UNIT 3 LEGENDRE, HERMITE AND LAGUERRE POLYNOMIALS

Structure **Page No.**

3.1 INTRODUCTION

In Unit 2, we discussed methods of finding series solutions of second order, linear, homogeneous differential equations with variable coefficients. In this unit and the unit to follow, we shall apply these methods to a few special second order equations occurring frequently in applied mathematics, engineering and physics. These equations are Legendre's, Bessel's, Hermite's and Laguerre's equations. The solution to these equations, that occur in applications, are referred to as **special functions**. They are called 'special' as they are different from the standard functions like sine, cosine, exponential, logarithmic etc. In this unit, we shall concentrate on Legendre, Hermite and Laguerre polynomials which are polynomial solutions to Legendre's, Hermite's and Laguerre's differential equations.

Legendre polynomials first arose in the problem of expressing the Newtonian potential of a conservative force field in an infinite series involving the distance variable of two points and their included central angle. Other similar problems dealing with either gravitational potential or electrostatic potentials and steady-state heat conduction problems in spherical solids, also lead to Legendre polynomials. Other polynomials which commonly occur in applications are the Hermite and Laguerre. They play an important role in quantum mechanics, and in probability theory.

We have started the unit by obtaining the power series solutions of Legendre's differential equation in Sec.3.2 and introduced Legendre polynomials and Legendre functions of both first and second kind. Various properties of Legendre polynomials are also discussed in this section. Polynomials solutions of Hermite's and Laguerre's equations and their properties are discussed in Sec.3.3. Applications of the Legendre and Hermite polynomials to physical situtations are discussed in Sec.3.4.

Objectives

After studying this unit you should be able to

- obtain the power series solutions of Legendre's differential equation;
- derive Rodrigue's formula, for Legendre polynomials;
- obtain Legendre polynomials through generating function;
- use recurrence relations for Legendre polynomials and its orthogonality property in various applications;
- derive Rodrigue's formula, generating function, recurrence relations and orthogonal property of Hermite and Laguerre polynomials and use them in various applications.

3.2 LEGENDRE POLYNOMIALS

The differential equation of the form

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \qquad (1)$$

is called **Legendre's differential equation** or simply **Legendre's equation**, where 'n' is a real constant. We can also write Legendre's equation in the form

$$\frac{d}{dx}\left\{(1-x)^2\frac{dy}{dx}\right\}+n(n+1)y=0 \qquad (2)$$

In Eqn.(1), if we divide by $(1-x^2)$ throughout then we can write the equation as

$$\frac{d^2y}{dx^2}-\frac{2x}{1-x^2}\frac{dy}{dx}+\frac{1}{1-x^2}n(n+1)y=0,\ x\neq\pm 1. \qquad (3)$$

Here $-\frac{2x}{1-x^2}$ and $\frac{n(n+1)}{1-x^2}$ are analytic functions for $x=0$. Also if we use Binomial expressions, then both these converge for $|x|<1$. Hence $x=0$ is an ordinary point of Legendre's Eqn.(1) and this suggests that Eqn.(1) has a power series solution about $x=0$.

Assume the series solution

$$y(x)=\sum_{k=0}^{\infty}c_k x^k \qquad (4)$$

Differentiting Eqn.(4) w.r. to x, we get

$$y'(x)=\sum_{k=1}^{\infty}c_k k\,x^{k-1}$$

and, $$y''(x)=\sum_{k=2}^{\infty}c_k k(k-1)\,x^{k-2}$$

Substituting for $y(x), y'(x)$ and $y''(x)$ in Eqn.(1), we get

$$(1-x^2)\sum_{k=2}^{\infty}c_k k(k-1)x^{k-2}-2x\sum_{k=1}^{\infty}c_k k\,x^{k-1}+n(n+1)\sum_{k=0}^{\infty}c_k x^k=0$$

i.e., $$\sum_{k=2}^{\infty}c_k\,k(k-1)x^{k-2}-\sum_{k=2}^{\infty}c_k k(k-1)x^k-2\sum_{k=1}^{\infty}c_k kx^k+n(n+1)\sum_{k=0}^{\infty}c_k x^k=0$$

i.e., $$\sum_{k=0}^{\infty}c_{k+2}(k+2)(k+1)x^k-\sum_{k=0}^{\infty}c_{k+2}(k+2)(k+1)x^{k+2}-2\sum_{k=0}^{\infty}c_{k+1}(k+1)x^{k+1}$$

$$+n(n+1)\sum_{k=0}^{\infty}c_k x^k=0 \qquad (5)$$

Equating the coefficients of various powers of x on both sides of Eqn.(5), we get

For x^0 : $c_2(2.1)+n(n+1)c_0=0$ i.e., $c_2=-\frac{n(n+1)}{2}c_0$ (6)

For x^1 : $c_3(3.2)-2c_1+n(n+1)c_1=0$ i.e., $c_3=\frac{2-n(n+1)}{3.2}c_1$ (7)

For x^k : $c_{k+2}(k+2)(k+1)-c_k\,k(k-1)-2c_k k+n(n+1)c_k=0$

$$\text{i.e., } c_{k+2}=\frac{k(k-1)+2k-n(n+1)}{(k+2)\,(k+1)}c_k,\ \text{ for } k\geq 0$$

$$\text{i.e., } c_k=\frac{(k-2)\,(k-1)-n(n+1)}{k\,(k-1)}c_{k-2},\ \text{for } k\geq 2$$

$$=\frac{-[n-(k-2)][n+(k-1)]}{k\,(k-1)}c_{k-2},\ \text{ for } k\geq 2 \qquad (8)$$

From recurrence relation (8), alongwith Eqns.(6) and (7), we observe that coefficients c_k for k even are multiples of c_0 while those for k odd are multiples of c_1. Thus, in general, we have

$$c_{2k} = \frac{-(n-2k+2)(n+2k-1)}{2k(2k-1)} c_{2k-2}, \text{ for } k \geq 1$$

$$= (-1)^k \frac{(n-2k+2)(n-2k+4)\ldots(n-2).n(n+1)(n+3)\cdots(n+2k-1)}{(2k)!} c_0, \quad \text{for } k \geq 1 \qquad (9)$$

and,

$$c_{2k+1} = \frac{-(n-2k+1)(n+2k)}{(2k+1)(2k)} c_{2k-1}, \text{ for } k \geq 1$$

$$= (-1)^k \frac{(n-2k+1)(n-2k+3)\ldots(n-1)(n+2)(n+4)\ldots(n+2k)}{(2k+1)!} c_1, \quad \text{for } k \geq 1 \qquad (10)$$

Substituting from relations (9) and (10) in relation (4), the power series solution of Eqn.(1) can be written as

$$y(x) = c_0\left[1 - \frac{n(n+1)}{(n)!}x^2 + \frac{(n+3)(n+1)n(n-2)}{(4)!}x^4 - \cdots\right]$$

$$+ c_1\left[x - \frac{(n+2)(n-1)}{(3)!}x^3 + \frac{(n+4)(n+2)(n-1)(n-3)}{(5)!}x^5 - \cdots\right]$$

$$= c_0\left[1 + \sum_{k=1}^{\infty}(-1)^k \frac{(n-2k+2)(n-2k+4)\cdots(n-2).n(n+1)(n+3)\ldots(n+2k-1)}{(2k)!}x^{2k}\right]$$

$$= c_1\left[x + \sum_{k=1}^{\infty}(-1)^k \frac{(n-2k+1)(n-2k+3)\cdots(n-1).(n+2)(n+4)\ldots(n+2k)}{(2k+1)!}x^{2k+1}\right]$$

$$= c_0\, y_1(x) + c_1\, y_2(x) \qquad (11)$$

where,

$$y_1(x) = 1 + \sum_{k=1}^{\infty}(-1)^k \frac{(n-2k+2)(n-2k+4)\ldots(n-2).n(n+1)(n+3)\ldots(n+2k-1)}{(2k)!}x^{2k} \qquad 12)$$

and

$$y_2(x) = x + \sum_{k=1}^{\infty}(-1)^k \frac{(n-2k+1)(n-2k+3)\ldots(n-1).n(n+2)(n+4)\ldots(n+2k)}{(2k+1)!}x^{2k+1} \qquad (13)$$

when n is not an integer both $y_1(x)$ and $y_2(x)$ converge for $|x|<1$.

Since c_0 and c_1 are arbitrary constants, and y_1 and y_2 are linearly independent solutions of Eqn.(1), solution (11) can be considered as the general solution of Legendre's Eqn.(1). The functions defined by Eqns.(12) and (13) are called **Legendre Functions.**

In case, n is a non-negative integer, then $y_1(x)$ reduce to a polynomial of degree n, if n even; and the same is true for $y_2(x)$ if n is odd. For example, **for n even**, we have

For $n = 0$: $y_1(x) = 1$

For $n = 2$: $y_1(x) = 1 - 3x^2$

For $n = 4$: $y_1(x) = 1 - 10x^2 + \frac{35}{3}x^4$

and so on. Thus $y_1(x)$ reduces to a polynomial of even powers. For these values of $n, y_2(x)$ remains an infinite series.

In general, for $n = 2m$, where m is an integer,

$$y_1(x) = 1 + \sum_{k=1}^{m} (-1)^k \frac{(2m-2k+2)(2m-2k+4)\ldots 2m(2m+1)(2m+3)\ldots(2m+2k-1)}{(2k)!} x^{2k} \qquad (14)$$

and for $k = m+1, k = m+2, \ldots$ etc. the terms in the summation on the R.H.S. of Eqn.(14) are zero.

When n **is odd**, $y_1(x)$ remains an infinite power series and $y_2(x)$ reduces to polynomial of odd powers. For example,

For $n = 1$: $y_2(x) = x$,

For $n = 3$: $y_2(x) = x - \frac{5}{3}x^3$.

For $n = 5$: $y_2(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5$.

In general, for $n = 2m+1$, we may obtain

$$y_2(x) = x + \sum_{k=1}^{\infty} (-1)^k \frac{(2m-2k+2)(2m-2k+4)\ldots 2m.(2m+3)(2m+5)\ldots(2m+2k+1)}{(2k+1)!} x^{2k+1}. \qquad (15)$$

These polynomials multiplied by suitable constants are called **Legendre polynomials.** The Legendre polynomials are denoted by $P_n(x)$ where n denotes the order of the polynomial. Therefore, when n takes positive integral values, one of the linearly independent solutions of Eqn.(1) is a Legendre polynomial and the second solution is an infinite series. In order to explicitly write the expressions for the Legendre polynomials, we need to evaluate the multiplicative constants. The values of the multiplicative constants are obtained by setting $P_n(1) = 1$. It is too cumbersome to use the recurrence relation given by Eqn.(8) or the polynomials given above to determine the multiplicative constants. It is easy to use the Rodrigue's formula which we shall discuss now to find the expressions for the Legendre Polynomials.

3.2.1 Rodrigue's Formula

We shall now show that a polynomial of degree n, obtained by applying the binomial theorem to $(x^2-1)^n$ and differentiating it n times, is a solution of the Legendre's differential equation. The expression for $P_n(x)$ can then be obtained by using $P_n(1) = 1$.

Consider the function

$$f(x) = (x^2-1)^n. \qquad (16)$$

Differentiating it w.r.t. to x, we get

$$f'(x) = n(x^2-1)^{n-1}.2x$$

$$\Rightarrow (x^2-1)f'(x) - 2n\,f(x)\,x = 0. \qquad (17)$$

Differentiating Eqn.(17), $(n+1)$ times by using the Leibnitz rule, we get

$$\left[(x^2-1)f^{(n+2)}(x) + (n+1)(2x)f^{(n+1)}(x) + \frac{n(n+1)}{2}.2\,f^{(n)}(x)\right]$$

$$-2n[x\,f^{n+1}(x) + (n+1).1.f^{(n)}(x)] = 0,$$

or, $(x^2-1)f^{(n+2)}(x) + 2x\,f^{(n+1)}(x) - n(n+1)f^{(n)}(x) = 0$,

or, $(1-x^2)\frac{d^2}{dx^2}[f^{(n)}(x)] - 2x\frac{d}{dx}[f^{(n)}(x)] + n(n+1)[f^{(n)}(x)] = 0$.

This shows that $f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)] = \frac{d^n}{dx^n}[(x^2-1)^n]$ satisfies Lagendre's Eqn.(1).

Thus Legendre polynomial $P_n(x)$ must be a constant multiple of $\frac{d^n}{dx^n}[(x^2-1)^n]$, i.e.,

$P_n(x) = A\frac{d^n}{dx^n}[(x^2-1)^n]$, where A is a constant. The constant A is determined by setting $P_n(1)=1$.

Now $P_n(x) = A\,D^n[(x^2-1)^n] = A\,D^n[(x-1)^n(x+1)^n]$, where $D = \frac{d}{dx}$

$$= A\sum_{r=0}^{n} {}^nC_r\,[D^{n-r}\{(x-1)^n\}].[D^r\{(x+1)^n]$$

$$= (n)!A(x+1)^n + \text{terms with } (x-1) \text{ as factor.}$$

Thus, $P_n(1) = 1 \Rightarrow A.(n)!.2^n = 1$

i.e., $A = \frac{1}{(n)!2^n}$.

Hence, $$\mathbf{P_n(x) = \frac{1}{(n)!2^n}\frac{d^n}{dx^n}\{(x^2-1)^n\}}. \qquad (18)$$

It is called **Rodrigue's Formula** for Legendre Polynomials.
Using the Binomial theorem we can write

$$(x^2-1)^n = \sum_{r=0}^{n} {}^nC_r(x^2)^{n-r}(-1)^r = \sum_{r=0}^{n}\frac{(-1)^r n!x^{2n-2r}}{r!(n-r)!}$$

Differentiating the above expression n times and substituting in Eqn.(18), we get

$$P_n(x) = \frac{1}{n!2^n}\sum_{r=0}^{n}\frac{(-1)^r n!}{r!(n-r)!}\frac{d^n(x^{2n-2r})}{dx^n}$$

$$= \mathbf{\frac{1}{2^n}\sum_{r=0}^{N}\frac{(-1)^r(2n-2r)!x^{n-2r}}{r!(n-r)!(n-2r)!}} \qquad (19)$$

where, $N = n/2$ or $(n-1)/2$ whichever is an integer. In notation we usually write

$N = [n/2]$, where $\left[\frac{n}{2}\right] = \begin{cases} n/2, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$.

