UNIT 5 GREEN'S FUNCTION METHOD

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5.1 INTRODUCTION

In the theory of differential equations an important role is played by certain mathematical objects known as Green's functions. These functions, besides being of fundamental importance in many applications, serve as mathematical characterization of important physical concepts. Consequently, for introducing Green's functions we have considered, in this unit two approaches: (i) physical and (ii) mathematical.

Since the motivation for their study can be brought to sharper focus through the first approach, therefore in Sec.5.2, we shall start with an example from mechanics, whose solution turns out to be an example of Green's function. In Sec. 5.3 we shall define Green's function for a boundary value problem (bvp) and show how it can be constructed for a given bvp. We shall also give here a method of solving a bvp using its Green's function. Since the construction of Green's function for a bvp is quite an involved process we have given in Sec.5.4 an alternative procedure for the construction of Green's functions for linear, non-homogeneous, second order differential equations by establishing a relationship between the method of variation of parameters and Green's functions.

Objectives

After studying this unit you should be able to

- define and construct Green's functions for a boundary value problem;
- obtain solution of a given boundary value problem using its Green's function.

5.2 DEFLECTION OF A PERFECTLY FLEXIBLE ELASTIC STRING UNDER THE ACTION OF A CONCENTRATED LOAD

We shall now determine the deflection of a perfectly flexible elastic string under the action of a concentrated load. Before we embark upon the task of obtaining the solution to the given problem we may mention that the problem to be dealt here is a mathematical idealisation of the corresponding problem for the determination of deflection under the action of distributed load. Just as the introduction of point charges, point masses, point heat source facilitate the solution of the problems in their respective fields, in the same way, the introduction of point force/concentrated load for solving physical problems related to beams and strings has turned out to be quite fruitful. It is a mathematical fiction, which cannot be realised physically as any non-zero load concentrated at a single point implies an infinite pressure, which would immediately cut through the string.

For the problem under consideration we have a perfectly flexible elastic string stretched to a length ℓ under tension T. Further, the string $0A = \ell$, is under the action of a concentrated load L as shown in Fig 1.

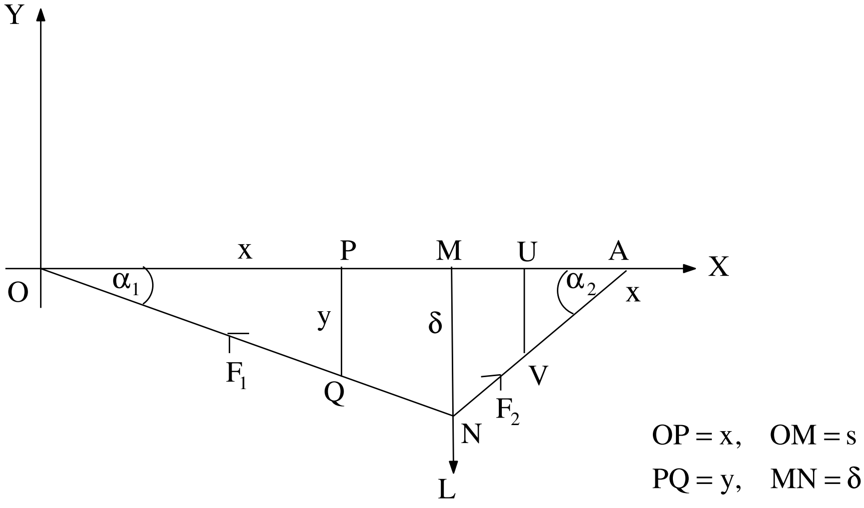

Fig.1

For determining the deflection $y(x)$ of the string at any point P at a distance x from O, we determine the governing equations, keeping in view, that the system is in equilibrium. Thus, through the balance of horizontal and vertical components of the forces, we have

$$F_1 \cos\alpha_1 = F_2 \cos\alpha_2 = T \tag{1}$$

$$F_1 \sin\alpha_1 + F_2 \sin\alpha_2 = -L \tag{2}$$

From Eqns.(1) and (2), we can write

$$\tan\alpha_1 + \tan\alpha_2 = -\frac{L}{T} \tag{3}$$

Substituting the values of $\tan\alpha_1$ and $\tan\alpha_2$ from the similar triangles OMN and AMN , we get

$$\frac{-\delta}{s} + \frac{-\delta}{\ell - s} = -\frac{L}{T}$$

or,

$$\delta = \frac{L}{T\ell}(\ell - s)\,s \tag{4}$$

In the above equation δ denotes the transverse deflection of the point of the string that was located at a distance s from the origin.

For determining the deflection $y(x)$ at any point x, we have from triangles OPQ and OMN: $\frac{y}{x} = \frac{\delta}{s}$.

Substituting the value of δ from Eqn.(4) in the above equation, we get

$$y(x) = \frac{L}{T\ell}(\ell - s)x, \qquad 0 \le x \le s$$

Similarly, if the position of the point P where deflection is observed is changed to point U then from triangles AUV and AMN we have, $\frac{y}{\delta} = \frac{\ell - x}{\ell - s}$.

or, $$y = \frac{L}{T\ell}(\ell - x)s, \; s \le x \le \ell.$$

Thus
$$y(x,s) = \begin{cases} \frac{L}{T\ell}(\ell - s)x, & 0 \le x \le s \quad (5a) \\ \frac{L}{T\ell}(\ell - x)s, & s \le x \le \ell \quad (5b) \end{cases}$$

It may be **noted** that in writing Eqns.(5a) and (5b) we have replaced $y(x)$ by $y(x,s)$ which indicates that the deflection y depends both on points s where the load is applied and the point x where deflection is observed.

If the roles of x and s are interchanged, that is, the load L is now at x and the deflection is measured at $s(>x)$ then the expression for deflection is obtained from Eqn.(5 b) with x and s, interchanged. Thus, we get $y(x,s)=\frac{L}{T\ell}(\ell-s)x$.

Further, when L assumes the value unity, the deflection $y(x,s)$ becomes $g(x,s)$ where $g(x,s)$ is given by

$$g(x,s)=\begin{cases}\dfrac{(\ell-s)x}{T\ell}, & 0\le x\le s \quad (6a)\\[2ex] \dfrac{(\ell-x)s}{T\ell}, & s\le x\le \ell \quad (6b)\end{cases}$$

The function $g(x,s)$ is called an **influence function**. The word influence is used in the sense that the function $g(x,s)$ describes the influence which a unit load concentrated at the point s has at any point x of the given string.

Now we shall show that the influence function **g(x,s)** associated with the stretched string with fixed ends satisfies the following properties

(i) It is symmetric, that is, $g(x,s)=g(s,x)$.

(ii) It satisfies the boundary conditions $g(0,s)=g(\ell,s)=0$, for all values of s satisfying $0\le s\le \ell$.

