is typified by the occurrences of two kinds of events counteracting the effect of each other on the overall change, like births and deaths. Such a process with death rate zero leads to a birth process which is, in another sense, a generalization of the Poisson process.

5.6 SOLUTIONS/ANSWERS

E1) We need to compute $P\{X(t)=k \mid X(t+s)=k+n\}$. Notice that by definition of conditional probability,

$$\begin{aligned} &P\{X(t)=k \mid X(t+s)=k+n\} \\ &= P\{X(t)=k,\ X(t+s)=k+n\} / P\{X(t+s)=k+n\} \\ &= P(X(t)=k,\ X(t+s)-X(t)=n\} / P(X(t+s)=k+n\} \\ &= P\{X(t)=k\}\ P\{X(t+s)-X(t)=n\} / P(X(t+s)=k+n), \end{aligned}$$

since in a Poisson process, the number of events in disjointed intervals are independent.

$$\begin{aligned} &= \{e^{-\lambda t}(\lambda t)^k / k!\}\ \{e^{-\lambda s}(\lambda s)^n\} / \{e^{-\lambda(t+s)}[\lambda(t+s)]^{k+n} / (k+n)!\} \\ &= \{(k+n)!/k!\,n!\}\ \{t/(t+s)\}^k\ \{s/(t+s)\}^n . \end{aligned}$$

E2) We may represent S_m as $S_m = T_1 + T_2 + \cdots + T_m$, where T_k is the time between the $(k-1)$-th and k-th event, $1 \le k \le m$. From Theorem 2, we know that the inter-arrival times $T_1, T_2, \ldots$ are independently distributed each having an exponential distribution with mean λ^{-1}.

Clearly,

$$\begin{aligned} &P(S_m \le t) \\ &= P\{X(t) \ge m\} \\ &= \sum_{k=m}^{\infty} e^{-\lambda t}(\lambda t)^k / k!, \end{aligned}$$

so that the pdf of S_m, which is the time to the m-th event is

$$\begin{aligned} &\{d/dt\}\ P(S_m \le t) \\ &= -\lambda e^{-\lambda t}\sum_{k=m}^{\infty}(\lambda t)^k / k! + e^{-\lambda t}\lambda\sum_{k=m}^{\infty} k(\lambda t)^{k-1} / k! \\ &= -\lambda e^{-\lambda t}\sum_{k=m}^{\infty}(\lambda t)^k / k! + \lambda e^{-\lambda t}\sum_{k=m-1}^{\infty}(\lambda t)^k / k! \\ &= e^{-\lambda t}\,\lambda^m\, t^{m-1} / (m-1)!,\ t \ge 0 . \end{aligned}$$

Now,

$$\begin{aligned} &P\{T_1 \le t \mid X(s+t)=n\} \\ &= P\{X(t) \ge 1 \mid X(t+s)=n\} \\ &= P\{X(t) \ge 1,\ X(t+s)=n\} / P\{X(t+s)=n\} \\ &= \sum_{k=1}^{\infty} P\{X(t)=k,\ X(t+S)=n\} / P\{X(t+s)=n\} \end{aligned}$$

$$=\sum_{k=1}^{\infty} P\{X(t)=k,\ X(t+S)-X(t)=n-k\}/P\{X(t+s)=n\}$$

$$=\sum_{k=1}^{\infty} P\{X(t)=k\}\ P\{X(t+S)-X(t)=n-k\}/P\{X(t+s)=n\}$$

Since, in a Poisson process, the number of events in disjoint intervals are independent.

$$=\sum_{k=1}^{\infty} \{e^{-\lambda t}(\lambda t)^k/k!\}\ \{e^{-\lambda s}(\lambda s)^{n-k}/(n-k)!\}$$

$$x\{n!/\{e^{-\lambda(t+s)}[\lambda(t+s)]^n\}$$

$$=\sum_{k=1}^{\infty} \frac{n!}{k!(n-k)!}\left(\frac{t}{t+s}\right)^k\left(\frac{s}{t+s}\right)^{(n-k)}$$

$$=\left\{\left(\frac{t}{t+s}\right)+\left(\frac{s}{t+s}\right)\right\}^n-\left(\frac{s}{t+s}\right)^n$$

$$=1-\left(\frac{s}{t+s}\right)^n.$$

E3) We need to argue that

(i) $Z(0)=0$

(ii) $\{Z(t):t>\}$ has independent increments

(iii) for any $h>0$, $Z(t+h)-Z(t)$, which is the number of events (birth) in the interval $(t,\ t+h]$ of length h, follows the Poisson distribution with mean $(\alpha+\beta)h$.