This function $P_n(x)$ as given by Eqn.(19) is called **Legendre function of the first kind** or **Legendre polynomial of degree n.**

Example 1: Show that

$$P_n(0) = \begin{cases} (-1)^{n/2}\dfrac{1.3.5.\dots(n-1)}{2.4.\dots n}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}.$$

Solution: From Rodrigue's formula for Legendre polynomials ,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

$$= \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x+1)^n(x-1)^n\right]$$

$$= \frac{1}{2^n n!}\Big[n!(x+1)^n + {}^nC_1 n(x+1)^{n-1}(x-1)n! + {}^nC_2 n(n-1)(x+1)^{n-2}\frac{n!}{2}(x-1)^2$$
$$+\cdots + n!(x-1)^n\Big]$$

$$= \frac{1}{2^n n!}\Big[n!(x+1)^n + ({}^nC_1)^2(x+1)^{n-1}(x-1)n! + ({}^nC_2)^2(x+1)^{n-2}(x-1)^2 n!$$
$$+\cdots + n!(x-1)^n\Big]$$

Putting $x = 0$, we get

$$P_n(0) = \frac{1}{2^n n!} n!\left[({}^nC_0)^2 - ({}^nC_1)^2 + ({}^nC_2)^2 + \cdots + (-1)^n ({}^nC_n)^2\right]$$

$$= \frac{1}{2^n n!} n! \begin{cases} 0 & , \text{ if n is odd} \\ (-1)^{n/2} \cdot {}^nC_{n/2}, & \text{if n is even} \end{cases} \quad [\because {}^nC_k = {}^nC_{n-k}]$$

$\therefore \quad P_n(0) = 0$ if n is odd

and $\quad P_n(0) = \frac{1}{2^n}(-1)^{n/2}\ {}^nC_{n/2}$, if n is even

$$= \frac{1}{2^n}(-1)^{n/2} \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}$$

$$= (-1)^{n/2} \frac{1.3.5.\ldots(n-1)}{2.4.6.\ldots n}$$

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You may now try the following exercises.

E1) Using Rodrigue's Formula, show that

a) $\int_{-1}^{1} f(x)P_n(x)dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} (x^2-1)^n f^{(n)}(x)dx.$

b) $\int_{-1}^{1} x^n P_n(x)dx = \frac{2^{n+1}(n!)^2}{(2n+1)!}.$

E2) Prove that all the roots of $P_n(x) = 0$ are real, distinct and lie between -1 and $+1$.

E3) Show that $\int_{-1}^{1} P_n(x)\,dx = 0$, except when $n = 0$, in which case, the value of the integral is 2.

Thus you have seen that Legendre polynomials are obtained either as polynomial solutions of Legendre's equation or by Rodrigue's formula. We shall now discuss the historical discovery of the Legendre polynomials through expansion of the **generating function** into a particular type of series. We shall show that these polynomials can also be obtained as coefficients of t^n in the expansion of $(1-2xt+t^2)^{-1/2}$.

3.2.2 Generating Function

The subject of potential theory is concerned with the forces of attraction due to the presence of a gravitational field. Central to the discussion of problems of gravitational attraction is **Newton's law of universal gravitation.**

Every particle of matter in the universe attracts every other particle with a force whose direction is that of the line joining the two, and whose magnitude is directly as the product of their masses and inversely as the square of their distance from each other.

The force field generated by a single particle is usually considered to be conservative. That is, there exists a potential function V such that the gravitational force **F** at a point of free space (i.e., free of point masses) is related to the potential function according to

$$\mathbf{F} = -\nabla V \qquad (20)$$

where the minus sign is conventional. If r denotes the distance between a point mass and a point of free space, the potential function can be shown to have the form

$$V(r) = \frac{k}{r} \tag{21}$$

where k is a constant whose numerical value does not concern us. Because of spherical symmetry of the gravitational field, the potential function V depends on only the radial distance r .

To obtain Legendre's results, let us suppose that a particle of mass m is located at point P , which is 'a' units from the origin of our coordinate system (see Fig.1).

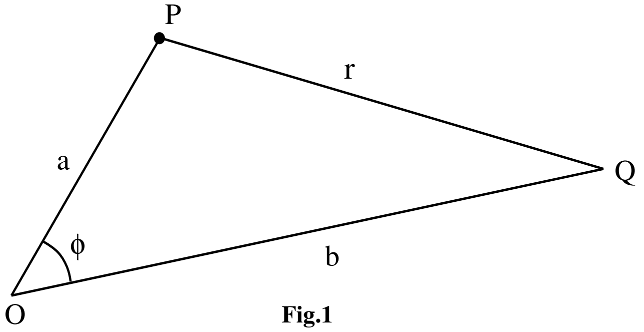

Fig.1

Let the point Q represent a point of free space r units from P and b units from the origin O . Let us assume that $b > a$, then, from the law of cosines, we find the relation

$$r^2 = a^2 + b^2 - 2ab\cos\phi \tag{22}$$

where ϕ is the central angle between segments OP and OQ . By rearranging the terms and factoring out b^2, it follows that

$$r^2 = b^2\left[1 - 2\frac{a}{b}\cos\phi + \left(\frac{a}{b}\right)^2\right], \quad a < b \tag{23}$$

We introduce the parameters

$$t = \frac{a}{b}, \quad x = \cos\phi \tag{24}$$

and taking the square root of Eqn.(23), we obtain

$$r = b\left(1 - 2xt + t^2\right)^{1/2} \tag{25}$$

Finally, substituting Eqn.(25) into Eqn.(21), we obtain the potential function

$$V = \frac{k}{b}\left(1 - 2xt + t^2\right)^{-1/2}, \quad 0 < t < 1 \tag{26}$$

We refer to the function $w(x,t) = (1 - 2xt + t^2)^{-1/2}$ as the **generating function** of the Legendre polynomials. We shall now develop $w(x,t)$ in a power series in the variable t and show that

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x), \qquad t \neq 1 \tag{27}$$

We write

$$(1-2xt+t^2)^{-1/2} = [1-(2x-t)t]^{-1/2} = (1-u)^{-1/2}$$

where, $u = (2x - t)t$

Expanding in a binomial series, we get

$$(1-u)^{-1/2} = 1 + \frac{1}{2}u + \frac{(1/2)(3/2)}{2!}u^2 + \frac{(1/2)(3/2)(5/2)}{3!}u^3 + \cdots$$

$$= \frac{1}{2} + \frac{2!}{(1!)2^2}u + \frac{4!}{(2!)^2 2^4}u^2 + \frac{6!}{(3!)^2 2^6}u^3 + \cdots + \frac{(2n)!}{(n!)^2 2^{2n}}u^n + \cdots$$

Substituting $u = (2x - t)t$, we get

$$(1-2xt+t^2)^{-1/2} = 1 + \frac{2!}{(1)^2 2^2}(2x-t)t + \frac{4!}{(2!)^2 2^4}(2x-t)^2 t^2 + \cdots$$
$$+ \frac{(2n-2r)!}{[(n-r)!]^2 2^{2n-2r}}(2x-t)^{n-r} t^{n-r} + \cdots + \frac{(2n)!}{(n!)^2 2^{2n}}(2x-t)^n t^n + \cdots \quad (28)$$

Since we are interested in the coefficients of t^n in the above series, we consider the $(n-r)^{th}$ term of the series and expand the product $(2x-t)^{n-r} t^{n-r}$ in powers of t.

We have

$$t^{n-r}[\,{}^{n-r}C_r (-t)^r (2x)^{n-2r}\,] = \frac{(n-r)!2^{n-2r}}{r!(n-2r)!}(-1)^r t^n x^{n-2r}$$

The $(n-r)^{th}$ term of series (28) then becomes

$$\frac{(2n-2r)!}{[(n-r)!]^2 2^{2n-2r}} \cdot \frac{(n-r)!2^{n-2r}}{r!(n-2r)!}(-1)^r t^n x^{n-2r} = \frac{(-1)^r (2n-2r)! t^n x^{n-2r}}{2^n (n-r)! r! (n-2r)!} \cdots \quad (29)$$

Summing the r.h.s of Eqn(29) for $r = 0, 1, \ldots, N$, where $N = \frac{n}{2}$ or $\frac{(n-1)}{2}$, whichever is an integer, we get on using Eqn.(19)

$$\sum_{r=0}^{N} \frac{(-1)^r (2n-2r)! x^{n-2r}}{2^n r!(n-r)!(n-2r)!} t^n = P_n(x) t^n \quad (30)$$

Therefore, $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n, t \neq 1$

Thus $P_n(x)$ can be obtained by the generating function. You may **observe** here that when n is an even number the polynomial $P_n(x)$ is an even function; and when n is odd, the polynomial is an odd function and the relation

$$P_n(-x) = (-1)^n P_n(x), \quad (31)$$

holds for $n = 0, 1, 2, \ldots$.

In Fig.2 below, we have given the graphs of $P_n(x)$, $n = 0, 1, 2, 3, 4$, over the interval $-1 \le x \le 1$.

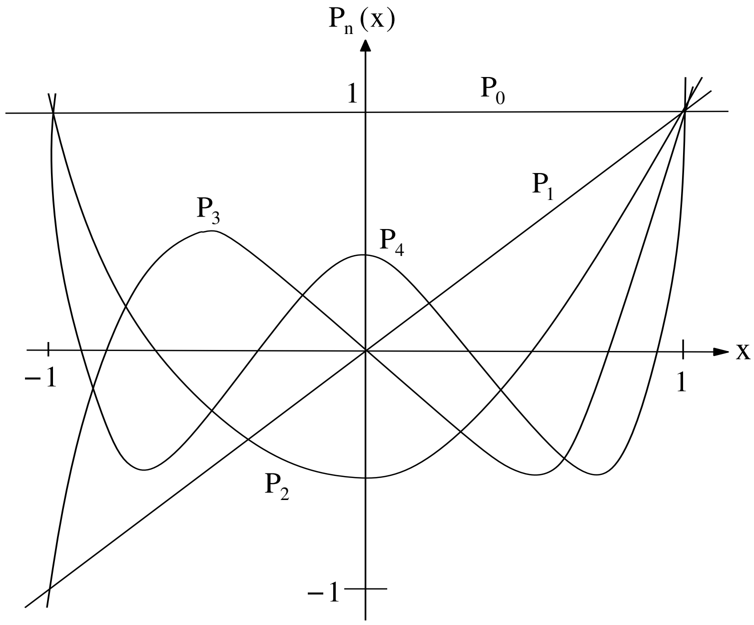

Fig.2: Graph of $P_n(x), n = 0, 1, 2, 3, 4$

Further, considering Eqn.(26) with $x = \cos\phi$ and $t = a/b$, we find that the potential function has the series expansion

$$V = \frac{k}{b}\sum_{n=0}^{\infty} P_n(\cos\phi)\left(\frac{a}{b}\right)^n, \quad a < b \qquad (32)$$

In terms of the argument $\cos\phi$, the Legendre polynomials can be expressed as trigonometric polynomials of the form shown below

$$P_0(\cos\phi) = 1$$

$$P_1(\cos\phi) = \cos\phi$$

$$P_2(\cos\phi) = \frac{1}{2}(3\cos^2\phi - 1) = \frac{1}{4}(3\cos 2\phi + 1)$$

$$P_3(\cos\phi) = \frac{1}{2}(5\cos^3\phi - 3\cos\phi) = \frac{1}{8}(5\cos 3\phi + 3\cos\phi)$$

Let us now take up the following examples.

Example 2: Show that

(a) $P_n(-x) = (-1)^n P_n(x)$

(b) $P_n(1) = 1$

(c) $P_n(-1) = (-1)^n$

Solution: From Generating Function for Legendre Polynomials, we get

(a) $$\sum_{n=0}^{\infty} P_n(-x)h^n = (1 + 2xh + h^2)^{\frac{-1}{2}}$$

$$= \left[1 - 2x(-h) + (-h)^2\right]^{\frac{-1}{2}}$$

$$= \sum_{n=0}^{\infty} P_n(x)(-h)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n P_n(x)h^n$$

$$\therefore P_n(-x) = (-1)^n P_n(x)$$

(b) $$\sum_{n=0}^{\infty} P_n(1)h^n = (1 - 2h + h^2)^{-\frac{1}{2}}$$

$$= (1-h)^{-1}$$

$$= 1 + h + h^2 + h^3 + \cdots + h^n + \cdots$$

$$= \sum_{n=0}^{\infty} h^n$$

$$\therefore \; P_n(1) = 1.$$

(c) $$\sum_{n=0}^{\infty} P_n(-1)h^n = (1 + 2h + h^2)^{-\frac{1}{2}}$$

$$= (1+h)^{-1}$$

$$= \sum_{n=0}^{\infty} (-1)^n h^n$$

$$\therefore \; P_n(-1) = (-1)^n$$

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Example 3: Using the generating function for the Legendre polynomial show that

$$P_0(x) = 1,\; P_1(x) = x,\; P_2(x) = \frac{3x^2 - 1}{2},\; P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

Solution: From generating function for Legendre polynomials we know that

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1-2hx+h^2)^{-\frac{1}{2}} = [1-h(2x-h)]^{-1/2}$$

$$= 1 + \frac{1}{2}h(2x-h) + \frac{1.3}{2.4}h^2(2x-h)^2 + \frac{1.3.5}{2.4.6}h^3(2x-h)^3 + \frac{1.3.5.7}{2.4.6.8}h^4(2x-h)^4 + \cdots$$

$$\Rightarrow P_0(x) + hP_1(x) + h^2P_2(x) + h^3P_3(x) + h^4P_4(x) + \cdots$$

$$= 1 + x.h + \frac{1}{2}(3x^2-1)h^2 + \frac{1}{2}(5x^3-3x)h^3 + \frac{1}{8}(35x^4-30x+3)h^4 + \cdots$$

Equating the coefficients of like powers of h, we get

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2-1), P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3).$$

***

Example 4: Express $f(x) = x^4+3x^3+4x^2-x+2$ in terms of Legendre polynomials.