(iii) It is a continuous function of x on the interval $[0,\ell]$.

(iv) $g_x(x,s)$ is continuous for $0\le x<s$ and $s<x\le \ell$ but has a jump $\left(-\frac{1}{T}\right)$ at $x=s$.

We examine these properties for $g(x,s)$ defined in (6a) and (6b).

For proving property (i), we interchange the role of x and s in (6a) and (6b) and obtain $g(s,x)=g(x,s)$.

For property (ii), put $x=0$ in (6 a) and $x=\ell$ in (6 b) to get $g(0,s)=g(\ell,s)=0$.

For property (iii), we equate left handed limit = right handed limit = value of the function at $x=s$

Thus, $\lim\limits_{x\to s^-} g(x,s)=\lim\limits_{x\to s^-}\frac{(\ell-s)x}{T\ell}=\frac{(\ell-s)s}{T\ell}$

$$\lim_{x\to s^+} g(x,s)=\lim_{x\to s^+}\frac{(\ell-x)s}{T\ell}=\frac{(\ell-s)s}{T\ell}$$

At $x=s$, $g(x,s)=g(s,s)=\left(\frac{\ell-s}{T\ell}\right)s$

Hence, $g(x,s)$ is a continuous function of x on the interval $[0,\ell]$.

Regarding property (iv) we have to show that

$$\lim_{x\to s^+} g_x(x,s)-\lim_{x\to s^-} g_x(x,s)=-\frac{1}{T}$$

or,

$$\left(\frac{-s}{T\ell}\right)-\left(\frac{\ell-s}{T\ell}\right)=-\frac{1}{T}.$$

Alternatively, the above property can be expressed as $g_x(x,s)$ is continuous for $0\le x<s$ and $s<x\le \ell$ but has a jump discontinuity $\left(-\frac{1}{T}\right)$ at $x=s$.

Thus the influence function $g(x,s)$ defined by Eqns(6a) and (6b) satisfies properties (i) – (iv) listed above. With little modifications in the properties for $g(x,s)$, we can define a new class of function called the **Green's function** which we shall discuss in the next section but before that we give here another application of influence function.

Application of Influence Function

Let us take up the application of influence function $g(x,s)$ for determining the deflection of the string under any piecewise continuous distributed load. **Refer** to Fig.1 once again where instead of concentrated load L, we have the distributed load.

Let $w(x)$ be the distributed load per unit length. Through the balance of horizontal and vertical components of the forces we obtain

$$F_1 \cos\alpha_1 = F_2 \cos\alpha_2 = T \tag{7}$$

$$F_2 \sin\alpha_2 = F_1 \sin\alpha_1 - w(x)\Delta x \tag{8}$$

Combining Eqns(7) and (8), we get

$$\tan\alpha_2 = \tan\alpha_1 - \frac{w(x)\Delta x}{T} \tag{9}$$

Eqn(9) can be written in the following form

$$\frac{\left(\frac{dy}{dx}\right)_{x+\Delta x} - \left(\frac{dy}{dx}\right)_x}{\Delta x} = -\frac{w(x)}{T}$$

Taking limit as $\Delta x \to 0$, we have

$$\lim_{\Delta x \to 0} \frac{\left(\frac{dy}{dx}\right)_{x+\Delta x} - \left(\frac{dy}{dx}\right)_x}{\Delta x} = -\frac{w(x)}{T}$$

or, $$T y'' = -w(x) \tag{10}$$

Eqn.(10) is the differential equation satisfied by the deflection curve of the string.

A straight forward procedure to solve this problem is to solve Eqn.(10) under the boundary conditions that the string is fixed at both the ends.

But here we shall describe the procedure for obtaining the desired deflection in terms of the influence function $g(x,s)$, without solving the governing differential Eqn.(10).

Let us divide the interval $[0,\ell]$ into n equal parts each of length $\Delta s_i = s_i - s_{i-1}$ at the points $0 = s_0, s_1, s_2, \ldots, s_n = \ell$. Further, assume ξ_i to be an arbitrary point in Δs_i. If $w(x)$ is the distributed load per unit length then the portion of the distributed load acting on Δs_i is $w(\xi_i)\Delta s_i$, concentrated at the point ξ_i.

Keeping in view the definition of $g(x,s)$ one can write the expression for the deflection produced at the point x by the load as $w(\xi_i)\Delta s_i\, g(x,s)$. In writing down this expression we have used the formula that the deflection produced at any point x by the load $w(\xi_i)\Delta s_i$ is the product of the load and the deflection produced at x by a unit load at the point ξ_i.

For obtaining the deflection due to the distributed load we sum up all the deflections to get

$$y(\Delta s_i) = \sum_{i=1}^{n} w(\xi_i)\, g(x,\xi_i)\Delta s_i \tag{11}$$

Taking the limit $\Delta s_i \to 0$, Eqn.(11) assumes the following form:

$$y(x) = \int_0^{\ell} w(s)\, g(x,s)\, ds \tag{12}$$

Eqn.(12) reveals that the deflection of the string under any piecewise continuous distributed load can be obtained by evaluating the integral (12), once we have the knowledge of $g(x,s)$.

You may now try the following exercises.

E1) Find the deflection curve of a string of length ℓ bearing a load per unit length as given below

$$w(x) = \begin{cases} 0, & 0 \le x < \dfrac{\ell}{2} \\ -1, & \dfrac{\ell}{2} < x \le \ell \end{cases}.$$

E2) Determine the deflection curve of a string of length ℓ bearing concentrated load P at $x = \frac{\ell}{4}$, $-2P$ at $x = \frac{\ell}{2}$ and $3P$ at $x = \frac{3\ell}{4}$.

Let us now see how the properties satisfied by the influence function $g(x,s)$ can be modified to define the Green's functions for boundary value problems.

5.3 GREEN'S FUNCTIONS

In the previous section we derived the expression for the influence function $g(x,s)$ for determining the deflection of the stretched string of negligible mass, tension T and length ℓ due to the concentrated unit force/load. Further, the properties listed there-in are satisfied by $g(x,s)$ of the above said problem. However, with little modification in the properties for $g(x,s)$ we can enlarge the scope of the function $g(x,s)$ in the sense that we can arrive at a larger class of functions $G(x,s)$ associated with linear differential equation with constant or variable coefficients. Accordingly, we examine the following boundary value problem

$$p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)\, y = 0 \tag{13}$$

$$\alpha y(a) - \beta y'(a) = 0, \quad \gamma y(b) - \delta y'(b) = 0 \tag{14}$$

where α, β, γ and δ are non-zero.