Since $Z(t)=0 \equiv X(0)=0,\ Y(0)=0$, (i) is obvious since $X(t)$ and $Y(t)$ themselves are Poisson processes. Again, because of the same reason, (ii) easily follows. Now, $Z(t+h)-Z(t)=\{X(t+h)-X(t)\}+\{Y(t+h)\}$ and each of the components on the right hand side is a Poisson random variable and they are independent as $X(t)$ and $Y(t)$ are also independent. Hence, their sum, i.e. $Z(t+h)-z(t)$ is a Poisson random variable for all $t\geq 0,\ h\geq 0$. This establishes that $\{Z(t):t\geq 0\}$ is a Poisson process.

E4)* Let $X(t)$ denotes the number of Bus No.2 that pass through the bus stand by time t and similarly let, $Y(t)$ the number of Bus No.2 that pass through the same bus stand; thus, since either of the two kinds of buses are fine for the person, the process of interest for the person would be $Z(t)=X(t)+Y(t)$. Again since both $X(t)$ and $Y(t)$ are Poisson processes, so is $Z(t)$ with intensity 10, assuming $X(t)$ and $Y(t)$ to be independent. As such, from theory,

$$P(X\geq u)=e^{-\lambda u} \text{ for } 0<u<t_0,$$
$$=0 \qquad \text{for } u\geq t_0$$

where t_0 is the time that the person arrived at the bus stand. Also, from theory, W has an exponential distribution with intensity 10. Accordingly, $P(W\geq 10)=e^{-10/10}=e^{-1}$.

E5) Let us try $P\{N(t)=k\}=P\{X(t+s_0)-X(t)=k\}=e^{-\lambda s_0}(\lambda s_0)^k/k!\}$ for all t. The right hand side is free of t!. Thus, $N(t)$ cannot be construed as a Poisson process.
Let us now look at $U(t).P\{U(t)=k\}=P\{X(t+s_0)-X(s_0)=k\}$

$$=e^{-\lambda t}(\lambda t)^k/k!.$$

Also, by definition $U(0)=0$. Finally, since $X(t)$ has independent increments, so does $U(t)$ as well. Hence, $U(t)$ is a Poisson process.

E7) Let $Y(t)$ denotes the number of persons added through immigration to the population so that $X(t)=Y(t)+N$. Now, for $k\geq N$, since $Y(t)$ is a Poisson process with parameter λ,

$$\begin{aligned} p_k(t+h)=P(X(t+h)=k &= \sum_{j=0}^{k} P\{X(t+h)-X(t)=k-j,\ X(t)=j\} \\ &= P\{Y(t+h)-Y(t)=0,\ k-N\}+ \\ &\quad +P\{Y(t+h)-Y(t)=1,\ Y(t)=k-N-1\}+0(h) \\ &= P\{Y(t+h)-Y(t)=0\}\ P\{Y(t)=k-N\}+ \\ &\quad +P\{Y(t+h)-Y(t)=1\}\ P\{Y(t)=k-N-1\}+0(h) \\ &= P\{Y(t+h)-Y(t)=0\}\ P\{X(t)=k\}+ \\ &\quad +P\{Y(t+h)-Y(t)=1\}\ P\{X(t)=k-1\}+0(h) \\ &= e^{-\lambda h}\,p_k+\lambda h\,e^{-\lambda h}\,p_{k-1}(t)+0(h) \\ &= \{1-\lambda h+0(h)\}\ p_k(t) \\ &\quad +\lambda h\{1-\lambda h+0(h)\}\,p_{k-1}(t)+0(h), \text{ so that} \end{aligned}$$

$\{p_k(t+h)-p_k(t)\}/h=-\lambda p_k(t)+\lambda p_{k-1}(t)+0(1)$, and hence, as $h\to\infty$,

$$p'_k(t)=-\lambda p_k(t)+\lambda p_{k-1}(t),\ k\geq N. \qquad (5.7.1)$$

For $k=N$, $p'_N(t)=-\lambda p_N(t)+\lambda p_{N-1}(t)=-\lambda p_N(t)$, since $p_{N-1}(t)=0$, as the population can never be smaller than the number of original individuals. Solving the differential equation, $p'_N(t)=-\lambda p_N(t)$, we find $e^{\lambda t}p_N(t)=C$, a constant. But, $p_N(0)=1$, since, again the original population size is N, so that $C=1$ and as such $p_N(t)=e^{-\lambda t}$. Now from Eqn. (5.7.1),

$p'_{N+1}(t)+\lambda p_{N+1}(t)=\lambda p_N(t)=\lambda e^{-\lambda t}$, i.e. $\frac{\partial}{\partial t}\{e^{\lambda t}p_{N+1}(t)\}=\lambda$, i.e.

$e^{\lambda t}p_{N+1}(t)=\lambda t+\text{constant}.$

Again, $p_{N+1}(0)=0$, so that the Constant $=0$ and as such,

$$p_{N+1}(t)=\lambda t\,e^{-\lambda t};$$

solving successively, $p_{N+k}(t)=(\lambda t)^k e^{-\lambda t}/k!,\ k=0,\ 1,\ 2,\ \ldots.$

---x---