Solution: We have shown in Example 3, that

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{3x^2-1}{3},\ P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3).$$

The above results can also be obtained by substituting $n = 0, 1, 2, 3,$ and 4 in relation (19).

From these relations we can write

$$x^2 = \frac{P_0}{3} + \frac{2}{3}.P_2(x),\ \ x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) \text{ and}$$

$$x^4 = \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}$$

$$= \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0$$

Thus, $x^4+3x^3+4x^2-x+2$

$$= \left(\frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{P_0}{5}\right) + 3\left(\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right) + 4\left(\frac{2}{3}P_2(x) + \frac{P_0}{3}\right) - P_1(x) + 2P_0(x)$$

$$= \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) + \frac{68}{21}P_2(x) + \frac{4}{5}P_1(x) + \frac{53}{15}P_0(x).$$

***

You may now attempt the following exercises.

E4) Prove that

$$P_n\left(-\frac{1}{2}\right) = P_0\left(-\frac{1}{2}\right)P_{2n}\left(\frac{1}{2}\right) + P_1\left(-\frac{1}{2}\right)P_{2n-1}\left(\frac{1}{2}\right) + \cdots + P_{2n}\left(-\frac{1}{2}\right)P_0\left(\frac{1}{2}\right).$$

E5) Show that

a) $P_n'(1) = \frac{1}{2}n(n+1)$

b) $P_n'(-1) = (-1)^{n+1}\frac{1}{2}n(n+1)$

E6) Show that

a) $P_{2n+1}(0) = 0$, $n = 0, 1, 2, \ldots$

b) $P_{2n}(0) = \frac{(-1)^n 2n!}{2^{2n}(n!)^2}, \; n = 0, 1, 2, \ldots$

E7) Verify that

a) $P_0(\cos\phi) = 1$

b) $P_1(\cos\phi) = \cos\phi$

c) $P_2(\cos\phi) = \frac{1}{4}(3\cos 2\phi + 1)$

d) $P_3(\cos\phi) = \frac{1}{8}(5\cos 3\phi + 3\cos\phi)$

We now give some relations between Legendre polynomials of different order.

3.2.3 Recurrence Relations

We shall now establish a few useful relations involving Legendre polynomials of different order or between Legendre polynomials and their derivatives, which are called recurrence relations.

From Rodrigue's formula, we have

$$P_{n+1}(x) = \frac{1}{2^{n+1}(n+1)!} D^{(n+1)}[(x^2-1)^{n+1}], \text{ where } D = \frac{d}{dx} \quad (33)$$

Differentiating Eqn.(33) w.r. to x, we get

$$P'_{n+1}(x) = \frac{1}{2^{n+1}(n+1)!} D^{n+1}[(n+1)(x^2-1)^n . 2x] \quad (34)$$

$$= \frac{1}{2^n (n)!} D^n [(x^2-1)^n + 2nx^2 (x^2-1)^{n-1}]$$

$$= \frac{1}{2^n (n)!} D^n [(x^2-1)^n + 2n(x^2-1+1)(x^2-1)^{n-1}]$$

$$= \frac{1}{2^n} \frac{1}{n!} D^n [(2n+1)(x^2-1)^n + 2n(x^2-1)^{n-1}]$$

$$= \frac{2n+1}{2^n n!} D^n (x^2-1)^n + \frac{1}{2^{n-1}(n-1)!} D\{D^{n-1}(x^2-1)^{n-1}\}$$

$$= (2n+1) P_n(x) + P'_{n-1}(x)$$

Thus $\mathbf{P'_{n+1}(x) = (2n+1) P_n(x) + P'_{n-1}(x)}$ (35)

Now we use Leibnitz rule on the right hand side of Eqn.(34) to obtain

$$P'_{n+1}(x) = \frac{1}{2^n (n)!} [\{D^{n+1}(x^2-1)^n\} x + (n+1)\{D^n (x^2-1)^n\}.1]$$

$$= x.D\left[\frac{1}{2^n (n)!} D^n \{(x^2-1)^n\}\right] + (n+1).\frac{1}{2^n (n)!}\{D^n (x^2-1)^n\}$$

$$= x P'_n(x) + (n+1) P_n(x)$$

$$\Rightarrow \quad \mathbf{P'_{n+1}(x) = x P'_n(x) + (n+1) P_n(x)} \quad (36)$$

Subtracting Eqn.(36) from Eqn.(35), we get

$$\mathbf{n P_n(x) = x P'_n(x) - P'_{n-1}(x)} \quad (37)$$

Changing $(n+1)$ to n in relation (35) and using it in relation (37), we get

$$n P_n(x) = x[(2n-1) P_{n-1}(x) + P'_{n-2}(x)] - P'_{n-1}(x)$$

$$= (2n-1) x P_{n-1}(x) + [x P'_{n-2}(x) - P'_{n-1}(x)]$$

$$= (2n-1) x P_{n-1}(x) - (n-1) P_{n-2}(x) \quad (\text{using relation (37)})$$

We can thus write

$$\mathbf{(n+1)\,P_{n+1}(x) = (2n+1)x\,P_n(x) - n\;P_{n-1}(x)} \qquad (38)$$

which is a three-term recurrence relation.

From Eqn.(36) and (37) following relations can also be obtained

$$\mathbf{(1-x^2)\,P_n'(x) = n\{\,P_{n-1}(x) - x\;P_n(x)\,\}} \qquad (39)$$

and $$\mathbf{(1-x^2)\;P_n(x) = (n+1)\{x\;P_n(x) - P_{n+1}(x)\,\}} \qquad (40)$$

We are leaving them here for you to do it yourself.

E8) Establish relations (39) and (40).

E9) Derive the identity
$(1-x^2)\,P_n'(x) = (n+1)[x\,P_n(x) - P_{n+1}(x)]\,,\ n = 0, 1, 2, \ldots.$

Legendre polynomials belong to an important class of orthogonal polynomials which we shall discuss now.

3.2.4 Orthogonality Property

We shall now show that the Legendre polynomials $P_n(x)$ satisfy the following orthogonal property

$$\int_{-1}^{+1} P_m(x)\,P_n(x)\,dx = \begin{cases} 0 & ,\text{if } \ m \neq n \\ \dfrac{2}{2n+1} & ,\text{if } \ m = n \end{cases}, \qquad (41)$$

Consider the Legendre equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0.$$

This can be written as

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0 \qquad (42)$$

Legendre Polynomials $P_m(x)$ and $P_n(x)$ satisfy Eqn.(42), thus

$$\frac{d}{dx}\left[(1-x^2)\;P_n'\right] = -n(n+1)\,P_n \qquad (43)$$

and $$\frac{d}{dx}\left[(1-x^2)\;P_m'\right] = -m(m+1)\,P_m \qquad (44)$$

Multiplying Eqn.(43) by P_m and Eqn.(44) by P_n and substracting the resulting expressions, we get

$$\frac{d}{dx}\left[(1-x^2)(P_m\,P_n' - P_n P_m')\right] = \left[m(m+1) - n(n+1)\right]P_n P_m$$

Integrating both sides w.r. to x from $x = -1$ to $x = 1$, we get

$$\left|(1-x^2)(P_m P_n' - P_n P_m')\right|_{-1}^{+1} = [m(m+1) - n(n+1)]\int_{-1}^{1} P_n(x)\,P_m(x)\,dx$$

The left hand term vanishes at $x = \pm 1$

$$\therefore \qquad (n+m+1)(m-n)\int_{-1}^{+1} P_m(x)\,P_n(x)dx = 0$$

Since m and n are non-negative integers, $\therefore n+m+1 \neq 0$. Thus if $m \neq n$, i.e., $m - n \neq 0$, then from above

$$\int_{-1}^{+1} P_m(x)P_n(x)dx = 0, \quad \text{for } m \neq n.$$

To prove the property for $m = n$, we use Rodrigue's formula

$$P_n(x) = \frac{1}{(n)!2^n} \frac{d^n}{dx^n} \{(x^2-1)^n\}$$

Let $f(x)$ be any function with at least n continuous derivatives on the interval $-1 \le x \le 1$ and consider the integral

$$I = \int_{-1}^{1} f(x) P_n(x) dx.$$

Using Rodrigue's formula

$$I = \frac{1}{2^n n!} \int_{-1}^{1} f(x) \frac{d^n}{dx^n} (x^2-1)^n dx$$

Integrating by parts, we get

$$I = \frac{1}{2^n n!} \left[f(x) \frac{d^{n-1}(x^2-1)^n}{dx^{n-1}} \right]_{-1}^{1} - \frac{1}{2^n n!} \int_{-1}^{1} f'(x) \frac{d^{n-1}(x^2-1)^n}{dx^{n-1}} dx$$

The first expression in the above equation on the right hand side, vanishes at both limits, so

$$I = -\frac{1}{2^n n!} \int_{-1}^{1} f'(x) \frac{d^{n-1}(x^2-1)^n}{dx^{n-1}} dx$$

and continuing to integrate by parts, we obtain

$$I = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} f^{(n)}(x) (x^2-1)^n dx.$$

Now if we put $f(x) = P_n(x)$ in the above integral then since $P_n^{(n)}(x) = \frac{2n!}{2^n n!}$, it follows that

$$I = \int_{-1}^{1} P_n^2(x) dx = \frac{2n!}{2^{2n}(n!)^2} \int_{-1}^{1} (1-x^2)^n dx$$

$$= \frac{2(2n)!}{2^{2n}(n!)^2} \int_{0}^{1} (1-x^2)^n dx$$

Using the substitution $x = \sin\theta$, and recalling the reduction formula,

$$\int \cos^{2n+1}\theta\, d\theta = \frac{1}{2n+1} \cos^{2n}\theta \sin\theta + \frac{2n}{2n+1} \int \cos^{2n-1}\theta\, d\theta .$$

Then the definite integral in I becomes

$$\int_0^1 (1-x^2)^n dx = \int_0^{\pi/2} \cos^{2n+1}\theta\, d\theta = \frac{2n}{2n+1} \int_0^{\pi/2} \cos^{2n-1}\theta d\theta$$

$$= \frac{2n}{2n+1} \frac{2n-2}{2n-1} \cdots \frac{2}{3} \int_0^{\pi/2} \cos\theta\, d\theta$$

$$= \frac{2^n n!}{1.3\ldots(2n-1)(2n+1)} = \frac{2^{2n}(n!)^2}{(2n)!(2n+1)}$$

Thus, we conclude that

$$I = \frac{2(2n)!}{2^{2n}(n!)^2} \int_0^1 (1-x^2)^n dx$$

$$= \frac{2(2n)!}{2^{2n}(n!)^2} \frac{2^{2n}(n!)^2}{(2n)!(2n+1)} = \frac{2}{2n+1}$$

or, $$\int_{-1}^{1} P_n^2(x) dx = \frac{2}{2n+1} \qquad (45)$$

Example 5: Assuming the orthogonality property of Legendre polynomials, show that

$$\int_{-1}^{1} (1-x^2) P_n^{(1)}(x) P_m^{(1)}(x)\,dx = \begin{cases} 0 & , \text{ if } m \neq n \\ \dfrac{2n(n+1)}{2n+1}, & \text{ if } m = n \end{cases}$$

where m and n are integers.

Solution: $\text{L.H.S.} = \int_{-1}^{1} (1-x^2) P_n^{(1)}(x)\, P_m^{(1)}(x)\,dx$

$$= \left| (1-x^2) P_m^{(1)}(x).P_n(x) \right|_{-1}^{1} - \int_{-1}^{1} P_n(x) \left[\frac{d}{dx} \left\{ (1-x^2) P_m^{(1)}(x) \right\} \right] dx$$

Since $P_m(x)$ is a solution of Legendre's equation, thus

$$\frac{d}{dx}\left[(1-x^2) P_m^{(1)}(x) \right] = -m(m+1) P_m(x)$$

$$\therefore \ \text{L.H.S.} = m(m+1) \int_{-1}^{1} P_n(x) P_m(x)\,dx = \begin{cases} 0 & , \text{ if } n \neq m \\ n(n+1) \int_{-1}^{1} P_n^2(x)\,dx, & \text{ if } n = m \end{cases}$$

$$= \begin{cases} 0 & , \text{ if } n \neq m \\ \dfrac{2n(n+1)}{2n+1}, & \text{ if } n = m \end{cases} \quad \text{[using result (41)]}$$

***

Many problems of potential theory depend on the possibility of expanding a given function in a series of Legendre polynomials. It is easy to see that this can always be done when the given function is itself a polynomial. For example, any third-degree polynomials $p(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$ can be written as

$$p(x) = b_0 P_0(x) + b_1 P_1(x) + b_2 \left[\frac{1}{3} P_0(x) + \frac{2}{3} P_2(x) \right] + b_3 \left[\frac{3}{5} P_1(x) + \frac{2}{3} P_3(x) \right] \text{(see Example 4)}$$

$$= \left(b_0 + \frac{b_2}{3} \right) P_0(x) + \left(b_1 + \frac{3b_3}{5} \right) P_1(x) + \frac{2b_2}{3} P_2(x) + \frac{2b_3}{5} P_3(x)$$

$$= \sum_{n=0}^{3} a_n P_n(x)$$

More generally, since $P_n(x)$ is a polynomial of degree n for every positive integer n, it can easily be seen that x^n can always be expressed as a linear combination of $P_0(x), P_1(x), \ldots, P_n(x)$, so any polynomial $p(x)$ of degree k has an expansion of the form

$$p(x) = \sum_{n=0}^{k} a_n P_n(x) \tag{46}$$

But when the given function is not a polynomial then it may not be easy to expand an 'arbitrary' function $f(x)$ in series of the form:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \tag{47}$$

We then need a new procedure for calculating the coefficients a_n in Eqn.(47), and the key lies in formula (41). We multiply Eqn.(47) by $P_m(x)$ and integrate term by term from -1 to 1 and obtain

$$\int_{-1}^{1} f(x) P_m(x)\,dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_m(x) P_n(x)\,dx$$

Using formula (41), it reduces to

$$\int_{-1}^{1} f(x)P_m(x)\,dx = \frac{2a_m}{2m+1}$$

Coefficients a_n in Eqn.(47) are then obtained as

$$a_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_n(x)\,dx \qquad (48)$$

The above manipulations are easy to justify if $f(x)$ is known in advance to have a series expansion of the form (47) and this series is integrable term by term on the interval $-1 \le x \le 1$. Both the conditions are obviously satisfied when $f(x)$ is a polynomial. However, we shall not discuss here the conditions under which the representation of form (47) and (48) is valid for any other type of functions. For now it suffices to say that for certain functions the series (47) will converge throughout the interval $-1 \le x \le 1$, even at points of finite discontinuities of the given function. Series of this type are called **Legendre series.**

In practice, the evaluation of integrals like (48) is performed numerically. However, if the function f is not too complicated, we can sometimes use various properties of Legendre polynomials to evaluate such integrals in closed form. We illustrate this through the following example.