A function $G(x,\xi)$ with the following properties is termed as the **Green's function** of the boundary-value problem (13) and (14), if

1) $G(x,\xi)$ satisfies the differential equation for $a \le x < \xi$, and $\xi < x \le b$.
2) $G(x,\xi)$ satisfies the homogeneous boundary conditions that is, $\alpha G(a,\xi) - \beta G'(a,\xi) = 0,\ \gamma G(b,\xi) - \delta G'(b,\xi) = 0$.
3) $G(x,\xi)$ is a continuous function of x for all x satisfying $a \le x \le b$.
4) For any ξ in [a, b], the function $G(x,\xi)$ has continuous derivative $G_x(x,\xi)$ in each of the half intervals $[\ a, \xi\ [$ and $]\xi, b]$ and the derivative $G_x(x,\xi)$ satisfies the condition

$$G_x(\xi^+,\xi) - G_x(\xi^-,\xi) = \frac{-1}{p_0(\xi)}, \quad \text{if} \quad x = \xi.$$

Similarly, for a bvp of the form:

$$Ly = p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)\, y = f(x) \tag{15}$$

$$\alpha y(a) - \beta y'(a) = 0 \text{ and } \gamma y(b) - \delta y'(b) = 0 \tag{16}$$

a Green's function $G(x, \xi)$, is the solution of

$$\delta(x-\xi) = \begin{cases} \infty, & \text{if } x = \xi \\ 0, & \text{if } x \neq \xi \end{cases}$$

$$LG = \delta(x - \xi) \tag{17}$$

satisfying properties 2) – 4) above. The only change in this case is that the source function $f(x)$ is replaced by a concentrated unit source at $x = \xi$. The function $\delta(x-\xi)$, called the Dirac-delta function, is the functional that maps a function $f(x)$ continuous at $x = \xi$, onto its value at $x = \xi$ such that $\int_0^\infty \delta(x-\xi)\,dx = 1$.

We shall now see how such a function $G(x, \xi)$, satisfying properties 1) – 4) can be obtained for a given bvp.

5.3.1 Construction of Green's Functions for boundary Value Problems

We shall illustrate through examples the construction of Green's functions for bvps.

Example 1: Using the definition of Green's function, construct Green's function for the boundary-value problem

$$\frac{d^2y}{dx} = 0, \quad y(0) = y'(1); \quad y'(0) = y(1)$$

Solution: We have to obtain $G(x,\xi)$ satisfying properties 1) - 4) listed above. Since $G(x,\xi)$ satisfies the homogeneous differential equation $\frac{d^2G}{dx^2} = 0$, therefore its solution can be written as

$$G(x,\xi) = \begin{cases} Ax + B, & 0 \le x \le \xi \quad (18a) \\ Cx + D, & \xi \le x \le 1. \quad (18b) \end{cases}$$

$G(x,\xi)$ satisfies the homogeneous boundary conditions, that is,

$$G(0,\xi) = G'(1,\xi); \qquad G'(0,\xi) = G(1,\xi)$$

Using the above conditions, we get,

$$B = C \text{ and } A = C + D \tag{19}$$

Further at $x = \xi$, $\quad A\xi + B = C\xi + D$ (20)

From the jump condition $\frac{\partial G}{\partial x}(1,\xi) - \frac{\partial G}{\partial x}(0,\xi) = -1$, we get

$$C - A = -1 \tag{21}$$

Hence we have four equation for the four unknowns A, B, C and D. From Eqn.(20)

$$(C - A)\xi = B - D \tag{22}$$

Combining Eqns.(19) and (21), we get $D = 1$.

Further, from Eqns.(21), (22) and (19), we get $B = C = (1-\xi)$.

$\therefore A = C + 1 = -(\xi - 2)$. Hence,

$$G(x,\xi) = -\begin{cases} (\xi-2)x + (\xi-1), & 0 \le x \le \xi \\ (\xi-1)x + (-1), & \xi \le x \le 1. \end{cases}$$

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Example 2: Construct Green's function for the differential equation

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$, under the conditions $y(x)$ is bounded as $x \to 0$ and $y(1) = \alpha y'(1), \ \alpha \neq 0$.

Solution: Since $G(x,\xi)$ satisfies the homogeneous differential equation we have

$$x\frac{d^2G}{dx^2}+\frac{dG}{dx}=0$$

or, we can write $x^2\frac{d^2G}{dx^2}+x\frac{dG}{dx}=0$ (23)

Treating Eqn.(24) as Euler's equation, its solution can be written by transforming it into a differential equation with constant coefficient viz.,

$$D(D-1)G+DG=0,\quad \text{where } x=e^z,\quad D=\frac{d}{dz}$$

or,

$$D^2G=0 \tag{24}$$

Solving Eqn.(24) we can write

$$G(x,\xi)=\begin{cases}A+Bz=A+B\ln x\ , & 0<x\le\xi\\ C+Dz=C+D\ln x\ , & \xi\le x\le 1\end{cases}$$

Using the condition of continuity at $x=\xi$, we get

$$A+B\ln\xi=C+D\ln\xi$$

or,

$$(A-C)+(B-D)\ln\xi=0 \tag{25}$$

Using the jump condition, we get

$$\lim_{x\to\xi^+}G_x(x,\xi)-\lim_{x\to\xi^-}G_x(x,\xi)=-\lim_{x\to\xi}\left(\frac{1}{x}\right)$$

or, $\frac{D}{\xi}-\frac{B}{\xi}=\frac{-1}{\xi}\Rightarrow D-B=-1$ (26)

Further, boundary condition that $G(x,\xi)$ is bounded as $x\to 0$ gives

$$B=0, \tag{27}$$

also $G(1,\xi)=\alpha G'(1,\xi)$ yields, $C=\alpha D$ (28)

Since $B=0$; $D=-1$ gives $C=-\alpha$, using these values of B, C, D in Eqn(25), we obtain

$$A=-(\alpha+\ln\xi) \tag{29}$$

Hence, $G(x,\xi)=-\begin{cases}\alpha+\ln\xi\ , & 0<x\le\xi\\ \alpha+\ln x\ , & \xi\le x\le 1\end{cases}$

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Example 3: Construct Green's function for the boundary value problem

$$\frac{d^2y}{dx^2}-y=x\ ,\qquad y(0)=y(1)=0$$

Solution: Keeping in view that $G(x,\xi)$ satisfies the homogeneous differential equation $\frac{d^2G}{dx^2}-G(x,\xi)=0$, $G(x,\xi)$ can be written as follows:

$$G(x,\xi)=\begin{cases}Ae^x+Be^{-x}\ , & 0\le x\le\xi & (30a)\\ Ce^x+De^{-x}\ , & \xi\le x\le 1 & (30b)\end{cases}$$

Since $G(x,\xi)$ satisfies the homogeneous boundary conditions

$$\therefore\quad G(0,\xi)=A+B=0\Rightarrow B=-A \tag{31}$$

and $G(1,\xi)=Ce+De^{-1}=0,\Rightarrow D=-Ce^2$ (32)

$$\therefore \quad G(x,\xi)=\begin{cases} A\,(e^{x}-e^{-x}) \Rightarrow 2A\sinh x\ , & 0\le x\le \xi \\ C\,(e^{x}-e^{2-x})\ , & \xi\le x\le 1 \end{cases}$$