Example 6: Find the Legendre series for

$$f(x) = \begin{cases} -1, & -1 \le x < 0 \\ 1\ , & 0 < x \le 1 \end{cases}.$$

Solution: The given function f is an odd function. Hence $f(x)P_n(x)$ is an odd function when n is even, and in this case

$$a_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_n(x)\,dx = 0, \quad n = 0, 2, 4, \ldots$$

For n odd, the product $f(x)P_n(x)$ is an even function, and therefore

$$a_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_n(x)\,dx$$

$$= (2n+1)\int_{0}^{1} f(x)P_n(x)\,dx = (2n+1)\int_{0}^{1} P_n(x)\,dx, \quad n = 1, 3, 5, \ldots$$

Now let us use the identity (35)

$$P_n(x) = \frac{1}{2n+1}\left[P'_{n+1}(x) - P'_{n-1}(x)\right]$$

and set $n = 2k+1$ for $k = 0, 1, 2, \ldots$

$$a_{2k+1} = (4k+3)\int_{0}^{1} P_{2k+1}(x)\,dx$$

$$= \int_{0}^{1} [P'_{2k+2}(x) - P'_{2k}(x)]\,dx$$

$$= \left[P_{2k+2}(x) - P_{2k}(x)\right]_0^1$$

$$= P_{2k}(0) - P_{2k+2}(0) \qquad [\because P_n(1) = 1 \text{ for all } n]$$

Using the result of E6), we have

$$a_{2k+1} = \frac{(-1)^k\,(2k)!}{2^{2k}(k!)^2} - \frac{(-1)^{k+1}(2k+2)!}{2^{2k+2}\left[(k+1)!\right]^2}$$

$$= \frac{(-1)^k (2k)!}{2^{2k}(k!)^2}\left[1 + \frac{(2k+2)(2k+1)}{2^2 (k+1)^2}\right]$$

$$= \frac{(-1)^k (2k)!}{2^{2k}(k!)^2}\left(1 + \frac{2k+1}{2k+2}\right) = \frac{(-1)^k (2k)!(4k+3)}{2^{2k+1} k!(k+1)!}$$

and thus

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!(4k+3)}{2^{2k+1} k!(k+1)!} P_{2k+1}(x) \,, \quad -1 \le x \le 1 \,.$$

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You may try a few exercises now.

E10) If $f(x)$ is a polynomial of degree less then n, prove that

$$\int_{-1}^{1} f(x) P_n(x)\, dx = 0 .$$

E11) Express $f(x) = x^3 + a x^2 + b x + c$ as a linear combination of Legendre polynomials.

E12) Show that $(2n+1)(x^2 - 1)P_n' = n(n+1)\,(P_{n+1} - P_{n-1})$.

E13) Prove that $\displaystyle\int_{-1}^{1} \frac{P_n(x)}{\sqrt{1 - 2xt + t^2}}\, dx = \frac{2t^n}{2n+1}$.

E14) Prove that $1 + \frac{1}{2} P_1(\cos\theta) + \frac{1}{3} P_2(\cos\theta) + \cdots = \ln \left\{\left(1 + \sin\frac{\theta}{2}\right) \Big/ \sin\frac{\theta}{2}\right\}$.

E15) Make the change of variable $x = \cos\phi$ in the DE

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi \frac{dy}{d\phi}\right) + n(n+1)\,y = 0$$

and show that it reduces to Legendre's Eqn.(1).

E16) Determine the values of n for which $y = P_n(x)$ is a solution of

a) $(1 - x^2)y'' - 2xy' + n(n+1)\,y = 0, \ y(0) = 0, \ y(1) = 1$

b) $(1 - x^2)y'' - 2xy' + n(n+1)\,y = 0, \ y'(0) = 0, \ y(1) = 1$.

3.2.5 Legendre Function of the Second Kind

We have shown that for positive integral values of n the Legendre polynomials $P_n(x)$ represents one of the linearly independent solutions of Legendre's Eqn.(1), viz.,

$$(1 - x^2)\, y'' - 2xy' + n(n+1)y = 0 \quad .$$

Because the equation is of second order, we know from the theory of differential equations that there exists a second linearly independent solution $Q_n(x)$ (say), such that

$$y = c_1 P_n(x) + c_2 Q_n(x) \qquad (49)$$

where c_1 and c_2 are arbitrary constants, is a general solution of Eqn.(1).

Also from the theory of second-order linear differential equations it is well known that if $y_1(x)$ is a non-trivial solution of

$$y'' + a(x)\, y' + b(x)\, y = 0 \qquad (50)$$

then a second linearly independent solution can be defined by

$$y_2(x) = y_1(x) \int \frac{\exp\left[-\int a(x)\, dx\right]}{y_1^2(x)}\, dx \quad \text{[ref. Unit 7, MTE-08].} \qquad (51)$$

Hence, if we express Eqn.(1) in the form

$$y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{(1-x^2)}y = 0$$

and let $y_1(x) = P_n(x)$, it follows that

$$y_2(x) = P_n(x)\int \frac{dx}{(1-x^2)\,[P_n(x)]^2} \tag{52}$$

is a second solution, linearly independent of $P_n(x)$. Because any linear combination of solutions is also a solution of a homogeneous differential equation, we defined the second solution of Eqn.(1) as follows:

$$Q_n(x) = P_n(x)\left\{A_n + B_n \int \frac{dx}{(1-x^2)\,[P_n(x)]^2}\right\}, \tag{53}$$

where A_n and B_n are constants to be chosen for each n. We refer to $Q_n(x)$ as the **Legendre function of the second kind** of integral order.

Accordingly, when $n = 0$, we choose $A_0 = 0$ and $B_0 = 1$, and hence

$$Q_0(x) = \int \frac{dx}{1-x^2}$$

which leads to

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad |x| < 1 \tag{54}$$

For $n = 1$, we set $A_1 = 0$ and $B_1 = 1$, from which we obtain

$$Q_1(x) = x\int \frac{dx}{(1-x^2)x^2}$$

$$= x\int \left(\frac{1}{1-x^2} + \frac{1}{x^2}\right) dx$$

$$= \frac{1}{2}x\ln\frac{1+x}{1-x} - 1 \tag{55}$$

or, $\quad Q_1(x) = x\,Q_0(x) - 1, \; |x| < 1 \quad (56)$

If we continue in this fashion then we will have to evaluate more difficult integrals. Since it is assumed that all solutions of Legendre's equation satisfy the recurrence formulas for $P_n(x)$, hence we select the Legendre functions $Q_n(x)$ so that,

$$Q_{n+1}(x) = \frac{2n+1}{n+1}x\,Q_n(x) - \frac{n}{n+1}Q_{n-1}(x) \tag{57}$$

for $n = 1, 2, 3, \ldots$ with $Q_0(x)$ and $Q_1(x)$ already defined. The substitution $n = 1$ into Eqn.(57) yields

$$Q_2(x) = \frac{3}{2}x\,Q_1(x) - \frac{1}{2}Q_0(x)$$

$$= \frac{1}{2}(3x^2 - 1)\,Q_0(x) - \frac{3}{2}x$$

or, $\quad Q_2(x) = P_2(x)\,Q_0(x) - \frac{3}{2}x, \; |x| < 1 \quad \left[\because P_2(x) = \frac{1}{2}(3x^2 - 1)\right] \quad (58)$

For $n = 2$, we find

$$Q_3(x) = P_3(x)\,Q_0(x) - \frac{5}{2}x^2 + \frac{2}{3}, \quad |x| < 1 \tag{59}$$

whereas in general, we state the result

$$Q_n(x) = P_n(x)\,Q_0(x) - \sum_{k=0}^{[(n-1)/2]} \frac{(2n-4k-1)}{(2k+1)(n-k)}P_{n-2k-1}(x), \quad |x| < 1 \tag{60}$$

for $n = 1, 2, 3, \ldots$.

Because of the logarithmic term in $Q_0(x)$, $Q_n(x)$ has infinite discontinuities at $x = \pm 1$. However, within the interval $-1 < x < 1$, these functions are well defined.

Without giving the details, we state that the Legendre functions $Q_n(x)$ satisfy all recurrence relations given in Sec.3.2.3 for $P_n(x)$. In addition, there are several relations that directly involve both $P_n(x)$ and $Q_n(x)$. For example, if $|t| < |x|$, then

$$\frac{1}{x-t} = \sum_{n=0}^{\infty} (2n+1)\, P_n(t)\, Q_n(x) \tag{61}$$

From this result, it can be easily shown that

$$Q_n(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)}{x-t}\, dt,\ n = 0, 1, 2, \ldots \tag{62}$$

which is called the **Neumann formula.**

You may now try the following exercises.

E17) Verify result (62).

E18) Find the general solution of the following differential equations in terms of $P_n(x)$ and $Q_n(x)$

a) $(1-x^2)\, y'' - 2xy' = 0$

b) $(1-x^2)\, y'' - 2xy' + 12y = 0$

c) $(1-x^2)\, y'' - 2xy' + 2y = 0$

d) $(1-x^2)\, y'' - 2xy' + 30y = 0$.

E19) Given $P_0(x) = 1$ and $Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x^2}$, find $W[P_0(x), Q_0(x)]$ where , W is the wronskian.

E20) Using Eqn.(53), show that the wronskian of $P_n(x)$ and $Q_n(x)$ is given by

$$W\left(P_n(x), Q_n(x)\right) = \frac{B_n}{1-x^2}\ ,\ n = 0, 1, 2, \ldots.$$

Other polynomials which commonly occur in applications are the Hermite and Laguerre polynomials which we shall be discussing now.

3.3 HERMITE AND LAGUERRE POLYNOMIALS

Let us first consider the Hermite polynomials.

Hermite Polynomials

The Hermite polynomials play an important role in problems involving Laplace's equations in cylinderical coordinates, in various problems in quantum mechanics and in probability theory.

The Hermite polynomials are polynomial solutions to **Hermite's equation**

$$y'' - 2xy' + 2n\, y = 0 \tag{63}$$

where n is a constant.

Using the power series method it can be easily shown that $y(x) = a_0 y_1(x) + a_1 y_2(x)$ is the general solution of Eqn.(63), where

$$y_1(x) = 1 - \frac{2n}{2!}x^2 + 2^2 n\frac{(n-2)}{4!}x^4 - \frac{2^3 n(n-2)\,(n-4)}{6!}x^6 + \cdots \tag{64}$$

and

$$y_2(x) = x - \frac{2(n-1)}{3!}x^3 + \frac{2^2(n-1)\,(n-3)}{5!}x^5 - \frac{2^3(n-1)(n-3)(n-5)}{7!}x^7 + \cdots \tag{65}$$

and both the series converge for all x. We shall not be going into the details of the series solution here and leave it for you to verify it yourself.

If n is a non-negative integer, then one of these series terminates and is thus a polynomial- $y_1(x)$ if n is even, and $y_2(x)$ if n is odd, while the other remains an infinite series. It can be easily verified that for $n = 0, 1, 2, 3, 4, 5,$ these polynomials are

$$1, x, 1-2x^2, x-\frac{2}{3}x^3, 1-4x^2+\frac{4}{3}x^4, x-\frac{4}{3}x^3+\frac{4}{15}x^5, \text{ respectively.}$$

The polynomial solutions of Hermite's Eqn.(63) are constant multiples of these polynomials. The constant multiples with the property that the terms containing the highest powers of x are of the form $2^n x^n$ are denoted by $H_n(x)$ and called the **Hermite polynomials.**

Let us now define the Hermite polynomials

Generating Function

We define the Hermite polynomials $H_n(x)$ by means of the relation

$$\mathbf{e^{(2xt-t^2)} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}}, \quad \text{for all finite } x \text{ and } t. \tag{66}$$

The function on the left hand side is called the generating function of the Hermite polynomials. This definition of Hermite polynomials is used often in statistical applications.

We have

$$e^{(2xt-t^2)} = e^{2xt}.e^{-t^2} = \left[\sum_{m=0}^{\infty} \frac{(2xt)^m}{m!}\right]\left[\sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!}\right]$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2x)^{n-2k}}{k!\,(n-2k)!} t^n \tag{67}$$

where the last step follows from the index change $m = n-2k$ and $[n/2]$ is the standard notation to the greatest integer $\le n/2$. From Eqns.(66) and (67) it follows that

$$\mathbf{H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k\, n!}{k!(n-2k)!} (2x)^{n-2k}} \tag{68}$$

where $\left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$

Eqn.(68) shows that $H_n(x)$ is a polynomials of degree n in x and that

$$H_n(x) = 2^n x^n + \pi_{n-2}(x),$$

where $\pi_{n-2}(x)$ is a polynomial of degree $(n-2)$ in x. Further, it follows that it is an even function of x for even n and an odd function of x for odd n. Thus, it follows that

$$\mathbf{H_n(-x) = (-1)^n H_n(x)} \tag{69}$$

The first few Hermite polynomials are listed in Table-1.

Table 1

$H_0(x) = 1$
$H_1(x) = 2x$
$H_2(x) = 4x^2 - 2$
$H_3(x) = 8x^3 - 12$
$H_4(x) = 16x^4 - 48x^2 + 12$
$H_5(x) = 32x^5 - 160x^3 + 120x$

In addition to series (68), the Hermite polynomials can be defined by the Rodrigues formula also.

The Rodrigue's Formula

We prove the formula $H_n(x) = (-1)^n e^{x^2} \dfrac{d^n}{dx^n}(e^{-x^2})$ (70)

We have the relation

$$e^{2tx-t^2} = e^{x^2-(t-x)^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

If we examine this relation in the light of Maclaurin's theorem, we see that

$$\frac{\partial^n}{\partial t^n}(e^{2tx-t^2})\Big|_{t=0} = e^{x^2} \frac{\partial^n}{\partial t^n}(e^{-(t-x)^2})\Big|_{t=0} = H_n(x)$$

or, $$\frac{\partial^n}{\partial t^n}[e^{-(t-x)^2}]\Big|_{t=0} = e^{-x^2} H_n(x)$$

If we introduce a new variable $z = x - t$ and use the fact that $\dfrac{\partial}{\partial t} = -\dfrac{\partial}{\partial z}$, then since $t = 0$ corresponds to $z = x$, the above expression reduces to

$$(-1)^n \frac{d^n}{dz^n}(e^{-z^2})\Big|_{z=x} = (-1)^n \frac{d^n(e^{-x^2})}{dx^n} = e^{-x^2} H_n(x)$$

$$\therefore \mathbf{H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})}$$

which proves the Rodrigue's formula (70).

Hermite polynomials satisfy a number of properties, we list some of these here without proving them. You can verify them yourself.

Recurrence Relations

Using the generating function for Hermite polynomials it can be shown that

$$\mathbf{x H'_n(x) = n\ H'_{n-1}(x) + n\ H_n(x)} \qquad (71)$$

The relation

$$e^{(2xt-t^2)} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} \qquad (72)$$

on differentiating with respect to x, yields

$$2t\, e^{(2xt-t^2)} = \sum_{n=0}^{\infty} \frac{H'_n(x)\,t^n}{n!} \qquad (73)$$

From Eqns.(72) and (73), we get

$$\sum_{n=0}^{\infty} \frac{2H_n(x)\,t^{n+1}}{n!} = \sum_{n=0}^{\infty} \frac{H'_n(x)\ t^n}{n!}$$

which with a shift of index on the left, yields $\mathbf{H'_0(x) = 0}$, and for $n \ge 1$,

$$\mathbf{H'_n(x) = 2n\, H_{n-1}(x)} \qquad (74)$$

Combining relations (71) and (74), we obtain

$$\mathbf{H_n(x) = 2x\ H_{n-1}(x) - H'_{n-1}(x)} \qquad (75)$$

Further, using relations (74) and (75), we obtain the relations

$$\mathbf{H_n(x) = 2x\ H_{n-1}(x) - 2(n-1)\ H_{n-2}(x)} \qquad (76)$$

and Hermite's differential equation

$$\mathbf{H''_n(x) - 2x\ H'_n(x) + 2n\ H_n(x) = 0} \qquad (77)$$

Relation (77) can be used to prove the orthogonality property of Hermite polynomials.

Orthogonality

The Hermite polynomials $H_n(x)$ satisfy the following orthogonality property

$$\int_{-\infty}^{\infty} \mathbf{e^{-x^2}\ H_n(x)\ H_m(x)dx} = \begin{cases} \mathbf{0\ ,} & \mathbf{if\ \ m \neq n} \\ \mathbf{2^n n!\sqrt{\pi}\ ,} & \mathbf{if\ \ m = n} \end{cases} \tag{78}$$

Eqn.(77) can be written in the form

$$[e^{-x^2}\ H'_n(x)]' + 2n\,e^{-x^2}\ H_n(x) = 0 \tag{79}$$

Similarly, for a polynomial $H_m(x)$, we can write

$$[e^{-x^2} H'_m(x)]' + 2m\,e^{-x^2} H_m(x) = 0 \tag{80}$$

Multiplying Eqn.(79) by $H_m(x)$ and Eqn.(80) by $H_n(x)$ and adding the two, we get

$$\begin{aligned} 2(n-m)\,e^{-x^2} H_n(x) H_m(x) &= H_n(x)[e^{-x^2} H'_m(x)]' - H_m(x)[e^{-x^2} H'_n(x)]' \\ &= [e^{-x^2}\{H_n(x)\ H'_m(x) - H'_n(x)\ H_m(x)\}]' \end{aligned}$$

Therefore, it follows that

$$2(n-m)\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\ H_m(x)\,dx = [e^{-x^2}\{H_n(x) H'_m(x) - H'_n(x) H_m(x)\}]_{-\infty}^{\infty} \tag{81}$$

Since the product of any polynomial in x by $e^{-x^2} \to 0$ as $x \to \infty$ or as $x \to -\infty$, we conclude that

$$\int_{-\infty}^{\infty} e^{-x^2}\ H_n(x)\ H_m(x)\,dx = 0,\ \ m \neq n \tag{82}$$

This proves the first part of formula, that is, the Hermite polynomials form an orthogonal set over the interval $]-\infty,\ \infty[$ with weight function e^{-x^2}.

To establish the second part of (78), we use Rodrigue's formula (70) and integrate

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\ H_n(x)\ dx = (-1)^n \int_{-\infty}^{\infty} H_n(x) \frac{d^n e^{-x^2}}{dx^n}\ dx$$

by parts with $u = H_n(x),\ \ du = H'_n(x)\,dx$

$$dv = \frac{d^n}{dx^n}(e^{-x^2})\,dx,\quad v = \frac{d^{n-1}}{dx^{n-1}}(e^{-x^2})$$

Since uv is the product of e^{-x^2} and a polynomial, it vanishes at both limits and

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-x^2}[H_n(x)]^2 dx &= (-1)^{n+1}\int_{-\infty}^{\infty} H'_n(x)\frac{d^{n-1}}{dx^{n-1}}(e^{-x^2})\ dx \\ &= (-1)^{n+2}\int_{-\infty}^{\infty} H''_n(x)\frac{d^{n-2}}{dx^{n-2}}(e^{-x^2})\,dx \\ &= \cdots\cdots = (-1)^{2n}\int_{-\infty}^{\infty} H_n^{(n)}(x)\ e^{-x^2}\ dx \end{aligned}$$

Now the term containing the highest power of x in $H_n(x)$ is $2^n x^n$, so $H_n^{(n)}(x) = 2^n\ n!$ and the last integral is

$$2^n n!\int_{-\infty}^{\infty} e^{-x^2}\ dx = 2^n n! 2\int_{0}^{\infty} e^{-x^2}\ dx = 2^n\ n!\sqrt{\pi}$$

which gives the desired result

$$\int_{-\infty}^{\infty} e^{-x^2}\ H_n^2(x)\ dx = 2^n\ n!\sqrt{\pi}$$

These orthogonality properties can be used to expand an arbitrary function $f(x)$ in a series

$$\mathbf{f(x)=\sum_{n=0}^{\infty} a_n \; H_n(x)} \tag{83}$$

The coefficients a_n can be found by multiplying Eqn.(83) by $e^{-x^2} H_m(x)$ and integrating term by term from $-\infty$ to ∞. Using formula (78), this gives

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)\; f(x)\; dx = \sum_{n=0}^{\infty} a_n \int_{-\infty}^{\infty} e^{-x^2} H_m(x)\; H_n(x)\, dx = a_m 2^m m!\sqrt{\pi}$$

Replacing m by n, in the above equation, we get

$$\mathbf{a_n = \frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_n(x)\; f(x)\; dx} \tag{84}$$

The series (83), with its coefficients calculated by formula (84), is called the **Hermite series** and is useful in the theory of orthogonal functions.

Let us consider the following example.

Example 7: Express $f(x) = e^{2bx}$ in a Hermite series, and use this result to deduce the value of the integral

$$\int_{-\infty}^{\infty} e^{-x^2+2bx} \; H_n(x)\; dx\,.$$

Solution: Set $t = b$ in the generating function (66) to obtain

$$e^{2bx-b^2} = \sum_{n=0}^{\infty} \frac{b^n}{n!} H_n(x)$$

We can then write the series

$$f(x) = e^{2bx} = e^{b^2} \sum_{n=0}^{\infty} \frac{b^n}{n1} H_n(x) = \sum_{n=0}^{\infty} a_n H_n(x)\text{, where, } a_n = \frac{b^n}{n!} e^{b^2}$$

Also, from formula (84), we obtain

$$a_n = \frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2+2bx} H_n(x)\, dx, \quad n = 0, 1, 2, \ldots$$

Thus, it follows that

$$\int_{-\infty}^{\infty} e^{-x^2+2bx} H_n(x)\, dx = \sqrt{\pi}\, 2^n \frac{n!b^n e^{b^2}}{n!} = \sqrt{\pi}\,(2b)^n\, e^{b^2}, \quad n = 0, 1, 2, \ldots$$

***

You may now try the following exercises.

E21) Verify that $y_1(x)$ and $y_2(x)$ given by Eqns.(64) and (65) respectively, are two linearly independent solutions of Hermite's Eqn.(63).

E22) Use the generating function for the Hermite polynomials to find $H_0(x),\; H_1(x),\; H_2(x)$ and $H_3(x)$.

E23) Prove the following relations for Hermite polynomials $H_n(x)$.

a) $x H'_n(x) = n\; H'_{n-1}(x) + n\; H_n(x)$

b) $H_n(x) = 2x\; H_{n-1}(x) - 2(n-1)\; H_{n-2}\;(x)$

c) $H''_n(x) - 2x\; H'_n(x) + 2n\; H_n(x) = 0$

E24) Show that

a) $H_{2n}(0) = (-1)^n \dfrac{2n!}{n!}$ b) $H_{2n+1}(0) = 0$

c) $H'_{2n}(0) = 0$ d) $H'_{2n+1}(0) = (-1)^n \dfrac{(2n+2)!}{(n+1)!}$

E25) Use the generating function (66) to derive the relation

$$x^n = \sum_{k=0}^{[n/2]} \frac{n!H_{n-2k}(x)}{2^n\, k!(n-2k)!}, \quad n = 0,1,2,\ldots$$

E26) Expand $f(x) = x^3 - 3x^2 + 2x$ in a series of the form $\sum_{n=0}^{\infty} a_n\ H_n(x)$.

Let us now consider the Laguerre polynomials.

Laguerre Polynomials

The Laguerre polynomials $L_n(x)$ are polynomial solutions to Laguerre's equation

$$xy'' + (1-x)\,y' + ny = 0 \tag{85}$$

where n is a constant. These polynomial solutions are obtained when n is a non-negative integer. An important application involving Laguerre polnomials in quantum mechanics is to find the wave function associated with the electron in a hydrogen atom.

We now define the Laguerre polynomials and list some of their useful properties.

We shall not be proving these properties here. We shall leave them for you to verify yourself.

Generating Function

We define the Laguerre polynomials $L_n(x)$ by means of the relations

$$\mathbf{(1-t)^{-1} e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} L_n(x) t^n,\ \ |t|<1,\ \ 0 \le x \le \infty} \tag{86}$$

The function on the left hand side is called the generating function of the Laguerre polynomials.

We have

$$(1-t)^{-1} e^{\frac{-xt}{1-t}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (xt)^k (1-t)^{-k-1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (xt)^k \sum_{m=0}^{\infty} \binom{-k-1}{m} (-1)^m\ t^m \tag{87}$$

But since $\displaystyle\binom{-k-1}{m} = \frac{(-k-1)(-k-2)(-k-3)\ldots(-k-m)}{m!}$

$$= (-1)^m \frac{(k+1)(k+2)\ldots(k+m)}{m!} = (-1)^m \binom{k+m}{m},$$

Eqn.(87) becomes

$$(1-t)^{-1} e^{\frac{-xt}{1-t}} = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (k+m)!\, x^k}{(k!)^2\, m!}\ t^{k+m} \tag{88}$$

where we have reversed the order of summation.

Using the change of index $m = n - k$ in Eqn.(88) and comparing with Eqn.(86), the Laguerre polynomials are defined by

$$\mathbf{L_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \, n! \, x^k}{(k!)^2 (n-k)!}} \tag{89}$$

In Table-2, we have listed the first few polynomials $L_n(x)$

Table 2

$L_0(x) = 1$
$L_1(x) = -x + 1$
$L_2(x) = \frac{1}{2!}(x^2 - 4x + 2)$
$L_3(x) = \frac{1}{3!}(-x^3 + 9x^2 - 18x + 6)$
$L_4(x) = \frac{1}{4!}(x^4 - 16x^3 + 72x^2 - 96x + 24)$

The Rodrigue's Formula

The Rodrigue's formula for Laguerre polynomials $L_n(x)$ is given by

$$\mathbf{L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x}), \quad n = 0, 1, 2, \ldots} \tag{90}$$

which can be verified by the application of the Leibniz formula

$$\frac{d^n}{dx^n}(fg) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k} f}{dx^{n-k}} \frac{d^k g}{dx^k}, \quad n = 1, 2, 3, \ldots \tag{91}$$

Recurrence Relations

We now list a few important properties satisfied by Laguerre polynomials

$$\mathbf{(n+1) L_{n+1}(x) + (x - 1 - 2n) L_n(x) + n \, L_{n-1}(x) = 0} \tag{92}$$

$$\mathbf{L'_n(x) - L'_{n-1}(x) + L_{n-1}(x) = 0} \tag{93}$$

$$\mathbf{x \, L'_n(x) = n L_n(x) - n \, L_{n-1}(x)} \tag{94}$$

and

$$\mathbf{x L''_n(x) + (1 - x) \, L'_n(x) + n \, L_n(x) = 0} \tag{95}$$

Relation (95) can be used to prove the following orthogonality property of Laguerre polynomials.

Orthogonality

The Laguerre polynomials $L_n(x)$ satisfy the orthogonality property

$$\mathbf{\int_0^{\infty} e^{-x} L_n(x) L_m(x) \, dx = 0, \; \text{if} \;\; m \neq n} \tag{96}$$

You may now try the following exercises.

E27) Determine the Laguerre polynomials $L_0(x), L_1(x), L_2(x)$ and $L_3(x)$.

E28) Prove the recurrence relations (92) – (95).

E29) Prove that $\int_0^{\infty} e^{-x} L_n(x) \, L_m(x) dx = 0$, if $m \neq n$.