Using the condition of continuity of $G(x,\xi)$ at $x=\xi$, we get

$$A(e^{\xi}-e^{-\xi})=C\{e^{\xi}-e^{2-\xi}\}$$

or, $$(C-A)\,e^{\xi}+(A-C\,e^{2})\,e^{-\xi}=0 \tag{33}$$

Using the jump condition

$$\lim_{x\to\xi^{+}} G_x\,(x,\xi)-\lim_{x\to\xi^{-}} G_x\,(x,\xi)=-1 \tag{34}$$

Substituting the value of G_x in Eqn.(34) and then taking the indicated limit, we get

$$(C-A)\,e^{\xi}-e^{-\xi}\,(A-C\,e^{2})=-1 \tag{35}$$

Solving Eqns.(33) and (35), we get

$$C-A=-\frac{1}{2}e^{-\xi} \tag{36}$$

$$A-C\,e^{2}=\frac{1}{2}e^{\xi} \tag{37}$$

Combining Eqns.(36) and (37), we get

$$A=\frac{1}{2(1-e^{2})}(e^{\xi}-e^{-\xi+2})\ ; \qquad C=\frac{1}{(1-e^{2})}\sinh\xi \tag{38}$$

Now $$\sinh 1=\frac{e-e^{-1}}{2}=\frac{e^{2}-1}{2e} \quad \text{or},\ \ (1-e^{2})=-2e\sinh 1 \tag{39}$$

Thus, from Eqns. (38) and (39), we can write

$$A=\frac{1}{-4\,e\sinh 1}(e^{\xi}-e^{-\xi+2})=-\frac{1}{2\sinh 1}\left(\frac{e^{\xi-1}-e^{-(\xi-1)}}{2}\right)$$

$$\therefore \qquad A=-\frac{1}{2\sinh 1}\sinh\,(\xi-1) \tag{40}$$

and $$C=-\frac{1}{2\,e\sinh 1}\ \sinh\xi \tag{41}$$

Thus, $$G(x,\xi)=\begin{cases} -\dfrac{1}{\sinh 1}\ \sinh\,(\xi-1)\sinh x\ , & 0\le x\le\xi \\ -\dfrac{1}{2\,e\sinh 1}\ \sinh\xi.\ (e^{x}-e^{2-x})\ , & \xi\le x\le 1 \end{cases}$$

or, $$G(x,\xi)=-\begin{cases} \dfrac{1}{\sinh 1}\ \sinh\,(\xi-1)\ \sinh x\ , & 0\le x\le\xi \\ \dfrac{1}{\sinh 1}\ \sinh\xi\ \sinh\,(x-1)\ , & \xi\le x\le 1 \end{cases} \tag{42}$$

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Example 4: Construct Green’s function for the following boundary value problem

$$\frac{d^{2}y}{dx^{2}}+k^{2}y=0 \quad , \qquad y(0)=y(1)=0$$

Solution: Keeping in view that $G(x,\xi)$ satisfies the homogeneous differential equation $\dfrac{d^{2}G}{dx^{2}}+k^{2}G(x,\xi)=0$, for $0\le x\le\xi$, $\xi\le x\le 1$, we can write

$$G(x,\xi)=\begin{cases} A\cos kx + B\sin kx, & 0\le x\le \xi \\ C\cos kx + D\sin kx, & \xi \le x \le 1 \end{cases} \qquad (43)$$

$G(x,\xi)$ satisfies the conditions $G(0,\xi)=0=G(1,\xi)$ (44)

Using conditions (44), we get

$$A=0, \text{ and } C\cos k + D\sin k = 0 \qquad (45)$$

Hence, $G(x,\xi)$ assumes the following form

$$G(x,\xi)=\begin{cases} B\sin kx, & 0\le x\le \xi \\ -C\dfrac{\sin k(x-1)}{\sin k}, & \xi \le x \le 1 \end{cases} \qquad (46)$$

Using the continuity condition at $x=\xi$, we get

$$B\sin k\xi = \frac{C}{\sin k}\sin k(1-\xi) = -\frac{C}{\sin k}\{\sin k\xi\cos k - \cos k\xi \sin k\}$$

$$\therefore \quad B = -\frac{C}{\sin k \sin k\xi}\sin k(\xi - 1)$$

$$\text{Hence, } G(x,\xi)=\begin{cases} \dfrac{C\sin k(1-\xi)\sin kx}{\sin k\sin k\xi}, & 0\le x\le \xi \\ \dfrac{C\sin k(1-x)\sin k\xi}{\sin k \sin k\xi}, & \xi \le x \le 1 \end{cases} \qquad (47)$$

The jump discontinuity condition to be satisfied is

$$\lim_{x\to\xi^+} G_x(x,\xi) - \lim_{x\to\xi^-}(x)\, G_x(x,\xi) = -1 \quad (\text{as } p_0(x)=1) \qquad (48)$$

Eqn.(48) yields

$$\frac{C}{\sin k\sin k\xi}\cos k(1-\xi)(-1)k\sin k\xi - C\frac{\sin k(1-\xi)}{\sin k \sin k\xi}k\cos k\xi = -1 \qquad (49)$$

simplifying Eqn.(49), we get

$$C=\frac{1}{k}\,\frac{\sin k\sin k\xi}{\sin k\xi\cos k(1-\xi)+\cos k\xi\sin k(1-\xi)}=\frac{\sin k\xi}{k} \qquad (50)$$

$$\therefore G(x,\xi)=\begin{cases} \dfrac{\sin k(1-\xi)\sin kx}{k\sin k}, & 0\le x\le \xi \\ \dfrac{\sin k\xi \sin k(1-x)}{k\sin k}, & \xi \le x \le 1 \end{cases}$$

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Example 5: Construct Green's function for the boundary value problem

$$\frac{d^3y}{dx^3}=0, \quad y(0)=y(1)=0, \quad y'(0)+y'(1)=0$$

Solution: We have to obtain $G(x,\xi)$ satisfying $\dfrac{d^3G}{dx^3}=0$, $G(0,\xi)=0$, $G(1,\xi)=0$, $G_x(0,\xi)+G_x(1,\xi)=0$, $G(\xi_-,\xi)=G(\xi_+,\xi)=G(\xi,\xi)$

Also, $\dfrac{\partial G}{\partial x}(1,\xi)-\dfrac{\partial G}{\partial x}(0,\xi)=1$

$$\frac{d^3G}{dx^3}=0 \Rightarrow G(x,\xi)=\begin{cases} Ax^2+Bx+C, & 0\le x\le \xi \\ Dx^2+Ex+F, & \xi \le x\le 1 \end{cases} \qquad (51)$$

Where A, B, C, D, E and F are to be determined using the above conditions:

$$G(0,\xi)=0 \Rightarrow C=0, \tag{52}$$

$$G(1,\xi)=0 \Rightarrow D+E+F=0 \tag{53}$$

Further, $\dfrac{\partial G(0,\xi)}{\partial x}+\dfrac{\partial G(1,\xi)}{\partial x}=0 \Rightarrow (2Ax+B)_{x=0}+(2Dx+E)_{x=1}=0$

or, $\quad B+2D+E=0 \tag{54}$

Further, continuity condition yields $A\xi^2+B\xi+C=D\xi^2+E\xi+F$

or, $\quad (A-D)\xi^2+(B-E)\xi+(C-F)=0 \tag{55}$

Finite jump condition gives $\dfrac{\partial G(1,\xi)}{\partial x}-\dfrac{\partial G(0,\xi)}{\partial x}=-1=2D+E-B \tag{56}$

Combining Eqns.(54) and (56) we get, $B=\dfrac{1}{2}$

Also, from Eqn.(52), $C=0$. Further, $2D+E=\dfrac{1}{2} \Rightarrow D-F=\dfrac{1}{2}$. It is not possible to solve explicitly for the remaining arbitrary constants A, D, E, F. In short, we remark that Green's function does not exist for the given bvp.

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Example 5 shows that construction of Green's function is not always possible for a given boundary value problem. But then the question arises: Is there a way to establish the existence of Green's function without going into the detailed working as done in Example 5? The **theorem on the existence of Green's function** guarantees the existence of Green's function for a given bvp. The application of this theorem to a given problem results in establishing the existence or non-existence of Green's function without going into the detailed working. We shall now state this theorem.

Theorem 1: Suppose we have a differential equation of order n given by

$$L(y)=p_0(x)\,y^n+p_1(x)\,y^{n-1}+\cdots+p_n(x)\,y=f(x), \tag{57}$$

where the functions $p_0(x), p_1(x), \ldots, p_n(x)$ and $f(x)$ are continuous on the closed interval $[a,b]$, $p_0(x)\neq 0$ on $[a,b]$ and the boundary conditions are

$$B_k(y)=\alpha_k^0\,y(a)+\alpha_k^1\,y'(a)+\alpha_k^2\,y''(a)+\cdots+\alpha_k^{n-1}\,y^{n-1}(a)$$
$$+B_k^0\,y(b)+B_k^1\,y'(b)+\cdots+B_k^{n-1}\,y^{n-1}(b)=0 \tag{58}$$
$$k=1,2,\ldots,n$$

where the linear forms $B_1, B_2, \ldots, B_n$, in $y(a), y'(a), \ldots y^{n-1}(a), y(b), y'(b), \ldots, y^{n-1}(b)$ are linearly independent. If the boundary value problem (57) – (58) has only the trivial solution $y(x)=0$, then for the operator L, Green's function exists and is unique.

We shall not be going into the details of this theorem here. To illustrate the theorem let us once again consider the bvp considered in Example 5.

We are given

$$\frac{d^3y}{dx^3}=0, \text{ and } \quad y(0)=y(1)=0, \quad y'(0)+y'(1)=0$$

$\dfrac{d^3y}{dx^3}=0 \Rightarrow y(x)=Ax^2+Bx^2+C$ and applying the boundary conditions we get

$$y(0)=C=0 \tag{59}$$

$$y(1)=A+B=0 \tag{60}$$

$$y'(x)=2Ax+B \quad ; \quad y'(0)=B \ , \quad y'(1)=2A+B$$

Thus, the third boundary condition $y'(0) + y'(1) = 0$, yields

$$B + 2A + B = 2(A + B) = 0 \tag{61}$$

Eqn.(61) is same as Eqn.(60). Thus, one of the boundary condition is redundant. Hence we have infinite number of solutions. While for the existence of Green's function we should have a trivial solution. Hence Green's function does not exist for the given bvp.

You may now try the following exercise.

E3) Check whether Green's function exists for the following boundary value problems and if it exist, construct it

a) $xy'' + y' = 0$, $y(0)$ is bounded, $y(\ell) = 0$.

b) $xy'' + y' - \frac{1}{x} y = 0$; $y(0)$ is finite, $y(1) = 0$.

c) $y'''(x) = 0$, $y(0) = y(1) = 0$, $y'(0) + y'(1) = 0$.

d) $y''(x) = 0$, $y(0) = y(1)$, $y'(o) = y'(1)$.

e) $y'' - k^2 y = 0$, $(k \neq 0)$; $y(0) = y(1) = 0$.

5.3.2 Solution of Boundary Value Problems via Green's Functions

Up till now we have been concerned with the development of procedure for the construction of Green's function for boundary value problem. We shall now state a theorem that helps in writing down the solution of the boundary value problem using its Green's function.

Theorem 2: If $G(x, \xi)$ is the Green's function of the homogeneous boundary-value problem

$$Ly(x) = 0 \quad , \quad B_1(y) = 0 \quad , \quad B_2(y) = 0, \ldots, B_n(y) = 0$$

where L and B_k are as defined in Theorem 1, then the solution of the boundary value problem

$$Ly = f(x); \quad B_k(y) = 0, \quad (k = 1, 2, 3, \ldots, n)$$

is given by

$$y(x) = \int_a^b G(x, \xi)\, f(\xi)\, d\xi \tag{62}$$

We shall now illustrate this theorem through an example.

Example 6: Using Green's function, solve the boundary-value problem

$$-\left(\frac{d^2 y}{dx^2} + y\right) = x \quad , \qquad y(0) = y\left(\frac{\pi}{2}\right) = 0$$

Solution: Since we have to solve the bvp via Green's function, we therefore, first prove the existence of Green's function for the boundary-value problem.
Solution of the homogeneous problem is given by

$$y(x) = A\cos x + B\sin x$$

Using boundary conditions, we obtain $y(0) = A = 0, \quad y\left(\frac{\pi}{2}\right) = B = 0$

Therefore, the homogeneous bvp possesses trivial solution $y(x) = 0$. Hence, the Green's function for the problem exists and can be expressed as

$$G(x, \xi) = \begin{cases} A\cos x + B\sin x \;, & 0 \le x \le \xi \\ C\cos x + D\sin x \;, & \xi \le x \le \dfrac{\pi}{2} \end{cases} \tag{63}$$

For obtaining A, B, C and D we use the properties of Green's function.

$$G(0, \xi) = A = 0 \;, \; G\left(\frac{\pi}{2}, \xi\right) = D = 0$$

Hence, $G(x,\xi)=\begin{cases} B\sin x \quad , & 0\le x\le \xi \\ C\cos x \quad , & \xi \le x\le \dfrac{\pi}{2}\end{cases}$

For continuity at $x=\xi$; $B\sin\xi = C\cos\xi$ (64)

Also, $\left(\dfrac{\partial G}{\partial x}\right)_{x=\frac{\pi}{2}} - \left(\dfrac{\partial G}{\partial x}\right)_{x=0} = \dfrac{-1}{p_0(x)} = 1$, yields

$$-C\sin\xi - B\cos\xi = .1 \tag{65}$$

Combining Eqns.(64) and (65) we get,

$$B = -\cos\xi \quad , \qquad C = -\sin\xi \tag{66}$$

Thus, $G(x,\xi)=\begin{cases} -\cos\xi\sin x \quad , & 0\le x\le \xi \\ -\sin\xi\cos x \quad , & \xi\le x\le \dfrac{\pi}{2}\end{cases}$ (67)

We now make use of Eqn.(62) to obtain the solution $y(x)$.