We shall now discuss some applications of these polynomials to physical situations.

3.4 APPLICATIONS TO PHYSICAL SITUATIONS

We shall first illustrate the role played by Legendre polynomials in solving certain boundary value problems of mathematical physics.

Steady-state temperature in a sphere

Consider a solid sphere of radius a . Let origin be its centre. Let the surface of the sphere be held at a prescribed temperature say, $g(\phi)$, which is assumed to be independent of coordinate θ , until within the sphere a steady state for temperature $T(\rho, \phi)$ is produced by the flow of heat. Here (ρ, θ, ϕ) are the spherical polar coordinates and we wish to find the steady-state temperature $T(\rho, \phi)$ which satisfies Laplace equation.

You may recall that Laplace equation for any function $T(x, y, z)$ is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0.$$

and in the spherical polar coordinates (ρ, θ, ϕ) where ,
$x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = \rho\cos\phi$, it reduces to

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial T}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial T}{\partial \phi}\right) + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 T}{\partial \theta^2} = 0 \qquad (97)$$

In the given situation since T does not depend on θ , Eqn.(97) reduces to

$$\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial T}{\partial \rho}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial T}{\partial \phi}\right) = 0 \qquad (98)$$

We have to find the solution of Eqn(98) subject to the boundary condition

$$T(a, \phi) = g(\phi) \qquad (99)$$

To solve Eqn.(98) , we use the method of separation of variable and take the product solution of the form

$$T(\rho, \phi) = R(\rho)\, P(\phi) \qquad (100)$$

Using Eqn.(100), Eqn.(98) becomes

$$P\frac{d}{d\rho}\left(\rho^2 \frac{dR}{d\rho}\right) + R\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi \frac{dP}{d\phi}\right) = 0$$

$$\Rightarrow \frac{1}{R}\frac{d}{d\rho}\left(\rho^2 \frac{dR}{d\rho}\right) = -\frac{1}{P}\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi \frac{dP}{d\phi}\right) \qquad (101)$$

The left side of Eqn.(101) is a function of ρ only whereas right hand side of Eqn.(101) is a function of ϕ only. This is possible only when each side is equal to the same constant, say . Then Eqn.(101) is equivalent to the equations.

$$\rho^2 \frac{d^2 R}{d\rho^2} + 2\rho \frac{dR}{d\rho} - \lambda R = 0 \qquad (102)$$

and, $$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi \frac{dP}{d\phi}\right) + \lambda P = 0 \qquad (103)$$

Eqn.(102) is an Euler equation, its auxiliary equation is

$$m(m-1) + 2m - \lambda = 0, \text{ or, } m^2 + m - \lambda = 0.$$

$$\Rightarrow m = \frac{-1 \pm \sqrt{1 + 4\lambda}}{2}.$$

General solution of Eqn.(102) can be written as

$$R = c_1 \rho^{\frac{-1}{2} + \sqrt{\frac{1}{4} + \lambda}} + c_2 \rho^{\frac{-1}{2} - \sqrt{\frac{1}{4} + \lambda}} \qquad (104)$$

In order that R is single-valued and bounded near $\rho = 0$, we put in Eqn.(104)

$c_2 = 0$ and $-\frac{1}{2}+\sqrt{\;+\frac{1}{4}} = n$, where n is a non-negative integer. It then follows that $= n(n+1)$ and Eqn.(104) reduces to

$$R = c_1\rho^n \tag{105}$$

and Eqn.(103) becomes $\frac{d^2P}{d\phi^2}+\frac{\cos\phi}{\sin\phi}\frac{dP}{d\phi}+n(n+1)P = 0$.

In this equation if we let $x = \cos\phi$, then we get

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+n(n+1)P = 0 \tag{106}$$

which is Legendre's equation.

By the physics of the problem, the function P must be bounded for $0 \le \phi \le \pi$ i.e., $-1 \le x \le 1$ and we know that, the solution of Eqn.(106) are constant multiples of Legendre polynomials, $P_n(x)$. If we combine this result with Eqn.(105), then we have particular solution of Eqn.(98) of the form

$$\alpha_n \rho^n P_n(\cos\phi)\text{, for each } n = 0, 1, 2, 3\ldots$$

where α_n are arbitrary constants.

The general solution of Eqn.(98) can be written as

$$T(\rho,\phi) = \sum_{n=0}^{\infty}\alpha_n\rho^n P_n(\cos\phi) \tag{107}$$

The boundary condition (99) now requires

$$g(\phi) = \sum_{n=0}^{\infty}\alpha_n a^n P_n(\cos\phi)$$

or equivalently,

$$g(\cos^{-1}x) = \sum_{n=0}^{\infty}\alpha_n a^n P_n(x)$$

Multiplying both sides of the above equation by $P_m(x)$ and integrating w.r.t. to x from $x = -1$ to $x = 1$ and using orthogonality property of Legendre polynomials, we get

$$\alpha_n = \frac{1}{a^n}\left(n+\frac{1}{2}\right)\int_{-1}^{1} g(\cos^{-1}x)P_n(x)\,dx \tag{108}$$

Hence relation (107) gives the required solution where coefficients are given by relation (108).

Let us look at another application of Legendre polynomials.

Electrostatic Dipole Potential

Consider two point charges of equal magnitude, say q, but of opposite sign.

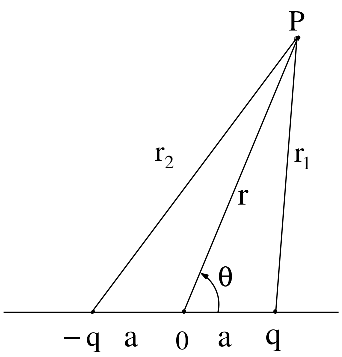

Fig.3

Let these charges be placed in a polar coordinate system (see Fig.3). With suitable units of measurement, the potential U at the point P is

$$U = \frac{q}{r_1}-\frac{q}{r_2} \tag{109}$$

where, $r_1 = \sqrt{r^2+a^2-2ar\cos\theta}$

and, $r_2 = \sqrt{r^2+a^2+2ar\cos\theta}$

From generating function relation for the Legendre's polynomials, we have

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty}t^n P_n(x) \tag{110}$$

when $r > a$, we can use Eqn.(110) to write

$$\frac{1}{r_1} = \frac{1}{r}\frac{1}{\sqrt{1-2(a/r)\cos\theta+(a/r)^2}} = \frac{1}{r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n \qquad (111)$$

and, similarly,

$$\frac{1}{r_2} = \frac{1}{r}\frac{1}{\sqrt{1-2(a/r)\cos(-\theta)+(a/r)^2}} = \frac{1}{r}\sum_{n=0}^{\infty} P_n(-\cos\theta)\left(\frac{a}{r}\right)^n \qquad (112)$$

From Eqns.(109), (110) and (111), we get

$$U = \frac{q}{r}\sum_{n=0}^{\infty}[P_n(\cos\theta) - P_n(-\cos\theta)]\left(\frac{a}{r}\right)^n \qquad (113)$$

From the properties of Legendre's polynomial $P_n(x)$, we know that $P_n(x)$ is even if n is even and it is odd if n is odd. Thus $[P_n(\cos\theta) - P_n(-\cos\theta)]$ equals 0 or, $2P_n(\cos\theta)$ according as n is even or odd. Thus relation (113) becomes

$$U = \frac{2q}{r}\sum_{n=0}^{\infty} P_{2n+1}(\cos\theta)\left(\frac{a}{r}\right)^{2n+1}$$

$$= \frac{2q}{r}\left[P_1(\cos\theta)\left(\frac{a}{r}\right) + P_3(\cos\theta)\left(\frac{a}{r}\right)^3 + \cdots\right]. \qquad (114)$$

When r is large compared with a, we assume that all terms except the first can be neglected and since $P_1(x) = x$, relation (114) becomes

$$U = 2aq\left(\frac{\cos\theta}{r^2}\right)$$

Physicists use this approximation for the dipole potential.

We shall now consider the application of Hermite polynomials and understand the role played by the Hermite polynomials $H_n(x)$ and the corresponding Hermite function $e^{-x^2/2}H_n(x)$ in the theory of the linear harmonic oscillator in quantum mechanics.

Simple Harmonic Oscillator

In wave mechanics, the basic equation for describing the location of a particle attracted by an energy potential $V(z)$ is the **Schrödinger's** (time-independent) **equation**

$$\frac{d^2\psi}{dz^2} + \frac{8m\pi^2}{h^2}[V(z) - E]\,\psi = 0 \qquad (115)$$

where m is the mass of the particle, E is the total energy, and h is Plank's constant. The unknown quantity ψ is called the **wave function** i.e. the amplitude of the wave whose intensity gives the probability of finding the particle at any given point in space. A fundamental problem in wave mechanics concerns the motion of a particle bounded in a potential well. It has been established that bounded solutions of Schrodinger's equation for such problems are obtained only for certain discrete energy levels of the particle within the well. A particular example of this important class of problems is the linear oscillator (also called the simple harmonic oscillator), the solution of which lead to Hermite polynomials.

If the restoring force on a particle displaced a distance z from its equilibrium position is $-kz$, where k can be associated with the spring constant of the classical oscillator, then its potential energy is $V(z) = \frac{1}{2}kz^2$. Substituting this expression into Eqn.(115) and introducing the dimensionless parameters

$$x = z\left(\frac{4mk\pi^2}{h^2}\right)^{1/4}, \quad \lambda = \frac{4\pi E}{h}\sqrt{\frac{m}{k}} = \frac{4\pi E}{h\omega}$$

where $\omega = \sqrt{k/m}$ is the angular frequency of the classical oscillator, we find that Eqn.(115) becomes

$$\psi'' + (\lambda - x^2)\psi = 0 \ , \quad -\infty < x < \infty \qquad (116)$$

where the primes denote differentiation with respect to x. In addition to Eqn.(116), the wave function ψ must satisfy the boundary condition

$$\lim_{|x|\to\infty} \psi(x) = 0 \qquad (117)$$

In order to look for bounded solutions of Eqn.(116), we start with the observation that λ becomes negligible compared with x^2 for large values of x. Thus, asymptotically we expect the solution of Eqn.(116) to behave like

$$\psi(x) \sim e^{\pm x^2/2} \ , \quad |x| \to \infty$$

where only the negative sign in the exponent is appropriate in order that Eqn.(117) be satisfied. Based on this observation, we make the assumption that Eqn.(116) has solutions of the form

$$\psi(x) = y(x)\ e^{-x^2/2} \qquad (118)$$

for suitable y. The substitution of Eqn.(118) into Eqn.(116) yields the differential equation

$$y'' - 2xy' + (\lambda - 1)\, y = 0 \qquad (119)$$

The boundary condition (117) suggests that whatever functional form y assumes, it must either be finite for all x or approach infinity at a rate slower than $e^{-x^2/2}$ approaches zero. It is seen that the only solution of Eqn.(119) satisfying this condition are those for which $\lambda - 1 = 2n$, or

$$\lambda = \lambda_n = 2n + 1, \ \ n = 0, 1, 2, \ldots \qquad (120)$$

called eigenvalues . In terms of E, we find

$$E_n = \frac{\left(n + \frac{1}{2}\right) h\omega}{2\pi}, \ n = 0, 1, 2, \ldots \qquad (121)$$

with $E_0 = h\omega / 4\pi$ being the lowest or minimum energy level. With λ so restricted, we see that Eqn.(119) becomes

$$y'' - 2xy' + 2ny = 0$$

which is Hermite's equation with solution $y = H_n(x)$. Hence, we conclude that to each eigenvalue λ_n given by Eqn.(120) there corresponds the solution of Eqn.(116) (called an eigenfunction) given by

$$\psi_n(x) = H_n(x)\ e^{-x^2/2}, \ n = 0, 1, 2, \ldots \qquad (122)$$

You may now try the following exercises.

E30) Show that the functions $\psi_n(x) = e^{-x^2/2}\ H_n(x)$ satisfy the relations.

a) $2n\,\psi_{n-1}(x) = x\,\psi_n(x) + \psi_n'(x)$

b) $2x\,\psi_n(x) - 2n\,\psi_{n-1}(x) = \psi_{n+1}(x)$

c) $\psi_n'(x) = x\,\psi_n(x) - \psi_{n+1}(x)$

E31) By writing $\psi_n(x) = c_n e^{-x^2/2} H_n(x)$, show that the normalized eigen function, that is $\int_{-\infty}^{\infty} [\,\psi_n(x)\,]^2\, dx = 1$, assumes the form

$$\psi_n(x) = \left[\sqrt{\pi}\, 2^n\, n!\right]^{-1/2} e^{-x^2/2} H_n(x), \ n = 0, 1, 2, \ldots.$$

We now end this unit by giving a summary of what we have covered in it.

3.5 SUMMARY

In this unit, we have learnt the following points:

1. The equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ is called **Legendre's Equation** where n is a real constant. The polynomial solutions of this equation for integral values of n are **Legendre polynomials .**
2. Legendre polynomial of degree n, having value 1 at $x = 1$, is denoted by $P_n(x)$ and is called **Legendre function of the first kind** and is given by
$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^n (k)!(n-k)!(n-2k)!} x^{n-2k}$$
where $[n/2] = \begin{cases} n/2 & , \text{ if } n \text{ is even} \\ \frac{n-1}{2} & , \text{ if } n \text{ is odd.} \end{cases}$
3. **Legendre function of the second kind** is denoted by $Q_n(x)$, when n is an integer and is given by
$$Q_n(x) = P_n(x)\left\{A_n + B_n \int \frac{dx}{(1-x^2)[P_n(x)]^2}\right.$$
where A_n and B_n are constants.
4. **Rodrigue's formula for Legendre's polynomials** is given by
$$P_n(x) = \frac{1}{2^n (n)!} \frac{d^n}{dx^n}[(x^2-1)^n].$$
5. **Orthogonality property** of the Legendre's polynomials $P_n(x)$ is
$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \begin{cases} 0 & , \text{ if } m \neq n \\ \frac{2}{2n+1}, & \text{ if } m = n \end{cases}$$
6. The function on the left side of
$$\frac{1}{\sqrt{1-2xt+t^2}} = P_0(x) + tP_1(x) + t^2 P_2(x) + \cdots + t^n P_n(x) + \cdots \mid x \mid < 1$$
is called the **generating function** of the Legendre polynomials.
7. Some of the **recurrence relations** for Legendre polynomials are
$$P'_{n+1}(x) = (2n+1)P_n(x) + P'_{n-1}(x)$$
$$P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x)$$
$$nP_n(x) = xP'_n(x) - P'_{n-1}(x)$$
$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$
$$(1-x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)]$$
$$(1-x^2)P_n(x) = (n+1)[xP_n(x) - P_{n+1}(x)]$$
8. The **Hermite polynomials** $H_n(x)$ are polynomial solutions to **Hermite's equation** $y'' - 2xy' + 2ny = 0$, where n is a constant, and are given by
$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!\,(n-2k)!}(2x)^{n-2k}$$
where $[n/2]$ is the greatest integer $\leq n/2$.
9. The function on the left side of equation $e^{(2xt-t^2)} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$, for all finite x and t is the **generating function** of the Hermite polynomials $H_n(x)$.
10. Some of the **recurrence** relations of Hermite polynomials are

$$\mathbf{x H'_n(x) = n\ H'_{n-1}(x) + n\ H_n(x)}$$
$$\mathbf{H'_n(x) = 2n\ H_{n-1}(x)}$$
$$\mathbf{H_n(x) = 2x\ H_{n-1}(x) - H_{n-1}(x)}$$
$$\mathbf{H_n(x) = 2x\ H_{n-1}(x) - 2(n-1)\ H_{n-2}(x)}$$

11. The **Rodrigues formula** for Hermite polynomials is given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

12. **Orthogonality property** of the Hermite polynomials $H_n(x)$ is

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\ H_m(x)\, dx = \begin{cases} 0 & , \text{ if } m \neq n \\ 2^n n! \sqrt{\pi}, & \text{if } m = n \end{cases}$$

13. The **Laguerre polynomials** $L_n(x)$ are polynomial solutions to **Laguerre's** equation $x y'' + (1-x) y' + ny = 0$, where n is a constant, and are given by

$$L_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k n! x^k}{(k!)^2 (n-k)!}$$

14. The function on the left hand side of the equation

$$(1-t)^{-1} e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} L_n(x) t^n,\ |t| < 1,\ 0 \le x < \infty$$

is the **generating function** of the Laguerre polynomials.

15. The **Rodrigues formula** for the Laguerre polynomials $L_n(x)$ is

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x})$$

16. Some of the **recurrence** relations of Laguerre polynomials are

$$\mathbf{(n+1)\ L_{n+1}(x) + (x - 1 - 2n)\ L_n(x) + n L_{n-1}(x) = 0}$$
$$\mathbf{L'_n(x) - L'_{n-1}(x) + L_{n-1}(x) = 0}$$
$$\mathbf{x L'_n(x) = n L_n(x) - n\ L_{n-1}(x)}$$

17. **Orthogonality property** of the Laguerre polynomials $L_n(x)$ is

$$\int_0^{\infty} e^{-x} L_n(x) L_m(x)\ dx = 0, \quad \text{if } m \neq n.$$

3.6 SOLUTIONS/ANSWERS

E1) a) Let $I = \int_{-1}^{1} f(x) P_n(x)\, dx$

$$= \frac{1}{2^n n!} \int_{-1}^{1} f(x) \frac{d^n}{dx^n}[(x^2-1)]\, dx$$

$$= \frac{1}{2^n n!} \left[f(x) \frac{d^{n-1}}{dx^{n-1}}\{(x^2-1)^n\} \Big|_{-1}^{1} - \int_{-1}^{1} f^{(1)}(x) \frac{d^{n-1}}{dx^{n-1}}\{(x^2-1)^n\}\, dx \right]$$

$$= \frac{(-1)}{2^n n!} \int_{-1}^{1} f^{(1)}(x) \frac{d^{n-1}}{dx^{n-1}}\{(x^2-1)^n\}\, dx$$

Proceeding like this

$$I = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} f^{(n)}(x).(x^2-1)^n\, dx.$$

b) Here $f(x) = x^n$, $f^{(n)}(x) = n!$

$$\therefore \int_{-1}^{1} x^n P_n(x)\,dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} n!.(x^2-1)^n\,dx$$

$$= \frac{(-1)^n}{2^n}\int_{-1}^{1}(x^2-1)^n\,dx$$

$$= \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

E2) By Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\{(x^2-1)^n\}$$

since $(x^2-1)^n$ vanishes n times at $x=1$ and $x=-1$, thus its n^{th} derivative and hence , $P_n(x)$ have all real zeroes lying between -1 and $+1$.

$\therefore$ the roots of $P_n(x)=0$ are real and lie between -1 and $+1$.

Let, if possible, the equation

$$P_n(x)=0$$

has a repeated root, say $'\alpha'$. Then

$$P_n(\alpha)=0 \quad \text{and} \quad P_n'(\alpha)=0 \tag{123}$$

since $P_n(x)$ is a solution of Legendre's equation, therefore,

$$(1-x^2)P_n^{(2)}(x) - 2xP_n^{(1)} + n(n+1)P_n(x) = 0 \tag{124}$$

Differentiating m times, by Leibnitz's theorem, we get

$$(1-x^2)P_n^{(m+2)}(x) - 2x(m+1)P_n^{(m+1)}(x) + (n-m)(n-m+1)P_n^{(m)}(x) = 0, \tag{125}$$

Putting $x=\alpha$ in Eqn.(124) and using Eqn.(123), we get

$$P_n^{(2)}(\alpha)=0$$

Putting $m=1,2,3,\ldots$ in Eqn.(125), we get

$$P_n^{(3)}(\alpha) = 0 = P_n^{(4)}(\alpha) = P_n^{(5)}(\alpha) = \cdots = P_n^{(n)}(\alpha)$$

But from definition, $P_n^{(n)}(\alpha) = \dfrac{2n}{2^n n!} \neq 0$.

Thus Eqn.(123) is not valid and hence $P_n(x)=0$ cannot have repeated roots. Hence all the roots of $P_n(x)=0$ are distinct.

E3) From Rodrigue's formula, we have

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\{(x^2-1)^n\}$$

$$\therefore \int_{-1}^{1} P_n(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{1}{2^n n!}\left|\frac{d^{n-1}}{dx^{n-1}}\{(x^2-1)^n\}\right|_{-1}^{+1}$$

$$= \frac{1}{2^n n!}\left|\frac{d^{n-1}}{dx^{n-1}}\left[(x-1)^n(x+1)^n\right]\right|_{-1}^{+1}$$

$$= \frac{1}{2^n n!}[0-0] = 0 \ \text{ if } \ n \neq 0$$

Now when $n=0$, then $P_0(x)=1$

$$\therefore \int_{-1}^{1} P_0(x)\,dx = \int_{-1}^{1} 1.dx = x\Big|_{-1}^{+1} = 2.$$

E4) We know that

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\,h^n \qquad (126)$$

Putting $x = \frac{1}{2}$ in Eqn.(126), we get

$$(1-h+h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n\left(\frac{1}{2}\right) h^n \qquad (127)$$

Again, putting $x = -\frac{1}{2}$ in Eqn.(126), we get

$$(1+h+h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n\left(-\frac{1}{2}\right) h^n \qquad (128)$$

Now put h^2 for h in Eqn.(128), we get

$$(1+h^2+h^4)^{-1/2} = \sum_{n=0}^{\infty} h^{2n} P_n\left(-\frac{1}{2}\right) \qquad (129)$$

But $(1+h^2+h^4)^{-1/2} = \{(1+h^2)^2 - h^2\}^{-1/2}$

$$= (1+h+h^2)^{-1/2}(1-h+h^2)^{-1/2}$$

Using Eqns.(127), (128) and (129) in the above, we get

$$\sum_{n=0}^{\infty} h^{2n} P_n(-1/2) = \left\{\sum_{n=0}^{\infty} h^n P_n(-1/2)\right\}\left\{\sum_{n=0}^{\infty} h^n P_n(1/2)\right\}$$

$$= [P_0(-1/2) + h\,P_1(-1/2) + h^2 P_2(-1/2) + \cdots] \times [P_0(1/2) + h\,P_1(1/2) + h^2 P_2(1/2) + \cdots]$$

Now equating the coefficient of h^{2n} on both sides, we get

$$P_n\left(-\frac{1}{2}\right) = P_0\left(-\frac{1}{2}\right)P_{2n}\left(\frac{1}{2}\right) + P_1\left(-\frac{1}{2}\right)P_{2n-1}\left(\frac{1}{2}\right) + \cdots + P_{2n-1}\left(-\frac{1}{2}\right)P_1\left(\frac{1}{2}\right)$$

$$+ P_{2n}\left(-\frac{1}{2}\right)P_0\left(\frac{1}{2}\right).$$

E5) Since $P_n(x)$ satisfies Legendre's equation, we have

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \qquad (130)$$

a) putting $x = 1$ in Eqn.(130), we get

$$-2P_n'(1) + n(n+1)\,P_n(1) = 0$$

i.e., $P_n'(1) = \frac{1}{2} n(n+1)\,P_n(1)$

$$= \frac{1}{2} n\,(n+1) \qquad [\because P_n(1) = 1]$$

b) putting $x = -1$ in Eqn.(130), we get

$$2P_n'(-1) + n\,(n+1)\,P_n(-1) = 0$$

i.e., $P_n'(-1) = -\frac{1}{2} n\,(n+1)\,P_n(-1)$

$$= (-1)^{n-1}\frac{1}{2}(n)\,(n+1) \qquad [\because P_n(-1) = (-1)^n\,]$$

E6) a) Putting $x = 0$ in Eqn.(27), we obtain

$$(1+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(0)\,t^n \qquad (131)$$

Expending left hand side of Eqn.(131) in binomial series

$$(1+t^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} t^{2n} \qquad (132)$$

Comparing terms on the right in series (131) and (132), we see that series (132) has only even powers of t. Thus, $P_n(0) = 0$ for $n = 1, 3, 5, \ldots$ or, equivalently, $P_{2n+1}(0) = 0, n = 0, 1, 2, \ldots$.

b) Since all odd terms in series (131) are zero, replace n by $2n$ in the series and compare with (132), we obtain

$$P_{2n}(0) = \binom{-1/2}{n} = \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)\ldots\left(\frac{-1}{2}-n+1\right)}{n!}$$

$$= \frac{(-1)^n \frac{1}{2}\left(\frac{1}{2}+1\right)\ldots\left(\frac{1}{2}+n-1\right)}{n!}$$

$$= \frac{(-1)^n\left(\frac{1}{2}+n-1\right)\left(\frac{1}{2}+n-2\right)\ldots\left(\frac{1}{2}+1\right)\frac{1}{2}}{n!}$$

$$= (-1)^n \binom{\frac{1}{2}+n-1}{n} = (-1)^n \binom{n-1/2}{n} = \frac{(-1)^n \Gamma\left(n+\frac{1}{2}\right)}{n!\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{(-1)^n (2n)!}{2^n (n!)^2}. \qquad \left[\because \Gamma\left(n+\frac{1}{2}\right) = \frac{2n!}{2^{2n} n!}\Gamma\left(\frac{1}{2}\right)\right].$$

E7) In the expressions for $P_0(x), P_1(x), P_2(x), P_3(x)$ in terms of x, as obtained in Example 3, put $x = \cos\phi$ and obtain the required results.

E8) a) Multiplying Eqn.(37) by x and then subtracting it from Eqn.(36), we get

$$(1-x^2)P_n'(x) = n\{P_{n-1}(x) - xP_n(x)\}.$$

b) Multiplying (36) by x and then subtracting it from (37), we get

$$(1-x^2)P_n'(x) = (n+1)\{xP_n(x) - P_{n+1}(x)\}.$$

E9) From relation (39), we have

$$(1-x^2)P_n'(x) = n\{P_{n-1}(x) - xP_n(x)\}$$

$$= (2n+1)xP_n(x) - (n+1)P_{n+1}(x) - nxP_n(x) \quad \text{[using (38)]}$$

$$= (n+1)[xP_n(x) - P_{n+1}(x)]$$

E10) We know that

$$P_n(x) = \frac{1.3.\ldots.(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \cdots\right]$$

Thus we see that $P_0(x)$ is a constant, $P_1(x)$ is a polynomial of degree 1, $P_2(x)$ is a polynomial of degree 2 and so on.