Hence, $y(x) = -\int_0^x (\cos x\sin\xi)\,\xi\, d\xi + \int_x^{\frac{\pi}{2}} (-\sin x\cos\xi)\xi\, d\xi$ (68)

$$= \cos x\,[\,(\xi\cos\xi)_0^x - \int_0^x \cos\xi\, d\xi\,] - \sin x\,[\,(\xi\sin\xi)_x^{\frac{\pi}{2}} - \int_x^{\frac{\pi}{2}} \sin\xi\, d\xi\,]$$

$$= \cos x\,(x\cos x - 0) - \cos x\sin x - \sin x\left(\frac{\pi}{2} - x\sin x\right) + \sin x\,(-\cos\frac{\pi}{2} + \cos x)$$

$$= x(\cos^2 x + \sin^2 x) - \frac{\pi}{2}\sin x$$

$$= x - \frac{\pi}{2}\sin x.$$

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You may now try the following exercise.

E4) Using Green's function, solve the boundary value problem

$$-\left(\frac{d^2y}{dx^2} - y\right) = x\ , \quad y(0) = y(1) = 0$$

You must have realised that the procedure discussed above for the construction of Green's function for a boundary value problem is quite an involved process. Consequently, in the next section, we look for an alternative procedure to construct Green's function by establishing a relationship between the method of variation of parameters and the Green's function approach for linear, non-homogeneous differential equation of second order with variable or constant coefficients.

5.4 METHOD OF VARIATION OF PARAMETERS AND GREEN'S FUNCTIONS

Consider the general form of second order, linear, non-homogeneous differential equation

$$p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = f(x) \tag{69}$$

where the functions $p_0(x), p_1(x), p_2(x)$ are continuous on the closed interval $[a,b], p_0(x) \ne 0$ on $[a,b]$ and the boundary conditions are $y(a) = 0 = y(b)$.

From your knowledge of the method of variation of parameters (Ref. Unit 7, Block-2, MTE-08) you know that if $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the associated homogeneous equation to Eqn.(69), then

$$y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x) \qquad (70)$$

is a solution of the non-homogeneous Eqn.(69) provided

$$\frac{du_1}{dx} = -\frac{y_2(x)f(x)}{p_0(x)W(x)} \quad , \quad \frac{du_2}{dx} = \frac{y_1(x)f(x)}{p_0(x)W(x)} \qquad (71)$$

Further, $W(x)$ is the Wronskian of the two solutions y_1 and y_2 and is given by

$$W(x) = y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x) \qquad (72)$$

For obtaining y as defined in Eqn.(70) we have to obtain the respective integrals of $\frac{du_1}{dx}$ and $\frac{du_2}{dx}$ between a to x and x to b. Thus,

$$u_1(x) = -\int_a^x \frac{y_2(t)f(t)}{W(t)p_0(t)}dt, \quad u_2(x) = -\int_x^b \frac{y_1(t)f(t)}{W(t)p_0(t)}dt$$

Hence,

$$y(x) = -y_1(x)\int_a^x \frac{y_2(t)f(t)}{W(t)p_0(t)}dt - y_2(x)\int_x^b \frac{y_1(t)f(t)}{W(t)p_0(t)}dt \qquad (73)$$

Solution (73) has to satisfy the conditions $y(a) = 0$, and $y(b) = 0$. Thus, for $y(a)$ to be zero, the first integral on the rhs of Eqn.(73) is zero and if for arbitrary values of function $f(x), y_2(x)$ is zero when $x = a$ then $y(a) = 0$. Similarly, for $y(b)$ to be zero the second integral on the rhs of Eqn.(73) is zero and the first integral vanishes for all arbitrary values of $f(x)$, if for $x = b$, $y_1(b) = 0$.

Assuming that $y_1(x)$ and $y_2(x)$ are so chosen that these solutions satisfy the conditions $y_2(a) = 0$, and $y_1(b) = 0$, we can write

$$y(x) = -\left[\int_a^x \frac{y_1(x)\,y_2(t)}{W(t)p_0(t)}f(t)\,dt + \int_x^b \frac{y_2(x)\,y_1(t)}{W(t)p_0(t)}f(t)\,dt\right] \qquad (74)$$

Eqn.(74) can be replaced by

$$y(x) = -\int_a^b G(x,t)\,f(t)\,dt \qquad (75)$$

$$\text{where,}\quad G(x,t) = \begin{cases} \dfrac{y_1(t)\,y_2(x)}{W(t)p_0(t)} \ , & x \le t \le b, \ \text{i.e.} \ \ a \le x \le t \\ \dfrac{y_1(x)\,y_2(t)}{W(t)p_0(t)} \ , & a \le t \le x, \ \text{i.e.} \ \ t \le x \le b \end{cases} \qquad (76)$$

The function $G(x,t)$ as defined in (76) satisfies all the properties of the Green's function. It is continuous at $x = t$, i.e., $\lim_{t\to t^-} G(t^-, t) = \lim_{t\to t^+} G(t^+, t) = G(t,t)$. Also $G(x,t)$ satisfies the boundary conditions, that is,

$$G(a,t) = \frac{y_1(t)\,y_2(a)}{p_0(t)\,W(t)} = 0 \ \text{as} \ y_2(a) = 0,$$

$$\text{similarly,}\ G(b,t) = \frac{y_1(b)\,y_2(t)}{W(t)\,p_0(t)} = 0 \ \text{as} \ y_1(b) = 0$$

Further, $G(x,t)$ satisfies the homogeneous differential equation $p_0(x)\,y''(x) + p_1(x)\,y'(x) + p_2(x)\,y(x) = 0$, as shown through method of variation of parameters.

To verify the jump condition $\left(\dfrac{\partial G}{\partial x}\right)_{x=t^+} - \left(\dfrac{\partial G}{\partial x}\right)_{t^{-1}} = \dfrac{-1}{p_0(t)}$

We have, $$G_x(x,t) = \begin{cases} \dfrac{y_1(t)\, y_2'(x)}{p_0(t)\, W(t)} \;, & a \le x \le t \\ \dfrac{y_1'(x)\, y_2(t)}{W(t)\, p_0(t)} \;, & t \le x \le b \end{cases} \tag{77}$$

Hence, $$\left(\frac{\partial G}{\partial x}\right)_{x=t^+} - \left(\frac{\partial G}{\partial x}\right)_{x=t^-} = \frac{y_1'(t)\, y_2(t) - y_1(t)\, y_2'(t)}{p_0(t)\, W(t)} = \frac{-1}{p_0(t)} \tag{78}$$

Also, $G(x,t) = G(t,x)$.

Thus, $G(x,t)$ defined by Eqn.(76)satisfies all the conditions that are necessary for it to be termed as Green's function.

Let us consider the following example.

Example 7: Determine the appropriate Green's function by using the method of variation of parameters for the boundary value problem

$$-\frac{d^2y}{dx^2} = f(x)\;,\quad y'(0) = 0\;,\quad y(1) = 0\;,$$

and express the solution as a definite integral.

Solution: We have to determine $y_1(x)$ and $y_2(x)$ as two linearly independent solutions of the associated homogeneous differential equation $\dfrac{d^2y}{dx^2} = 0$. So that $y_1'(0) = 0$, $y_2(1) = 0$.

We take, $y_1(x) = 1$ and $y_2(x) = 1 - x$

The Wronskian of these two functions is

$$W(x) = \begin{vmatrix} 1 & 1-x \\ 0 & -1 \end{vmatrix} = -1 \ne 0\,.$$

The Green's function can now be constructed by using Eqn.(76). We obtain

$$G(x,\xi) = \begin{cases} x-1 \;, & 0 \le x \le \xi \\ \xi - 1 \;, & \xi \le x \le 1 \end{cases} \tag{79}$$

Thus solution of a given bvp as a definite integral is given by

$$y(x) = \int_0^1 G(x,\xi)\, f(\xi)\; d\xi\,. \tag{80}$$

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There are certain **limitations** of formula (76) for obtaining Green's function. This method can be used only if we know the solution of the corresponding homogeneous differential equation, which we know is not always possible. Consequently, we give an alternative approach for developing Green's function expansion in a series of suitably chosen orthogonal functions. We take up an example that illustrate the use of eigen function expansion for obtaining Green's function for ODEs.

Example 8: Construct Green's function for the non-homogeneous problem

$$\frac{d^2y}{dx^2} = f(x),\; y(0) = y(\ell) = 0 \tag{81}$$

Solution: As per definition $G(x,\xi)$ satisfies

$$\frac{d^2G}{dx^2} = \delta(x-\xi) \tag{82}$$

$$G(0,\xi) = G(\ell,\xi) = 0 \tag{83}$$

Since $G(x,\xi)$ vanishes at both the ends of the interval $(0,\ell)$ therefore it can be expanded in terms of a series of suitably chosen orthogonal functions. Herein, we choose Fourier sine series for $G(x,\xi)$, that is

$$G(x,\xi)=\sum_{n=1}^{\infty} \gamma_n(\xi)\sin\frac{n\pi x}{\ell}, \tag{84}$$

where the coefficients γ_n depend on the parameter ξ.
From Eqn.(84), we get

$$\frac{\partial G}{\partial x}=\sum_{n=1}^{\infty}\left(\frac{n\pi}{\ell}\right)\gamma_n(\xi)\cos\frac{n\pi}{\ell}x.$$

$$\frac{\partial^2 G}{\partial x^2}=\sum_{n=1}^{\infty}-\left(\frac{n^2\pi^2}{\ell^2}\right)\gamma_n(\xi)\sin\frac{n\pi}{\ell}x \tag{85}$$

The r.h.s of Eqn.(82) can also be expressed as

$$\delta(x-\xi)=\sum_{n=1}^{\infty}\Delta_n\sin\frac{n\pi}{\ell}x \tag{86}$$

where the Fourier coefficients Δ_n are given by

$$\Delta_n(\xi)=\frac{2}{\ell}\int_0^{\ell}\delta(x-\xi)\sin\frac{n\pi}{\ell}x\,dx=\frac{2}{\ell}\sin\frac{n\pi}{\ell}\xi \tag{87}$$

Combining Eqns.(82), (85) and (86), we get

$$\sum_{n=1}^{\infty}-\left(\frac{n^2\pi^2}{\ell^2}\right)\gamma_n(\xi)\sin\frac{n\pi}{\ell}x=\sum_{n=1}^{\infty}\frac{2}{\ell}\sin\frac{n\pi}{\ell}\xi\,\sin\frac{n\pi}{\ell}x$$

or, $\qquad \gamma_n(\xi)=-\dfrac{2\ell}{n^2\pi^2}\sin\dfrac{n\pi}{\ell}\xi.$

Hence, $G(x,\xi)=-\dfrac{2\ell}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi\xi}{\ell}\sin\frac{n\pi}{\ell}x \qquad (88)$

Through the explicit formula (76) the Green's function for the given problems turns out to be

$$G(x,\xi)=\begin{cases}\dfrac{x(\ell-\xi)}{\ell} & (0\le x\le\xi)\\[2mm] \dfrac{\xi(\ell-x)}{\ell} & (\xi\le x\le\ell)\end{cases} \tag{89}$$

Though the two expressions given by Eqns.(88) and (89) for the Green's function appear to be different but they are same. You can verify it by expanding Eqn.(89) in a Fourier sine series. In order to obtain the solution of the non-homogeneous bvp (82) we use the formula

$$y(x)=\int_0^{\ell}G(x,\xi)f(\xi)\,d\xi. \tag{90}$$

On using the value of $G(x,\xi)$ from Eqn.(88), we get

$$y(x)=\frac{-2\ell}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{\ell}\int_0^{\ell}f(\xi)\sin\frac{n\pi\xi}{\ell}\,d\xi \tag{91}$$

On using the substitution

$$b_n=\frac{2}{\ell}\int_0^{\ell}f(\xi)\sin\frac{n\pi}{\ell}\xi\,d\xi$$

we get,

$$y(x)=-\frac{\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{b_n}{n^2}\sin\frac{n\pi}{\ell}x. \tag{92}$$

as the required solution of bvp (82).

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You may now try the following exercises.

E5) Solve each of the following boundary value problem by determining the appropriate Green's function by using the method of variation of parameters and expressing the solution as a definite integral.

a) $-y'' = f(x)$, $y(0) = 0$, $y(1) + y'(1) = 0$.

b) $-(y'' + y) = f(x)$, $y'(0) = 0$, $y(1) = 0$.

c) $-y'' = f(x)$, $y(0) = 0$, $y'(1) = 0$.

d) $-(y'' + y) = f(x)$, $y(0) = 0$, $y(1) = 0$.