Thus we can express x^m in terms of Legendre forms $P_0(x), P_1(x), \ldots P_m(x)$, and hence a polynomial $f(x)$ of degree $k < n$ can be expressed as

$$f(x) = A_0P_0(x) + A_1P_1(x) + \cdots + A_kP_k(x)$$

$$\therefore \int_{-1}^{1} f(x)P_n(x)dx = \sum_{r=0}^{k} A_r \int_{-1}^{1} P_r(x)P_n(x)dx$$

$$= \sum_{r=0}^{k} A_r.0, \text{ as } r \neq n$$

$$= 0.$$

E11) Proceeding as in Example 4.

$$f(x) = x^3 + ax^2 + bx + c$$
$$= \frac{2}{3}P_3(x) + \frac{3}{5}P_1(x) + a\left[\frac{P_0(x)}{3} + \frac{2}{3}P_2(x)\right] + b\,P_1(x) + c\,P_0(x)$$
$$= \frac{2}{3}P_3(x) + \frac{2}{3}a\,P_2(x) + \left(\frac{3}{5} + b\right)P_1(x) + \left(\frac{a}{3} + c\right)P_0(x).$$

E12) From E8) (a) and (b), we have

$$(1-x^2)\,P_n'(x) = n\,\{P_{n-1}(x) - x\,P_n(x)\}$$

and $(1-x^2)\,P_n'(x) = (n+1)\,\{x\,P_n(x) - P_{n+1}(x)\}$

Substituting for $xP_n(x)$ from the first relation in second relation, we get

$$(1-x^2)P_n'(x) = (n+1)\left[\left\{P_{n-1}(x) - \frac{(1-x^2)P_n'(x)}{n}\right\} - P_{n+1}(x)\right]$$

or, $$(1-x^2)\left[1 + \frac{n+1}{n}\right]P_n'(x) = (n+1)[P_{n-1}(x) - P_{n+1}(x)]$$

or, $$(2n+1)(1-x^2)\,P_n'(x) = n\,(n+1)[P_{n-1}(x) - P_{n+1}(x)]\,.$$

E13) From generating function of Legendre Polynomials, we have

$$[1-2xt+t^2]^{-1/2} = \sum_{m=0}^{\infty} t^m P_m(x)$$
$$\therefore \int_{-1}^{1} \frac{P_n(x)}{\sqrt{1-2xt+t^2}}\,dx = \int_{-1}^{1} P_n(x)\left[\sum_{m=0}^{\infty} t^m P_m(x)\right]dx$$
$$= t^n \int_{-1}^{1} [P_n(x)]^2\,dx\,, \qquad \text{[using formula (41)]}$$
$$= t^n \cdot \frac{2}{2n+1}\,.$$

E14) We know that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

Integrating both sides w.r.t t between the limit 0 to 1, we get

$$\int_0^1 (1-2xt+t^2)^{-1/2}\,dt = \sum_{n=0}^{\infty} P_n(x)\int_0^1 t^n\,dt$$

Putting $x = \cos\theta$, we get

$$\int_0^1 \frac{dt}{\sqrt{1-2\cos\theta . t + t^2}} = \sum_{n=0}^{\infty} P_n(\cos\theta).\left.\frac{t^{n+1}}{n+1}\right|_0^1$$

or, $$\int_0^1 \frac{dt}{\sqrt{(t-\cos\theta)^2 + \sin^2\theta}} = \sum_{n=0}^{\infty} P_n(\cos\theta).\frac{1}{n+1}$$

or, $$\ln\left[(t-\cos\theta) + \sqrt{\sin^2\theta + (t-\cos\theta)^2}\right]\Bigg|_0^1 = \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1}$$

or, $$\ln\left[\frac{(1-\cos\theta) + \sqrt{\sin^2\theta + (1-\cos\theta)^2}}{1-\cos\theta}\right] = \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1}$$

or, $$\ln\frac{\sqrt{2} + \sqrt{1-\cos\theta}}{\sqrt{1-\cos\theta}} = \sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1}$$

or, $$\ln \frac{\sqrt{2}+\sqrt{2\sin^2\frac{\theta}{2}}}{\sqrt{2\sin^2\frac{\theta}{2}}} = \sum_{n=0}^{\infty} \frac{P_n(\cos\theta)}{n+1}$$

or, $$\ln \frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}} = 1 + \frac{1}{2}P_1(\cos\theta) + \frac{1}{3}P_2(\cos\theta) + \frac{1}{4}P_3(\cos\theta) + \cdots .$$

E15) $x = \cos\phi$, therefore, $\frac{dx}{d\phi} = -\sin\phi = -\sqrt{1-x^2}$ and $\frac{dy}{d\phi} = -\sin\phi \frac{dy}{dx}$

$$\therefore \frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi \frac{dy}{d\phi}\right) + n(n+1)y = \frac{-1}{\sqrt{1-x^2}}\left[2x\sqrt{1-x^2}\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2}\left(-\sqrt{1-x^2}\right)\right] + n(n+1)y$$

$$= (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y .$$

E16) a) $y = P_n(x)$ satisfies the given equation and the boundary conditions are $y(0) = 0 \Rightarrow P_n(0) = 0$ and $y(1) = 1 \Rightarrow P_n(1) = 1$, (which is true).
Now $P_n(0) = 0$, for n odd (see Example 1)
$\therefore$ $y = P_n(x)$ is a solution for $n = 1, 3, 5, \ldots$.

b) $y = P_n(x)$ satisfies the given equation and the condition $y(1) = 1$ [$\because P_n(1) = 1$]
$y'(0) = 0 \Rightarrow P_n'(0) = 0$, but $P_n'(0) = n\,P_{n-1}(0)$ [using (39)]
$P_n'(0) = 0 \Rightarrow P_{n-1}(0) \Rightarrow n = 0, 2, 4, \ldots$.
$\therefore$ $y = P_n(x)$ is a solution for $n = 0, 2, 4, \ldots$.

E17) Multiplying both sides of Eqn.(61) with $P_m(t)$ and integrating from -1 to 1 w.r.t. t, we get

$$\int_{-1}^{1} \frac{1}{x-t} P_m(t)dt = \sum_{n=0}^{\infty} Q_n(x)(2n+1)\int_{-1}^{1} P_n(t)P_m(t)dt$$

$$\therefore \int_{-1}^{1} \frac{P_m(t)}{x-t}dt = Q_n(x)\ (2n+1)\ \frac{2}{2n+1}$$

or, $$Q_n(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)}{x-t}\,dt .$$

E18) a) Comparing the given equation with Eqn.(1), we see that given equations is a Legendre's equation with $n = 0$, so its general solution is $y = C_1P_0(x) + C_2Q_0(x)$.

b) Comparing the given equation with Eqn.(1), we find that $n(n+1) = 12$, therefore, $n = 3, -4$. Since n is non-negative, the general solution of the given equation is $y = C_1P_3(x) + C_2Q_3(x)$.

c) $y = C_1P_1(x) + C_2Q_1(x)$.

d) $y = C_1P_5(x) + C_2Q_5(x)$.

E19) $$W(P_0, Q_0) = \begin{vmatrix} P_0 & Q_0 \\ P_0' & Q_0' \end{vmatrix} = P_0Q_0' - Q_0P_0' = \frac{1}{1-x^2} .$$

E20) $W[P_n(x), Q_n(x)] = P_n Q_n' - Q_n P_n'$

Substituting in the above equation for Q_n and Q_n' from Eqn.(53), we get

$$W[P_n(x), Q_n(x)] = P_n P_n'\left[A_n + B_n \int \frac{dx}{(1-x^2)[P_n(x)]^2}\right] + [P_n(x)]^2 B_n \frac{1}{(1-x^2)[P_n(x)]^2}$$

$$- P_n P_n'\left[A_n + B_n \int \frac{dx}{(1-x^2)[P_n(x)]^2}\right]$$

$$= \frac{B_n}{1-x^2},\ n = 0, 1, 2, \cdots.$$

E21) Consider $y = \sum_{m=0}^{\infty} c_m x^m,\ c_0 \neq 0$. Substitute for y, y' and y'' in Eqn.(63), simplify and equate the coefficients of equal power of x to get two linearly independent solutions $y_1(x)$ and $y_2(x)$ [ref. E9) Unit 2].

E22) $$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} = H_0(x) + H_1(x)t + \frac{H_2(x)}{2!}t^2 + \frac{H_3(x)}{3!}t^3 + \cdots$$

Now, $$e^{2tx-t^2} = 1 + (2tx - t^2) + \frac{(2tx-t^2)^2}{2!} + \frac{(2tx-t^2)^3}{3!} + \cdots$$

$$= 1 + (2x)t + (2x^2 - 1)t^2 + \left(\frac{4x^3 - 6x}{3}\right)t^3 + \cdots$$

Comparing the two series, we have

$$H_0(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2,\ H_3(x) = 8x^3 - 12x.$$

E23) a) Let $w = e^{(2xt - t^2)} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$

then, $(x-t)\frac{\partial w}{\partial x} - t\frac{\partial w}{\partial t} = 0$

or, $$\sum_{n=0}^{\infty} x \frac{H_n'(x)}{n!} t^n - \sum_{n=0}^{\infty} \frac{n H_n(x) t^n}{n!} = \sum_{n=0}^{\infty} \frac{n H_{n-1}'(x)}{n!} t^n$$

Comparing coefficients of t^n on both the sides, weget

$$x H_n'(x) - nH_n(x) = n H_{n-1}(x).$$

b) From relation (75), we get $H_n(x) = 2x H_{n-1}(x) - [2(n-1) H_{n-2}(x)]$

[using relation (74)].

c) $H_n''(x) = 2n H_{n-1}'(x)$ [using Eqn.(74)]

$= 2n[2x H_{n-1}(x) - H_n(x)]$ [using Eqn.(75)]

$= 2x H_n'(x) - 2n H_n(x)$ [using Eqn.(74)].

E24) a) We have from Eqn.(68)

$$H_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k 2n!(2x)^{2n-2k}}{k!\,(2n-2k)!}$$

The last term in the above summation is a constant and therefore, we have

$$H_{2n}(0) = \frac{(-1)^n 2n!}{n!}.$$

b) From Eqn.(68), we have

$$H_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k 2n+1!(2x)^{2n+1-2k}}{k!\,(2n+1-2k)!}$$

The last term in the above summation is a function of x, hence we get

$$H_{2n+1}(0) = 0.$$

c) Differentiating both sides of Eqn.(68), we obtain

$$H_n'(x) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^k n!(2x)^{n-2k-1}.2}{k!(n-2k-1)!}$$

from which it follows that

$$H_{2n}'(x) = \sum_{k=0}^{n-1} \frac{(-1)^k (2n)!.2.(2x)^{2n-2k-1}}{k!\,(2n-2k-1)!}$$

Putting $x = 0$, we get

$H_{2n}'(0) = 0$

d) $H_{2n+1}'(x) = \sum_{k=0}^{n} \frac{(-1)^k (2n+1)!.2.(2x)^{2n-2k}}{k!\,(2n-2k)!}$

$$H_{2n+1}'(0) = \frac{(-1)^n\,(2n+1)!.2}{n!} = \frac{(-1)^n (2n+2)!}{(n+1)!}.$$

E25) From Eqn.(66), we have

$$e^{(2xt-t^2)} = \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!}$$

$$e^{2xt} = e^{t^2} \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!}$$

$$\therefore \sum_{n=0}^{\infty} \frac{(2x)^n t^n}{n!} = \sum_{m=0}^{\infty} \frac{t^{2m}}{m!} \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!}$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{H_{n-2k}(x)\; t^n}{k!(n-2k)!}$$

[by interchanging m and k and setting $k = n - 2k$].

Comparing the coefficients of t^n on both the sides, we have

$$x^n = \sum_{k=0}^{[n/2]} \frac{n!H_{n-2k}(x)}{2^n k!(n-2k)!}$$

E26) Using E22) required result is, $\frac{-3}{2}H_0 + \frac{7}{4}H_1 - \frac{3}{4}H_2 + \frac{1}{8}H_3$.

E27) We have, $L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$, then

$$L_0(x) = 1,\; L_1(x) = e^x \frac{d}{dx}(x\; e^{-x}) = 1 - x,$$

$$L_2(x) = e^x \frac{d^2}{dx^2}(x^2 e^{-x}) = 2 - 4x + x^2$$

$$L_3(x) = e^x \frac{d^3}{dx^3}(x^3 e^{-x}) = 6 - 18x + 9x^2 - x^3$$

E28) Let $w(x,t) = (1-t)^{-1} \exp\left(\frac{-xt}{1-t}\right)$. It satisfy the identity

$$(1-t)^2 \frac{\partial w}{\partial t} + (x - 1 + t)\, w = 0$$

Using Eqn.(86), the above equation reduces to

$$\sum_{n=1}^{\infty} [(n+1) L_{n+1}(x) + (x - 1 - 2n) L_n(x) + n\, L_{n-1}(x)]\, t^n = 0$$

Equating the coefficients of t^n to zero, relation (89) is proved.

Similarly, using Eqn.(86) into the identity $(1-t)\frac{\partial w}{\partial x} + tw = 0$, relation (90) is obtained.

Differentiating relation (89), using relation (90) in it and then eliminating $L'_{n+1}(x)$ and $L'_{n-1}(x)$ from it relation (91) is obtained.

Differentiating relation (91) and using relation (90), relation (92) is obtained.

E29) From Laguerre's differential equation we have for any two Laguerre polynomials $L_m(x)$ and $L_n(x)$

$$x L''_m + (1-x)L'_m + mL_m = 0$$

$$x L''_n + (1-x)L'_n + n L_n = 0$$

Multiplying these equations by L_n and L_m respectively and subtracting, we obtain

$$x[L_n L''_m - L_m L''_n] + (1-x)[L_n L'_m - L_m L'_n] = (n-m)L_m L_n$$

or, $\frac{d}{dx}[L_n L'_m - L_m L'_n] + \frac{(1-x)}{x}[L_n L'_m - L_m L'_n] = \frac{(n-m)}{x} L_m L_n$

or, $\frac{d}{dx}\{x e^{-x}[L_n L'_m - L_m L'_n]\} = (n-m)e^{-x} L_m L_n$, where we have multiplied by I.F. $e^{\int (1-x)/x\, dx} = e^{\ln x - x} = x e^{-x}$

Integrating from 0 to ∞

$$(n-m)\int_0^\infty e^{-x} L_m(x) L_n(x) dx = \left\{x e^{-x}[L_n L'_m - L_m L'_n]\right\}_0^\infty = 0$$

If $m \neq n$, $\int_0^\infty e^{-x} L_m(x) L_n(x)\, dx = 0$.

E30) a) Putting $\psi_n(x) = e^{-x^2/2} H_n(x)$ and $\psi'_n(x) = e^{-x^2/2} H'_n(x) - x e^{-x^2/2} H_n(x)$, in the given equation, it reduces to

$2nH_{n-1}(x) = H'_n(x)$ which is relation (74)

b) The given equation reduces to

$$2xH_n(x) - 2n H_{n-1}(x) = H_{n+1}(x)$$

replacing n by $(n-1)$ we get relation (76).

c) Substituting for $\psi_n(x)$ and its derivatives in the given equation, relation (75) is obtained.

E31) $\int_{-\infty}^{\infty} [\psi_n(x)]^2 dx = 1 \Rightarrow \int_{-\infty}^{\infty} c_n^2 e^{-x^2} [H_n(x)]^2 dx = 1$

$\Rightarrow c_n^2 2^n n! \sqrt{\pi} = 1$ or, $c_n = \frac{1}{\left[2^n n! \sqrt{\pi}\right]^{1/2}}$ [using result (78)]

$$\therefore \psi_n(x) = \left[\sqrt{\pi} 2^n n!\right]^{-\frac{1}{2}} e^{-x^2} H_n(x), \quad n = 0, 1, 2, \ldots.$$

—x—