We now end this unit by giving a summary of what we have learnt in it.

5.5 SUMMARY

In this unit, we have covered the following:

1. Introduced the concept of Green's function through physical consideration that has eventually led to the definition of influence function.
2. Employed the influence function for obtaining the deflection of the stretched string of negligible mass under piecewise continuous distributed load.
3. Enlarged the scope of the influence function by defining Green's function in the context of boundary value problems associated with linear ODE with constant or variable coefficients.
4. Constructed Green's functions for bvps and obtained the solution of bvps via Green's functions.
5. Outline a procedure for constructing Green's function via the method of variation of parameters.

5.6 SOLUTIONS/ANSWERS

E1) $$y(x) = \begin{cases} -\dfrac{\ell x}{8T}, & 0 \le x \le \ell \\ \dfrac{4x^2 - 5\ell x + \ell^2}{8T}, & \dfrac{\ell}{2} \le x \le \ell \end{cases}$$

E2) $$y(x) = \begin{cases} \dfrac{Px}{2T}, & 0 \le x \le \dfrac{\ell}{4} \\ \dfrac{P(\ell - 2x)}{4T}, & \dfrac{\ell}{4} \le x \le \dfrac{\ell}{2} \\ \dfrac{3P(\ell - x)}{2T}, & \dfrac{3\ell}{4} \le x \le \ell \end{cases}$$

E3) a) $$G(x,\xi) = \begin{cases} \ln\dfrac{\xi}{\ell}, & 0 \le x \le \xi \\ \ln\dfrac{x}{\ell}, & \xi \le x \le \ell \end{cases}$$

b) $$G(x,\xi) = \begin{cases} \dfrac{x}{2}\left(\xi - \dfrac{1}{\xi}\right), & 0 \le x \le \xi \\ \dfrac{\xi}{2}\left(x - \dfrac{1}{x}\right), & \xi \le x \le 1 \end{cases}$$

c) Green's function doesn't exist.

d) Green's function does not exist, bvp has infinite number of solutions.

e) $$G(x,\xi)=\begin{cases}\dfrac{\sinh k(\xi-1)\sinh x}{k\sinh k}, & 0\le x\le \xi\\ \dfrac{\sinh k\xi \sinh k(x-1)}{k\sinh k}, & \xi\le x\le 1\end{cases}$$

E4) Solution of given bvp can be written in the form

$$y(x)=\int_0^1 G(x,\xi)\,\xi\,d\xi$$

where we have from Example 3

$$G(x,\xi)=-\begin{cases}\dfrac{\sinh x\sinh(\xi-1)}{\sinh 1}, & 0\le x\le\xi\\ \dfrac{\sinh\xi\sinh(x-1)}{\sinh 1}, & \xi\le x\le 1\end{cases}$$

$$\therefore\quad -y(x)=\int_0^x \frac{\xi\sinh\xi\sinh\ (x-1)}{\sinh 1}\,d\xi+\int_x^1\frac{\xi\sinh x\sinh\ (\xi-1)}{\sinh 1}\,d\xi$$

$$=\frac{\sinh\ (x-1)}{\sinh 1}\int_0^x \xi\sinh\xi+\frac{\sinh x}{\sinh 1}\int_x^1 \xi\sinh\ (\xi-1)\ d\xi\,.$$

Now,

$$\int_0^x \xi\sinh\xi\ d\xi=(\xi\cosh\xi)_0^x-\int_0^x\cosh\xi\ d\xi=x\cosh x-\sinh x$$

and

$$\int_x^1 \xi\sinh\ (\xi-1)\ d\xi=1-x\cosh\ (x-1)+\sinh\ (x-1)$$

Thus

$$-y(x)=\frac{1}{\sinh x}\begin{cases}\sinh\ (x-1)\ [x\cosh x-\sinh x]+\\ \sinh x\ [1-x\cosh(x-1)+\sinh\ (x-1)]\end{cases}$$

$$=\frac{\sinh x}{\sinh 1}-x\,.$$

E5) Solution in all cases is given by $y(x)=\int_0^1 G(x,\xi)\ f(\xi)\ d\xi$.

a) We take $y_1(x)=x,\ y_2(x)=2-x$ which satisfy

$$y_1(0)=0,\ y_2(1)=2-1=1,\ y_2'(1)=-1.$$

$$\therefore\quad y_2(1)+y_2'(1)=0$$

$$W(\xi)=\begin{vmatrix}\xi & 2-\xi\\ 1 & -1\end{vmatrix}=-\xi-2+\xi=-2\ne 0$$

Hence, $$G(x,\xi)=\begin{cases}\dfrac{\xi(2-x)}{(-2)}, & 0\le x\le\xi\\ \dfrac{x(2-\xi)}{(-2)}, & \xi\le x\le 1\end{cases}$$

b) $y_1(x)=\cos x\ ,\ y_1'(0)=\sin 0=0$

$y_2(x)=\sin\ (x-1),\ y_2(1)=0$

$$W(\xi)=-\cos\xi\cos(-\xi+1)+\sin(1-\xi)\sin\xi=\cos(\xi-\xi+1)=\cos 1\ne 0$$

$$\text{Thus,}\quad G(x,\xi)=\begin{cases}\dfrac{\cos\xi\sin(x-1)}{\cos 1}, & 0\le x\le\xi\\[2mm] \dfrac{\cos x\sin(\xi-1)}{\cos 1}, & \xi\le x\le 1\end{cases}$$

c) $y_1(x)=x,\ \ y_2(x)=1$

$W(\xi)=\xi\times 0-1.1=-1\neq 0$

$$\text{Hence,}\quad G(x,\xi)=\begin{cases}\dfrac{\xi}{-1}, & 0\le x\le\xi\\[2mm] \dfrac{x}{-1}, & \xi\le x\le 1\end{cases}$$

d) $G(x,\xi)$ satisfies the homogeneous differential equation
$y''+y=0;\ \ y_1(x)=\sin x\ ;\ \ y_1(0)=0$
$y_2(x)=\sin(x-1),\ \ y_2(1)=0$

$$W(\xi)=\begin{vmatrix}\sin\xi & \sin(\xi-1)\\ \cos\xi & \cos(\xi-1)\end{vmatrix}=\sin\xi\cos(\xi-1)-\cos\xi\sin(\xi-1)$$

$$=\sin(\xi-\xi+1)=\sin 1\neq 0$$

$$\therefore G(x,\xi)=\begin{cases}\dfrac{\sin(x-1)\sin\xi}{\sin 1}, & 0\le x\le\xi\\[2mm] \dfrac{\sin x\sin(\xi-1)}{\sin 1}, & \xi\le x\le 1\end{cases}.$$

—x—